

# Non-local structural interfaces

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## SUMMARY:

Mechanical interfaces joining linear elastic solids, that can model a continuum/discrete transition are analyzed. These interfaces are characterized by a finite thickness and given structural properties. Problems concerning elliptical inclusions and cracks are analytically solved.

## 1. INTRODUCTION

There are many mechanical problems involving interfaces joining different parts of a continuous body. Examples taken from biomechanics include: fibrous joints (suture, syndesmosis and gomphosis) and web-like trabeculae suspending the brain within the meninges. In solid mechanics, discrete structures joining continuous bodies can be found in cracks bridged by fibers, in the description of atomic interactions in contact (Movchan et al. 2003) and fracture mechanics (Gao et al. 2001). These transition zones are characterized by well-defined microstructures; for instance, a truss-like structure comprised of glass fibres bridging a crack in a short glass-fibre-reinforced polypropylene is shown in Fig. 1.



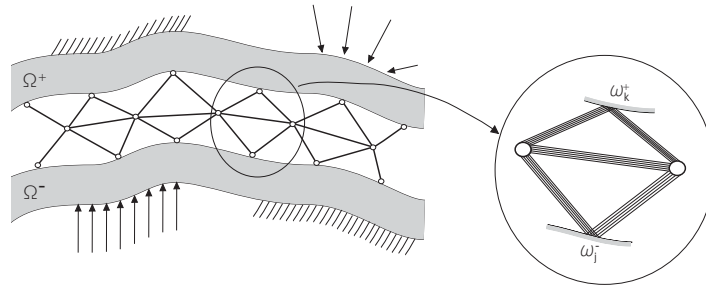
**Figure 1:** A bridged crack in short glass-fiber-reinforced polypropylene

Usually, mechanical interfaces are modeled by employing the concept of a zero-thickness imperfect interface. Interfacial non-linearity may be introduced to model different situations of interest [see for instance applications in: fragmentation and decohesion (Camacho and Ortiz, 1996; Needleman, 1992; Rice and Wang, 1989; Drugan, 2001), interactions between inclusions (Levy and Hardikar, 1999), bifurcation (Radi et al. 1999), composites (Levy and Dong, 1998; Lipton and Talbot, 2001), biomechanics (Mann et al. 1997; Gei et al. 2002)] and may avoid the interpenetration by the introduction of a suitable penalty in compression. However, mechanical interfaces in reality are

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characterized by a finite thickness and structural properties, which are often overly simplified by the zero-thickness model. Two different efforts have been made to provide more refined models of thick interfaces, namely, Rubin and Benveniste (2004) have introduced a ‘transition-layer interface’ based on a Cosserat-shell description, whereas Bigoni and Movchan (2002) have suggested modelling of the interface as a truly discrete structure. In particular, the concept of structural interface has been introduced, possessing a finite width and specific mechanical properties (see the sketch in Fig. 2, specialized to two-dimensional deformations for simplicity). The interest in this proposal is that the incorporation of a specific structure in general introduces *non-local effects* and that these follow from



**Figure 2:** A model of structural interface

the description in a natural and rational way. In other words, while a *zero-thickness* interface model is a phenomenological model, a structural interface provides a direct description of the relevant microstructure. The model proposed by Bigoni and Movchan (called ‘BM-model’ in the following) defines a quasi-local interface in the sense that non-locality is confined to opposite points on the interface. However, the BM-model is generalized in the present paper to incorporate the full non-locality induced by a generic truss structure joining two continuous media, comprised for simplicity of isotropic, linear elastic materials and loaded at their boundaries (Fig. 2). In this discrete/continuum problem, a difficulty arises in describing the contact between the structure and the elastic media. This is overcome by working with mean values of displacements and tractions at the joints and assuming that junction is characterized by a small dimension. The paper is organized as follows. The model of a structural interface is introduced in Section 1. We specialize then to the case of an elliptical inclusion connected by a structural interface to an infinite medium loaded by a uniform remote stress. This case can be solved analytically by employing complex potentials (Muskhelishvili, 1953), and it permits us to systematically investigate the effects of the interfacial nonlocality (Section 3).

