Tables and summary
“Numerical Methods for Dynamic System and Control”

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Fourier Series

\[ S_f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \cos \frac{k\pi x}{\ell} + b_k \sin \frac{k\pi x}{\ell} \right), \]
\[ a_0 = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \, dx, \]
\[ a_k = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{k\pi x}{\ell} \, dx, \]
\[ b_k = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{k\pi x}{\ell} \, dx. \]

Z and Laplace transform

**Z Transform Table:** \( \sum_{k=0}^{\infty} f_k z^{-k} \)

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<tr>
<th>( k \ell = k(k-1) \cdots (k-\ell+1) = \frac{k!}{(k-\ell)!} )</th>
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Constrained minima and Lagrange multiplier

Consider the constrained minimization problem

\[
\begin{align*}
\text{minimize:} & \quad f(x) \\
\text{subject to:} & \quad h_i(x) = 0 \quad i = 1, 2, \ldots, m
\end{align*}
\]

Solution algorithm

- Compute the Lagrangian function: 
  \[ L(x, \lambda) = f(x) - \sum_{k=1}^{m} \lambda_k h_k(x) \]
- Solve the nonlinear system \( \nabla_x L(x, \lambda) = 0^T \) with \( h(x) = 0 \).
- For each solution points \( (x^*, \lambda^*) \) compute \( \nabla h(x^*) \) and check it is full rank, or the rows are linearly independent.
- Compute the matrix \( K \) the kernel of \( \nabla h(x^*) \), i.e. \( \nabla h(x^*)K = 0 \).
- Compute the reduce Hessian
  \[ H = K^T \nabla^2_x L(x^*, \lambda^*)K, \]
  - Necessary condition: \( H \) is semi-positive definite.
  - Sufficient condition: \( H \) is positive definite.

The following theorem prove the sufficient condition.

**Theorem 1 (of Lagrange multiplier)** Let \( f \in C^2(\mathbb{R}^n, \mathbb{R}) \) a map and \( x^* \) a local minima of \( f(x) \) satisfying the constraints \( h \in C^2(\mathbb{R}^n, \mathbb{R}^m) \), i.e. \( h(x^*) = 0 \). If \( \nabla h(x^*) \) is full rank then there exists \( m \) scalars \( \lambda_k \) such that

\[
\nabla_x L(x^*, \lambda) = \nabla f(x^*) - \sum_{k=1}^{m} \lambda_k \nabla h_k(x^*) = 0^T \quad (A)
\]

moreover, for all \( z \in \mathbb{R}^n \) which satisfy \( \nabla h(x^*)z = 0 \) it follows

\[
z^T \nabla^2_x L(x^*, \lambda)z = z^T \left( \nabla^2 f(x^*) - \sum_{k=1}^{m} \lambda_k \nabla^2 h_k(x^*) \right)z \geq 0 \quad (B)
\]

in other words the matrix \( \nabla^2_x (f(x^*) - \lambda \cdot h(x^*)) \) is semi-SPD in the Kernel of \( \nabla h(x^*) \).
Proof. Let $x^*$ a local minima, then there exists $\varepsilon > 0$ such that
\[ f(x^*) \leq f(x), \quad \text{for all } x \in B \text{ with } h(x) = 0, \]  
where $B = \{ x | \| x - x^* \| \leq \varepsilon \}$. Consider thus, the functions sequence
\[ f_k(x) = f(x) + k\| h(x) \|^2 + \alpha \| x - x^* \|^2, \quad \alpha > 0 \]  
with the corresponding sequence of (unconstrained) local minima in $B$:
\[ x_k = \arg\min_{x \in B} f_k(x). \]

The sequence $x_k$ is contained in the compact ball $B$ and from compactness there exists a converging sub-sequence $x_{k_j} \to \bar{x} \in B$. The rest of the proof to verify that $\bar{x} = x^*$ and it a minimum.