## 2. GOVERNING EQUATIONS FOR A STRUCTURAL INTERFACE

Two elastic continuous bodies connected by a structural interface represent a model of a *multi-structure*, namely, an elastic multidimensional body. These types of structure are very common in many fields of engineering and have recently received much attention (Ciarlet et al. 1989; Ciarlet, 1990, 1997; Puel and Zuazua, 1993; Argatov and Nazarov, 1994; Conca and Zuazua, 1994; Kozlov and al. 1999, 2000; Nardocchi and Podio-Guidugli, 2001; Nardocchi, 2002). Inspired by the above formulations, a simple model for the analysis of two-dimensional multi-structures is developed below. This is based on the following assumptions.

- *Solid/structure junctions.* Let us denote with  $\Omega^-$  and  $\Omega^+$  (with boundary  $\partial\Omega^-$  and  $\partial\Omega^+$ , respective-ly) the two continuous linear elastic two-dimensional bodies connected by the structural interface. The  $j$ -th junction between a bar and the continuous body is represented by a contact region  $\omega_j^\pm$  on  $\partial\Omega$

$$\omega_j^\pm = \partial\Omega^\pm \cap B(\mathbf{x}_j^\pm, \rho), \quad (1)$$

where  $B(\mathbf{x}_j^\pm, \rho)$  is the disk of radius  $\rho$  centred at  $\mathbf{x}_j^\pm$  and  $\pm$  stands for either + or -. In addition, it is assumed that  $\partial\Omega^\pm$  possesses continuous curvature near  $\mathbf{x}_j^\pm$ , so that there exists a class  $C^2$  function

$$\mathbf{z}_j^\pm : [-s_j, s_j] \rightarrow \omega_j^\pm, \quad \mathbf{x}^\pm = \mathbf{z}_j^\pm(s), \quad (2)$$

mapping the segment  $[-s_j, s_j]$  into the contact region  $\omega_j^\pm$  and transforming  $s=0$  into the point  $\mathbf{x}_j^\pm$ .

At the junction, the load is transmitted as if the bar were a “linear, filamentary structure”, connecting the junctions of central points  $\mathbf{x}_k^+$  and  $\mathbf{x}_j^-$ , and defined by the direction singled out by the unit vector  $\mathbf{a}_{(kj)}$ ,

$$\mathbf{a}_{(kj)} = \frac{\mathbf{x}^+ - \mathbf{x}^-}{|\mathbf{x}^+ - \mathbf{x}^-|}, \quad \mathbf{x}^+ \in \omega_k^+, \quad \mathbf{x}^- \in \omega_j^-. \quad (3)$$

The traction  $\mathbf{t}^k$  transmitted at the  $k$ -junction is assumed to be a linear function of the bar elongation, so that

$$\mathbf{t}^k(\mathbf{x}^+) = k_{(jk)} [(\mathbf{u}(\mathbf{x}^+) - \mathbf{u}(\mathbf{x}^-)) \cdot \mathbf{a}_{(kj)}] \mathbf{a}_{(kj)}, \quad \mathbf{x}^+ \in \omega_k^+, \quad \mathbf{x}^- \in \omega_j^-, \quad (4)$$

where  $k_{(jk)}$  is the stiffness coefficient relative to the filament connecting points  $\mathbf{x}^+$  and  $\mathbf{x}^-$ .

In addition, the equilibrium of the filament  $kj$  requires that

$$\mathbf{t}^k(\mathbf{x}^+) = \mathbf{t}^j(\mathbf{x}^-), \quad \mathbf{x}^+ \in \omega_k^+, \quad \mathbf{x}^- \in \omega_j^-. \quad (5)$$

Since in a junction  $k$  different bars can converge, each characterized by a certain stiffness  $k_{jk}$  and inclination  $\mathbf{a}_{(kj)}$ , eqn. (4) is replaced by

$$\mathbf{t}^k(\mathbf{x}^+) = \sum_{j=1}^m k_{jk} [(\mathbf{u}(\mathbf{x}^+) - \mathbf{u}(\mathbf{x}^-)) \cdot \mathbf{a}_{(kj)}] \mathbf{a}_{(kj)}, \quad \mathbf{x}^+ \in \omega_k^+, \quad \mathbf{x}^- \in \omega_j^-. \quad (6)$$

- *The boundary value problem.* The stress field  $\sigma$  in the continuous elastic bodies satisfies