**Step 1:** $h(\bar{x}) = 0$. Notice that the sequence $x_k$ satisfy $f_k(x_k) \leq f(x^*)$, in fact
\[ f_k(x_k) \leq f_k(x^*) = f(x^*) + k\| h(x^*) \|^2 + \alpha \| x^* - x^* \|^2 = f(x^*). \]
and by definition (2) we have
\[ k_j\| h(x_{k_j}) \|^2 + \alpha \| x_{k_j} - x^* \|^2 \leq f(x^*) - f(x_{k_j}) \]
\[ \leq f(x^*) - \min_{x \in B} f(x) = C < +\infty \]  
so that
\[ \lim_{j \to \infty} \| h(x_{k_j}) \| = 0 \Rightarrow \| h \left( \lim_{j \to \infty} x_{k_j} \right) \| = \| h(\bar{x}) \| = 0 \Rightarrow h(\bar{x}) = 0. \]

**Step 2:** $\bar{x} = x^*$. From (3)
\[ \alpha \| x_{k_j} - x^* \|^2 \leq f(x^*) - f(x_{k_j}) - k_j\| h(x_{k_j}) \|^2 \leq f(x^*) - f(x_{k_j}) \]
and taking the limit
\[ \alpha \| \lim_{j \to \infty} x_{k_j} - x^* \|^2 \leq \alpha \| \bar{x} - x^* \|^2 \leq f(x^*) - \lim_{j \to \infty} f(x_{k_j}) \leq f(x^*) - f(\bar{x}) \]
From $\| h(\bar{x}) \| = 0$ it follows that from (1) that $f(x^*) \leq f(\bar{x})$ and
\[ \alpha \| \bar{x} - x^* \|^2 \leq f(x^*) - f(\bar{x}) \leq 0 \]
and, thus $\bar{x} = x^*$. 

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**Step 3: Build multiplier.** Cause $x_{kj}$ are *unconstrained local minima* for $f_k(x)$ it follows
\[
\nabla f_k(x_{kj}) = \nabla f(x_{kj}) + k_j \nabla \|h(x_{kj})\|^2 + \alpha \nabla \|x_{kj} - x^*\|^2 = 0
\]
remembering that
\[
\nabla \|x\|^2 = \nabla (x \cdot x) = 2x^T,
\]
\[
\nabla \|h(x)\|^2 = \nabla (h(x) \cdot h(x)) = 2h(x)^T \nabla h(x),
\]
it follows (doing transposition)
\[
\nabla f(x_{kj})^T + 2k_j \nabla h(x_{kj})^T h(x_{kj}) + 2\alpha (x_{kj} - x^*) = 0. \tag{4}
\]
Left multiplying by $\nabla h(x_{kj})$
\[
\nabla h(x_{kj}) \left[ \nabla f(x_{kj})^T + 2\alpha (x_{kj} - x^*) \right] + 2k_j \nabla h(x_{kj}) \nabla h(x_{kj})^T h(x_{kj}) = 0
\]
Cause $\nabla h(x^*) \in \mathbb{R}^{mn\times m}$ is full rank for $j$ large by continuity $\nabla h(x_{kj})$ is full rank and thus $\nabla h(x_{kj}) \nabla h(x_{kj})^T \in \mathbb{R}^{mn\times mn}$ are nonsingular, thus
\[
2k_j h(x_{kj}) = -\left( \nabla h(x_{kj}) \nabla h(x_{kj})^T \right)^{-1} \nabla h(x_{kj}) \left[ \nabla f(x_{kj})^T + 2\alpha (x_{kj} - x^*) \right]
\]
taking the limit for $j \to \infty$
\[
\lim_{j \to \infty} 2k_j h(x_{kj}) = -\left( \nabla h(x^*) \nabla h(x^*)^T \right)^{-1} \nabla h(x^*) \nabla f(x^*)^T = -\lambda \tag{5}
\]
and taking the limit of (4) with (5) we have $\nabla f(x^*)^T - \nabla h(x^*)^T \lambda = 0$.

**Step 4: Build a special sequence of $z_j$.** We needs a sequence $z_j \to z$ such that $\nabla h(x_{kj}) z_j = 0$ for all $j$. The sequence $z_j$ is built as the projection of $z$ into the Kernel of $\nabla h(x_{kj})$, i.e.
\[
z_j = z - \nabla h(x_{kj})^T \left[ \nabla h(x_{kj}) \nabla h(x_{kj})^T \right]^{-1} \nabla h(x_{kj}) z
\]
in fact
\[
\nabla h(x_{kj}) z_j = \nabla h(x_{kj}) z - \nabla h(x_{kj}) \nabla h(x_{kj})^T \left[ \nabla h(x_{kj}) \nabla h(x_{kj})^T \right]^{-1} \nabla h(x_{kj}) z
\]
\[
= \nabla h(x_{kj}) z - \nabla h(x_{kj}) z = 0
\]
consider now the limit
\[
\lim_{j \to \infty} z_j = z - \lim_{j \to \infty} \nabla h(x_{kj})^T \left[ \nabla h(x_{kj}) \nabla h(x_{kj})^T \right]^{-1} \nabla h(x_{kj}) z
\]
\[
= z - \nabla h(x^*)^T \left[ \nabla h(x^*) \nabla h(x^*)^T \right]^{-1} \nabla h(x^*) z
\]
and thus, if \( z \) is in the kernel of \( \nabla h(x^*) \), i.e. \( \nabla h(x^*)z = 0 \) we have

\[
\nabla h(x_k)z_j = 0 \quad \text{with} \quad \lim_{j \to \infty} z_j = z.
\]

**Step 5: Necessary conditions.** Cause \( x_{k_j} \) are unconstrained local minima for \( f_{k_j}(x) \) it follows that matrices \( \nabla^2 f_{k_j}(x_{k_j}) \) are semi positive defined, i.e.

\[
z^T \nabla^2 f_{k_j}(x_{k_j})z \geq 0, \quad \forall z \in \mathbb{R}^n
\]

moreover

\[
\begin{align*}
\nabla^2 f_{k_j}(x_{k_j}) &= \nabla^2 f(x_{k_j}) + k \nabla^2 \|h(x_{k_j})\|^2 + 2\alpha \nabla(h(x_{k_j}) - x^*) \\
&= \nabla^2 f(x_{k_j}) + k \nabla^2 \sum_{i=1}^{m} h_i(x_{k_j})^2 + 2\alpha \mathbf{I}
\end{align*}
\]

(6)

using the identity

\[
\nabla^2 h(x)^2 = \nabla(2h(x)\nabla h(x)^T) = 2\nabla h(x)^T \nabla h(x) + 2h(x)\nabla^2 h(x)
\]

in (8)

\[
\nabla^2 f_{k_j}(x_{k_j}) = \nabla^2 f(x_{k_j}) + 2k \sum_{i=1}^{m} \nabla h_i(x_{k_j})^T \nabla h_i(x_{k_j}) + 2k \sum_{i=1}^{m} h_i(x_{k_j}) \nabla^2 h_i(x_{k_j}) + 2\alpha \mathbf{I}
\]

Let \( z \in \mathbb{R}^n \) then

\[
0 \leq z^T \nabla^2 f_{k_j}(x_{k_j})z,
\]

i.e.

\[
0 \leq z^T \nabla^2 f(x_{k_j})z + \sum_{i=1}^{m} (2k \lambda_i h_i(x_{k_j})) z^T \nabla^2 h_i(x_{k_j})z + 2k \| \nabla h(x_{k_j})z \|^2 + 2\alpha \| z \|^2
\]