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma}(\mathbf{x}) = \mathbf{0}, & \mathbf{x} \in \Omega^\pm, \\ \boldsymbol{\sigma}(\mathbf{x})\mathbf{n}(\mathbf{x}) = \sum_{j=1}^m k_{jk} [(\mathbf{u}(\mathbf{x}^+) - \mathbf{u}(\mathbf{x}^-)) \cdot \mathbf{a}_{(kj)}] \mathbf{a}_{(kj)}, & \mathbf{x} \in \omega_k^\pm, \\ \text{prescribed tractions or displacements on } \partial\Omega^\pm \setminus (\cup_k \omega_k^\pm), & \end{cases} \quad (7)$$

where  $\mathbf{n}$  is the outward unit normal to the boundary of the two connected bodies and  $k$  is an integer ranging between 1 and the number of junctions  $N$ , possibly taking different values on  $\partial\Omega^+$  and  $\partial\Omega^-$ .

- *Problem formulation in terms of averaged tractions and displacements at the junctions.*  
A simplification of the problem (7) is pursued here by working with averaged quantities at the junctions between bars and bodies. To this purpose, we introduce the averaged tractions and displacements at junctions as (here the superscript  $\pm$  is omitted for conciseness)

$$\begin{aligned} \bar{\mathbf{t}}^j &= \frac{1}{|\omega_j|} \int_{\omega_j} \mathbf{t}(\mathbf{x}) = \frac{1}{|\omega_j|} \int_{-s_j}^{s_j} \mathbf{t}(z_j(s)) |z'_j(s)| ds, \\ \bar{\mathbf{u}}^j &= \frac{1}{|\omega_j|} \int_{\omega_j} \mathbf{u}(\mathbf{x}) = \frac{1}{|\omega_j|} \int_{-s_j}^{s_j} \mathbf{u}(z_j(s)) |z'_j(s)| ds, \end{aligned} \quad (8)$$

so that the boundary value problem (7) becomes

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma}(\mathbf{x}) = \mathbf{0}, & \mathbf{x} \in \Omega^\pm, \\ \boldsymbol{\sigma}(\mathbf{x})\mathbf{n}(\mathbf{x}) = \sum_{j=1}^m k_{jk} [(\bar{\mathbf{u}}^+ - \bar{\mathbf{u}}^-) \cdot \mathbf{a}_{(kj)}] \mathbf{a}_{(kj)}, & \mathbf{x} \in \omega_k^\pm, \\ \text{prescribed tractions or displacements on } \partial\Omega^\pm \setminus (\cup_k \omega_k^\pm). & \end{cases} \quad (9)$$

Since we will substitute the solution of problem (7) by the solution of the simpler problem (9), a digression is now needed to comment on the difference between the two solutions. In particular, we will assume that the extension of the junction zones will be small when compared to the dimension of the connected bodies, so that these zones “contract” on points  $\mathbf{x}_j$ .

It can be shown then that employing results due to von Mises (1945), Sternberg (1954) and Gurtin (1972) the elastic energy evaluated by solving (7) differs from that evaluated by solving (9) by terms  $O(s_j^2)$ .

Problem (9) is employed as the basis for analysing structural interfaces. We proceed as follows:

- the solution for a uniform traction distribution over the junction regions  $\omega_k$  are constructed (usually via a Green's function);
- for both the elastic solids connected by the interface, the displacements at the junctions points  $\mathbf{u}(\mathbf{x}_k)$  are written as functions of the (for the moment unknown) uniform tractions  $\mathbf{t}(\mathbf{x}_k)$  applied at joint regions and of the prescribed “external” boundary conditions, in terms of given tractions and displacements. This corresponds to the solution of eqns. (9), with the second equation replaced by

$$\boldsymbol{\sigma}(\mathbf{x})\mathbf{n}(\mathbf{x}) = \mathbf{t}(\mathbf{x}_j), \quad \mathbf{x} \in \omega_k^\pm. \quad (10)$$

- the conditions of equilibrium of all the nodes of the truss structure defining the interface are imposed. To this purpose, eqn. (9)<sub>2</sub> is employed to enforce equilibrium for the nodes located at the junctions with the solid bodies, whereas for nodes internal to the structure, equilibrium is imposed using classical methods of structural mechanics. All tractions at junctions and normal stresses within the bars are obtained in this way.

### 3. RESULTS

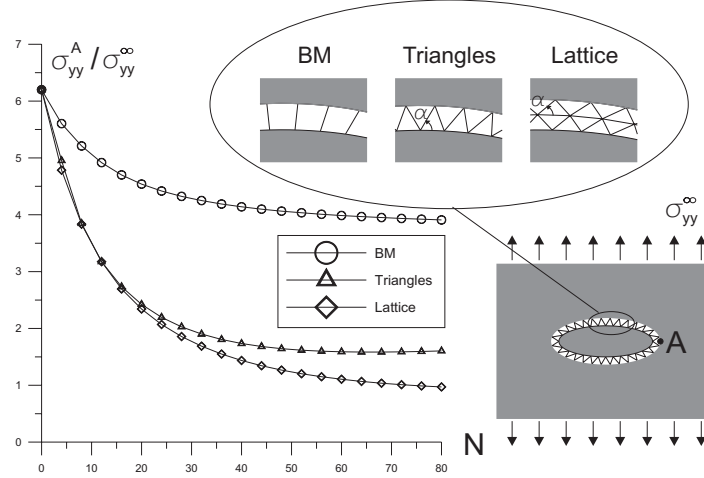
In order to apply the methodology described in Sect. 2 to the problem of an elliptical elastic inclusion connected by a structural interface to an infinite elastic sheet, we need some preliminary results. These are the three solutions corresponding to

- a self-equilibrated distribution of a number of uniform loadings acting on an elliptical elastic disk;
- a self-equilibrated distribution of a number of uniform loadings acting on an elliptical hole in an infinite elastic sheet;
- an elliptical hole in an infinite elastic sheet, loaded by a remote uniform stress.

The solution of the above problems has been found by employing the Kolosov-Muskhelishvili (1953) complex potentials technique.

#### 3.1. Elliptical inclusion connected to an infinite matrix by a structural interface

As a first example showing the capabilities of the proposed approach, we consider an elliptical inclusion with semi-axes  $a^-=12d$  and  $b^-=4d$ , connected to an infinite matrix with an elliptical hole with semi-axes  $a^+=a^-+d$  and  $b^+=b^-+d$ . The two materials forming the ellipse and the matrix are characterized by the same value of Poisson ratio  $\nu = 0.3$  and  $\mu^-/\mu^+=10$ , so that the matrix is more compliant than the inclusion. A uniform stress  $\sigma_{yy}^\infty=\mu^+/100$  is applied at infinity. Three different types of structural interface have been considered: the discrete BM-model, a warren truss structure and a hexagonal lattice structure. The bars have a stiffness  $k=\mu^-/b^- = 10$  in the case of the BM-model, whereas they have a stiffness equal to  $k/(2 \sin^2\alpha)$  (with  $\alpha$  shown in the detail of Fig. 3) for the triangular structures and  $k(1+\cos^2\alpha)/\sin^2\alpha$  for the hexagonal lattice, providing the same global stiffness in the radial direction. The stress concentration, namely,  $\sigma_{yy}$  at the point A, is plotted in Fig. 3 for different densities of interfacial bars  $N$  ( $2N$  in the case of triangles and  $5N$  for lattice structures, but with the same global stiffness).