Inequality is true for all \( z \in \mathbb{R}^n \) and thus for any \( z \) in the kernel of \( \nabla h(x^*) \). Choosing \( z \) in the kernel of \( \nabla h(x^*) \) from previous step the sequence \( z_j \) satisfy

\[
0 \leq z_j^T \nabla^2 f(x_{k_j})z_j + \sum_{i=1}^{m} (2k \lambda_i h_i(x_{k_j})) z_j^T \nabla^2 h_i(x_{k_j})z_j + 2\alpha \| z_j \|^2
\]

and taking the limit \( j \to \infty \) with (5)

\[
0 \leq z^T \nabla^2 f(x^*)z + \sum_{i=1}^{m} \lambda_i z^T \nabla^2 h_i(x^*)z + 2\alpha \| z \|^2
\]
cause $\alpha > 0$ can be chosen arbitrarily it follows

$$0 \leq z^T \nabla^2 f(x^*) z - \sum_{i=1}^{m} \lambda_i \left[ z^T \nabla^2 h_i(x^*) z \right]$$

which is the relation to be proved. □

**Inequality constraints**

It is possible to adapt theorem 1 for inequality constraints. Consider the NLP problem

- **minimize:** $f(x)$
- **subject to:** $h_i(x) = 0 \quad i = 1, 2, \ldots, m$
  $g_j(x) \geq 0 \quad i = 1, 2, \ldots, p$

introducing the slack variables $e_i, i = 1, 2, \ldots, p$ and $y^T = (x^T, e^T)$ the new problem

- **minimize:** $f(y) = f(x)$
- **subject to:** $h_i(y) = h_i(x) = 0 \quad i = 1, 2, \ldots, m$
  $h_{i+m}(y) = g_i(x) - e_i^2 = 0 \quad i = 1, 2, \ldots, p$

with the Lagrangian function:

$$\mathcal{L}(x, e, \lambda, \mu) = f(x) - \sum_{k=1}^{m} \lambda_k h_k(x) - \sum_{k=1}^{p} \mu_k \left( g_k(x) - e_k^2 \right)$$

The first order condition becomes

$$\nabla_x \mathcal{L}(x^*, e, \lambda, \mu) = \nabla f(x^*) - \sum_{k=1}^{m} \lambda_k \nabla h_k(x^*) - \sum_{k=1}^{p} \mu_k \nabla g_k(x^*) = 0^T,$$

$$\nabla e \mathcal{L}(x^*, e, \lambda, \mu) = 2(\mu_1 e_1, \ldots, \mu_p e_p) = 0^T,$$

$$h_k(x^*) = 0,$$

$$g_k(x^*) = e_k^2 \geq 0,$$

and second order condition become $z^T \nabla^2 \mathcal{L}_{(x,e)}(x^*, e, \lambda, \mu) z \geq 0$ for $z$ in the kernel of matrix

$$\begin{pmatrix}
    \nabla_x h(x^*) & 0 \\
    \nabla_x g(x^*) & 2 \text{diag}(e_1, \ldots, e_p)
\end{pmatrix} \quad (7)$$
where
\[
\nabla^2_{(x, e)} L(x^*, e, \lambda, \mu) z = \begin{pmatrix}
\nabla^2_{z} L(x^*, e, \lambda, \mu) & 0 \\
0 & \nabla^2_{z} L(x^*, e, \lambda, \mu)
\end{pmatrix}
\]
(8)
and \( \nabla_z \nabla^T_{(x, e)} L(x^*, e, \lambda, \mu) = 0 \), moreover
\[
\nabla^2_{z} L(x^*, e, \lambda, \mu) = 2 \text{diag}(\mu_1, \mu_2, \ldots, \mu_p).
\]

Notice that \( \mu_k e_k = 0 \) is equivalent of \( \mu_k e_k^2 = 0 \) and thus \( \mu_k g_k(x*) = 0 \). So that when \( g_k(x*) > 0 \) then \( \mu_k = 0 \). Up to a reordering we split
\[
g(x) = \begin{pmatrix}
g^{(1)}(x) \\
g^{(2)}(x)
\end{pmatrix}
\]
and thus (7) becomes
\[
\begin{pmatrix}
\nabla_x h(x*) & 0 & 0 \\
\nabla_x g^{(1)}(x*) & 0 & 0 \\
\nabla_x g^{(2)}(x*) & 0 & 0
\end{pmatrix}
\]
\begin{pmatrix}
2 \text{diag}(e_{k+1}, \ldots, e_p) = E.
\end{pmatrix}
(9)
and
\[
\nabla^2_{z} L(x^*, e, \lambda, \mu) = \begin{pmatrix}
M & 0 \\
0 & 0
\end{pmatrix}, \quad M = 2 \text{diag}(\mu_1, \mu_2, \ldots, \mu_r)
\]
(10)

The group of constraints \( g^{(1)}(x*) \) that are zeros are the active constraints. The kernel of (9) can be written as
\[
\mathcal{K} = \begin{pmatrix}
K & 0 \\
0 & I \\
-E^{-1} \nabla_x g^{(2)}(x*) K & 0
\end{pmatrix}, \quad K \text{ is the kernel of } \begin{pmatrix}
\nabla_x h(x*) \\
\nabla_x g^{(1)}(x*)
\end{pmatrix}
(11)
where \( K \) is the kernel of the matrix
\[
\begin{pmatrix}
\nabla_x h(x*) \\
\nabla_x g^{(1)}(x*)
\end{pmatrix}
\]
thus $z$ can be written as $Kd$ and thus second order condition $z^T \nabla^2_{(x,e)} \mathcal{L}(x^*, e, \lambda, \mu) z \geq 0$ become

$$0 \leq d^T \left[ K^T \nabla^2_{(x,e)} \mathcal{L}(x^*, e, \lambda, \mu) K \right] d, \quad d \in \mathbb{R}^s$$

and using (11) with (8) and (10)

$$\left[ K^T \nabla^2_{(x,e)} \mathcal{L}(x^*, e, \lambda, \mu) K \right] = K^T \begin{pmatrix} \nabla^2_{x} \mathcal{L}(x^*, e, \lambda, \mu) & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{pmatrix} K,$$

$$= \begin{pmatrix} K^T \nabla^2_{x} \mathcal{L}(x^*, e, \lambda, \mu) K & 0 \\ 0 & M \end{pmatrix}$$

Using the solution algorithm of the equality constrained problem we have

- Necessary condition: the matrices
  $$K^T \nabla^2_{x} \mathcal{L}(x^*, e, \lambda, \mu) K, \quad \text{and} \quad M$$
  must be semi-positive defined. This implies that $\mu_k \geq 0$ for $k = 1, 2, \ldots, p$

- Sufficient condition: the matrices
  $$K^T \nabla^2_{x} \mathcal{L}(x^*, e, \lambda, \mu) K, \quad \text{and} \quad M$$
  must be positive defined. This implies that $\mu_k > 0$ for the active constraints.
Constrained minima, NLP problem

Consider the constrained minimization problem

\[
\begin{align*}
    \text{minimize:} & \quad f(x) \\
    \text{subject to:} & \quad h_i(x) = 0 \quad i = 1, 2, \ldots, m \quad (12) \\
    & \quad g_i(x) \geq 0 \quad i = 1, 2, \ldots, p
\end{align*}
\]

Solution algorithm

- Compute the Lagrangian function:
  \[
  L(x, \lambda, \mu) = f(x) - \sum_{k=1}^{m} \lambda_k h_k(x) - \sum_{k=1}^{p} \mu_k g_k(x)
  \]

- Solve the nonlinear system
  \[
  \begin{align*}
  \nabla_x L(x, \lambda, \mu) &= 0^T \\
  \lambda_k h_k(x) &= 0 \quad k = 1, 2, \ldots, m \\
  \mu_k g_k(x) &= 0 \quad k = 1, 2, \ldots, p
  \end{align*}
  \]
  keep only the solutions with \(\mu^*_k \geq 0\) and \(g_k(x^*) \geq 0\).

- For each solution points \((x^*, \lambda^*, \mu^*)\) compute \(\nabla h(x^*)\) with \(\nabla g_k(x^*)\) where \(g_k(x^*) = 0\) are the active constraints with \(\mu_k > 0\) and check they are linearly independent.