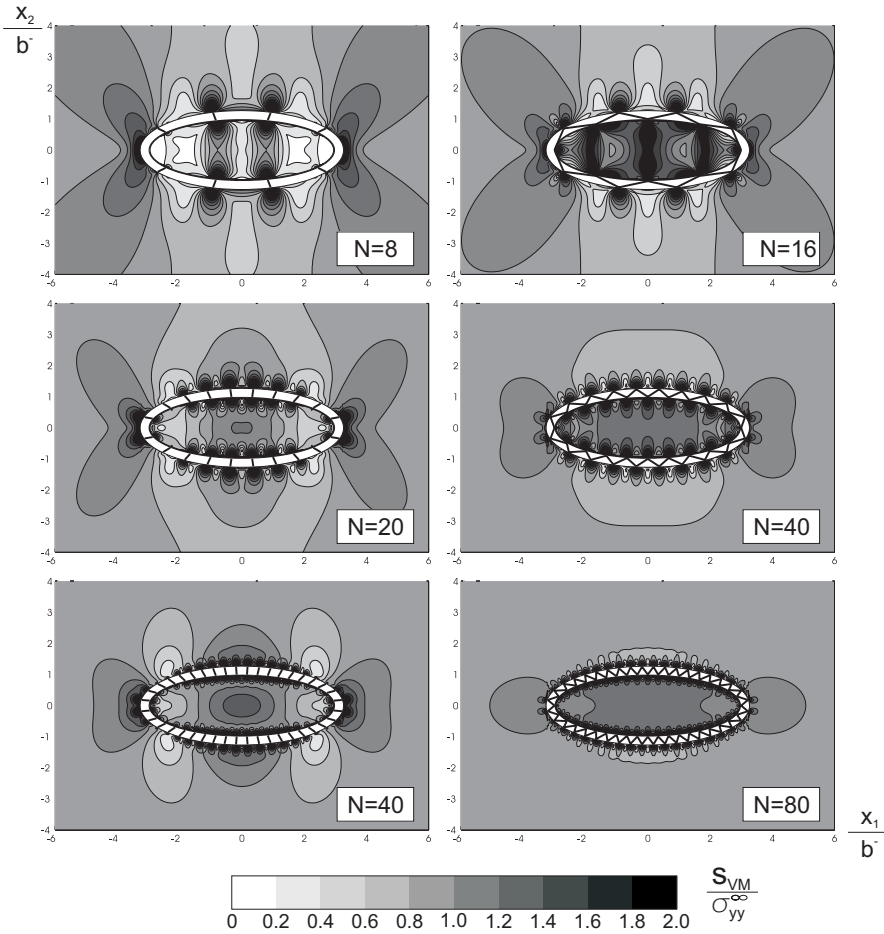
**Figure 3:** Stress concentration for an elliptical inclusion, loaded under uniaxial (vertical) stress, as a function of interfacial morphology and density of bars.

It can be seen from the figure that the stress concentration is stronger for the discrete BM-model than for the others, where the non-locality introduced by the inclination of the bars decreases the severity of the stress field. For an elliptical hole the maximum stress concentration is given by the well-known formula

$$\frac{\sigma_{yy}}{\sigma_{yy}^\infty} = 1 + 2\frac{a}{b}, \quad (11)$$

so that in our case it is equal to 6.2, the value approached in the graphs when the number of bars approaches zero, while at increasing number of bars, the stress intensity tends to an asymptotic value.

We note also that there is not much difference between results obtained employing the Warren truss structure and the hexagonal lattice, so that only the cases of the discrete BM-model and of the Warren truss structure are considered in Figs. 4, where level sets of the von Mises stress (made dimensionless through division by  $\sigma_{yy}^\infty$ ) are plotted in a region near the inclusion.

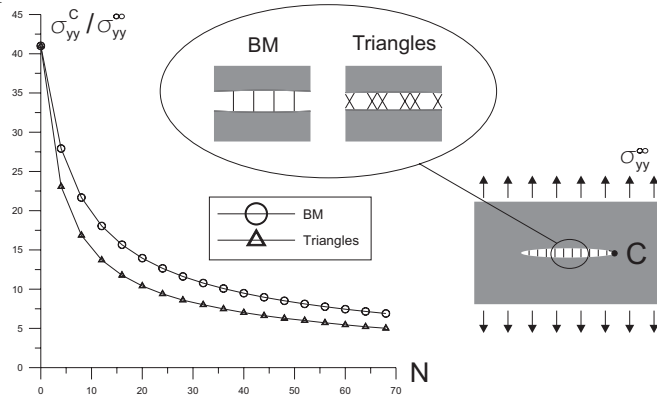


**Figure 4:** Von Mises stress distribution for an elliptical inclusion, loaded under remote uniaxial (vertical) stress, as a function of interfacial morphology (discrete BM-model left, Warren truss structure right) and density (number of bars is increasing up to down) of the bars.