- Compute matrix \(K\) the kernel of \(\nabla h(x^*)\) with \(\nabla g_k(x^*)\) where \(g_k(x^*) = 0\) are the active constraints with \(\mu_k > 0\).

- Compute the reduce Hessian
  \[
  H = K^T \nabla^2 \nabla L(x^*, \lambda^*) K,
  \]
  - Necessary condition: \(H\) is semi-positive definite.
  - Sufficient condition: \(H\) is positive definite and \(\mu_k > 0\) for all the active constraints.

Definition 1 The set

\[
\mathcal{F} = \{ x \in \mathbb{R}^n \mid h_k(x) = 0, \quad k = 1, 2, \ldots, m, \quad g_k(x) \geq 0, \quad k = 1, 2, \ldots, p, \}
\]

is called the feasible region or set of feasible points.
Definition 2 (Active set) The set \( \mathcal{A}(x) \) defined as
\[ \mathcal{A}(x) = \{ k \mid g_k(x) = 0 \} \]
is the set of active (unilateral) constraints.

Constrained minima general theorem and KKT

The following theorem (see [1]) give the necessary conditions for constrained minima. Notice that no condition on the constraints are necessary.

Theorem 2 (Fritz John) If the functions \( f(x), g_1(x), \ldots, g_p(x) \), are differentiable, then a necessary condition that \( x^\ast \) be a local minimum to problem:

\[
\begin{align*}
\text{minimize:} & \quad f(x) \\
\text{subject to:} & \quad g_i(x) \geq 0 \quad i = 1, 2, \ldots, p
\end{align*}
\]
is that there exist scalars \( \mu_0^\ast, \mu_1^\ast, \mu_p^\ast \), (not all zero) such that the following inequalities and equalities are satisfied:

\[
\nabla_x L(x^\ast, \mu^\ast) = 0^T \\
\mu_k^\ast g_k(x^\ast) = 0, \quad k = 1, 2, \ldots, p; \\
\mu_k^\ast \geq 0, \quad k = 0, 1, 2, \ldots, p;
\]

where

\[
L(x, \mu) = f(x) - \sum_{k=1}^{p} \mu_k g_k(x)
\]

In [2] Kuhn and Tucker showed that if a condition, called the first order constraint qualification, holds at \( x^\ast, \lambda^\ast \) then \( \lambda_0 \) can be taken equal to 1.

Definition 3 (Constraints qualification LI) Let be the unilateral and bilateral constraints \( g(x) \) and \( h(x) \), the point \( x^\ast \) is admissible if
\[
g_k(x^\ast) \geq 0, \quad h_k(x^\ast) = 0.
\]
The constraints \( g(x) \) and \( h(x) \) are qualified at \( x^\ast \) if the point \( x^\ast \) is admissible and the vectors

\[
\{ \nabla g_k(x^\ast) : k \in \mathcal{A}(x^\ast) \} \cup \{ \nabla h_1(x^\ast), \nabla h_2(x^\ast), \ldots, \nabla h_m(x^\ast) \}
\]
are linearly independent.
Definition 4 (Constraint qualification (Mangasarian-Fromovitz)) The constraints \( g(x) \) and \( h(x) \) are qualified at \( x^* \) if the point \( x^* \) is admissible and does not exist a linear combination

\[
\sum_{k \in A} \alpha_k \nabla g_k(x^*) + \sum_{k=1}^m \beta_k \nabla h_k(x^*) = 0
\]

with \( \alpha_k \geq 0 \) for \( k \in A(x^*) \) and \( \alpha_k \) with \( \beta_k \) not all 0. In other words, there not exists a non trivial linear combination of the null vector such that \( \alpha_k \geq 0 \) for \( k \in A(x^*) \).

The next theorems are taken from [3].

Theorem 3 (First order necessary conditions) Let \( f \in C^1(\mathbb{R}^n) \) and the constraints \( g \in C^1(\mathbb{R}^n, \mathbb{R}^p) \) and \( h \in C^1(\mathbb{R}^n, \mathbb{R}^m) \). Suppose that \( x^* \) is a local minima of (12) and that the constraints qualification LI holds at \( x^* \). Then there are Lagrange multiplier vectors \( \lambda \) and \( \mu \) such that the following conditions are satisfied at \((x^*, \lambda, \mu)\)

\[
\nabla_x L(x^*, \lambda^*, \mu^*) = 0^T
\]

\[
h_k(x^*) = 0, \quad k = 1, \ldots, m;
\]

\[
\mu_k^* g_k(x^*) = 0, \quad k = 1, \ldots, p;
\]

\[
\mu_k^* \geq 0, \quad k = 1, \ldots, p;
\]

where

\[
L(x, \lambda, \mu) = f(x) - \sum_{k=1}^m \lambda_k h_k(x) - \sum_{k=1}^p \mu_k^* g_k(x)
\]

Theorem 4 (Second order necessary conditions) Let \( f \in C^2(\mathbb{R}^n) \) and the constraints \( g \in C^2(\mathbb{R}^n, \mathbb{R}^p) \) and \( h \in C^2(\mathbb{R}^n, \mathbb{R}^m) \). Let \( x^* \) be a local minima of \( L(x, \lambda, \mu) \) satisfying the First order necessary conditions, a necessary condition for \( x^* \) be a local minima is that the \( m + p \) scalars (Lagrange Multiplier) of the first order necessary condition satisfy:

\[
d^T \nabla^2_x L(x^*, \lambda^*, \mu^*) d \geq 0
\]

for all \( d \) such that

\[
\nabla h_k(x^*) d = 0, \quad k = 1, \ldots, m
\]

\[
\nabla g_k(x^*) d = 0, \quad if k \in A(x^*) and \mu_k > 0
\]

\[
\nabla g_k(x^*) d \geq 0, \quad if k \in A(x^*) and \mu_k = 0
\]
Remark 1 The conditions
\[ \nabla g_k(x^\star) d = 0, \quad \text{if } k \in \mathcal{A}(x^\star) \text{ and } \mu_k > 0 \]
\[ \nabla g_k(x^\star) d \geq 0, \quad \text{if } k \in \mathcal{A}(x^\star) \text{ and } \mu_k = 0 \]
restrict the space of direction to be considered. If changed with
\[ \nabla g_k(x^\star) d = 0, \quad \text{if } k \in \mathcal{A}(x^\star) \]
theorems 4 is still valid cause necessary condition is tested in a smaller set.

Theorem 5 (Second order sufficient conditions) Let \( f \in C^2(\mathbb{R}^n) \) and the constraints \( g \in C^2(\mathbb{R}^n, \mathbb{R}^p) \) and \( h \in C^2(\mathbb{R}^n, \mathbb{R}^m) \). Let \( x^\star \) satisfying the First order necessary conditions, a sufficient condition for \( x^\star \) be a local minima is that the \( m + p \) scalars (Lagrange Multiplier) of the first order necessary condition satisfy:
\[ d^T \nabla^2_x \mathcal{L}(x^\star, \lambda^\star, \mu^\star) d > 0 \]
for all \( d \neq 0 \) such that
\[ \nabla h_k(x^\star) d = 0, \quad k = 1, 2, \ldots, m \]
\[ \nabla g_k(x^\star) d = 0, \quad \text{if } k \in \mathcal{A}(x^\star) \text{ and } \mu_k > 0 \]
\[ \nabla g_k(x^\star) d \geq 0, \quad \text{if } k \in \mathcal{A}(x^\star) \text{ and } \mu_k = 0 \]

Remark 2 The condition
\[ \nabla g_k(x^\star) d \geq 0, \quad \text{if } k \in \mathcal{A}(x^\star) \text{ and } \mu_k = 0 \]
restrict the space of direction to be considered. If omitted the theorems 5 is still valid cause sufficient condition is tested in a larger set.

References