### 3.2 Fiber reinforced elliptical hole

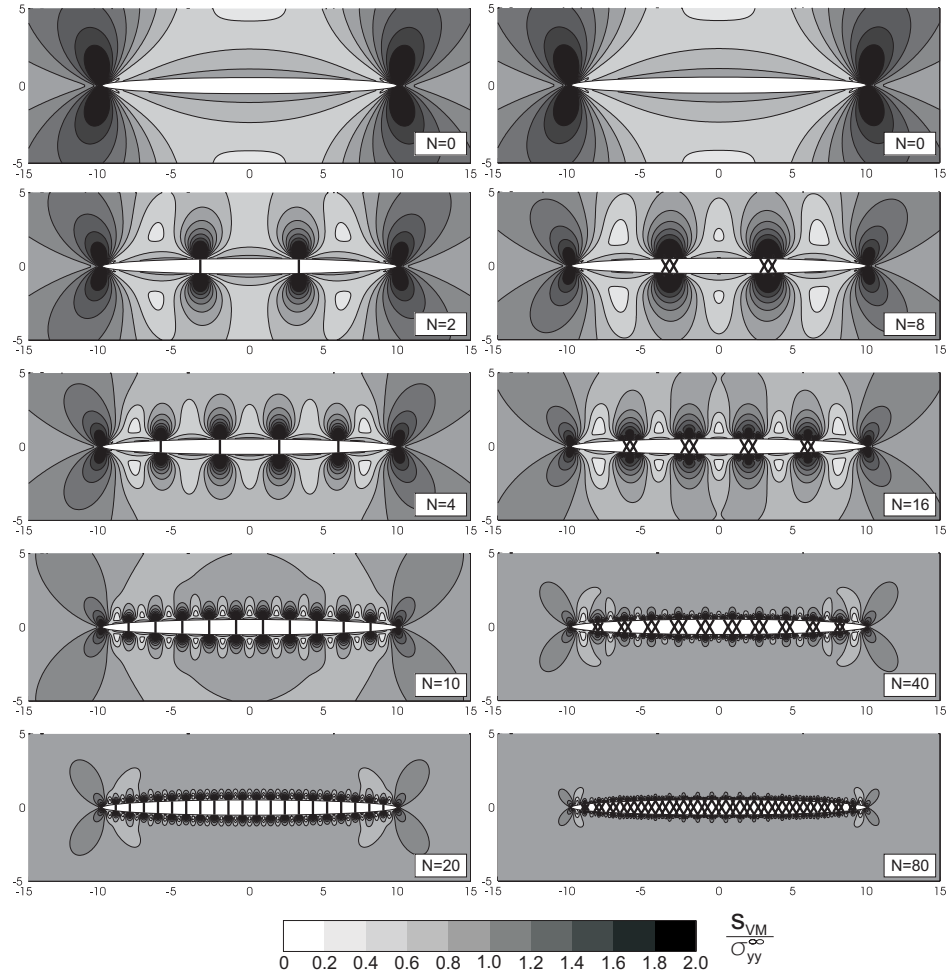
The case of an elliptical hole reinforced with discrete fibres (linear bars) can also be analysed within our framework. This case is particularly interesting in view of the implications on crack bridging and toughening of brittle materials with fibres. An infinite elastic matrix is considered,  $\nu = 0.3$ , with an elliptical hole of high aspect ratio  $a/b=20$ , so that the geometry is approaching that of a crack. A uniform uniaxial stress  $\sigma_{yy}^{\infty} = \mu/100$  is applied at infinity. Bars of stiffness  $k b / \mu = 10$  are considered. Two geometries of fibers have been considered: purely vertical and inclined. The latter geometry has been defined to provide the same vertical stiffness with two mirror truss elements inclined at  $30^\circ$  with respect to the vertical (see the detail in Fig. 5), corresponding to a stiffness

$k/(4\cos^2 \pi/6)$  of each bar. Obviously, the truss element also has a horizontal stiffness. The stress concentration for the elliptical hole is plotted for different bar densities in Fig. 5, ranging between 0 and 70 bars for the vertical fibre model and 0 and 280 for the truss fibre model. It is evident that the increase of bar density relieves the stress at the tip, although the two structures perform more or less equivalently in this representation.



**Figure 5:** Stress concentration for an elliptical hole reinforced with fibers and loaded under uniaxial (vertical) stress. Effect of morphology (vertical fibers are considered in the left part of the figure, while inclined on the right) and density of fiber.

The difference between the predictions of the two models appears evident in Fig. 6, where the level sets of the von Mises stress (divided by  $\sigma_{yy}^{\infty}$ ) are reported for different bar densities. In particular, the number N of bars is taken equal to  $\{0, 2, 4, 10, 20\}$  and  $\{0, 8, 16, 40, 80\}$ , respectively, in the left (vertical fibers) and right part (inclined fibers) of Fig. 6. It can be seen that the inclined bar model provides a better stress relief around the elliptical hole.



**Figure 6:** Von Mises stress (divided by remote stress) for the reinforced elliptical hole loaded by a uniaxial stress  $\sigma_{yy}^{\infty}$ . Parallel vertical fibers (left) and inclined fibers (right) are considered with the same global vertical stiffness.

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