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Mindlin second-gradient elastic properties from dilute two-phase Cauchy-elastic composites Part II: Higher-order constitutive properties and application cases

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Abstract

Starting from a Cauchy elastic composite with a dilute suspension of randomly distributed inclusions and characterized at first-order by a certain discrepancy tensor (see part I of the present article), it is shown that the equivalent second-gradient Mindlin elastic solid: (i.) is positive definite only when the discrepancy tensor is negative defined; (ii.) the non-local material symmetries are the same of the discrepancy tensor, and (iii.) the non-local effective behaviour is affected by the shape of the RVE, which does not influence the first-order homogenized response. Furthermore, explicit derivations of non-local parameters from heterogeneous Cauchy elastic composites are obtained in the particular cases of: (*a*) circular cylindrical and spherical isotropic inclusions embedded in an isotropic matrix, (*b*) *n*-polygonal cylindrical voids in an isotropic matrix, and (*c*) circular cylindrical voids in an orthortropic matrix.

Keywords: Dilute distribution of spherical and circular inclusions; n-polygonal holes; Higher-order elasticity; Effective non-local continuum; Composite materials.

1 Introduction

In part I of the present study (Bacca et al., 2013), a methodology has been presented to obtain an equivalent second-order Mindlin elastic material (Mindlin and Eshel, 1968), starting from a dilute suspension of randomly distributed elastic inclusions embedded in an elastic matrix, under symmetry assumptions for both the RVE and the inclusion. In particular, by imposing the vanishing of the elastic energy mismatch $\mathcal{G}$ between the heterogeneous Cauchy elastic and the Mindlin equivalent materials produced by the same second-order displacement boundary condition, the equivalent second gradient elastic (SGE) solid has been found to be defined (at first-order in the volume fraction $f \ll 1$ of the inclusion phase) by the sixth-order tensor

$$\mathsf{A}^{eq}_{ijhlmn} = -f\frac{\rho^2}{4}\left(\tilde{\mathsf{C}}_{ihln}\delta_{jm} + \tilde{\mathsf{C}}_{ihmn}\delta_{jl} + \tilde{\mathsf{C}}_{jhln}\delta_{im} + \tilde{\mathsf{C}}_{jhmn}\delta_{il}\right), \tag{1}$$

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where ρ is the radius of the sphere (or circle in 2D) of inertia of the RVE, and the discrepancy tensor $\tilde{\mathbf{C}}$ is introduced to define at the first-order in f the difference between the local constitutive tensors for the effective material $\mathbf{C}^{eq}$ and the matrix $\mathbf{C}^{(1)}$, so that

$$\mathbf{C}^{eq} = \mathbf{C}^{(1)} + f\tilde{\mathbf{C}}. \tag{2}$$

Note that $\mathbf{A}^{eq}$ is zero *either* when the inclusions are not present, $f = 0$, *or* when the inclusion has the same elastic properties of the matrix, $\tilde{\mathbf{C}} = \mathbf{0}$.

In the present part II of our study it is shown (Section 2) that the nonlocal material identified via second-order match of elastic energies through the constitutive tensor (1): (i.) is positive definite if and only if the discrepancy tensor is negative defined; (ii.) shares the same material symmetries with the discrepancy tensor (obtained as homogenized material at first-order); (iii.) is affected by the RVE shape, differently from the homogenized response at first-order. Moreover, a series of examples useful in view of applications are provided in Section 3, in particular, the material constants defining the nonlocal behaviour are explicitly obtained for dilute suspensions of isotropic elastic circular cylindrical inclusions, of cylindrical voids with n-polygonal cross section and of spherical elastic inclusions embedded in an isotropic matrix, and for dilute suspension of cylindrical voids with circular cross section distributed in an orthortropic matrix.

2 Some properties of the effective SGE solid

Some properties of the effective SGE solid are obtained below from the definition of the effective higher-order constitutive tensor $\mathbf{A}^{eq}$, eqn (1).

2.1 Heterogeneous Cauchy RVE leading to positive definite equivalent SGE material

Statement. For constituents characterized by a positive definite strain energy, a positive definite equivalent SGE material is obtained if and only if the first-order discrepancy tensor $\tilde{\mathbf{C}}$ is negative definite.

Proof. For constituents characterized by a positive definite strain energy, the first-order homogenization always leads to a positive definite equivalent fourth-order tensor $\mathbf{C}^{eq}$, so that a positive strain energy (see eqn (9) in Part I) is stored within the equivalent SGE material if and only if

$$\mathbf{A}^{eq}_{ijhlmn}\chi_{ijh}\chi_{lmn} > 0 \qquad \forall \boldsymbol{\chi} \neq \mathbf{0} \ \text{ with } \ \chi_{ijk} = \chi_{jik}, \tag{3}$$

where the summation convention over repeated indices is used henceforth. Considering the form (1) of $\mathbf{A}^{eq}$ (note the '$-$' sign), a positive definite equivalent SGE material is obtained when

$$\tilde{\mathbf{C}}_{ijhk}\chi_{lij}\chi_{lhk} < 0 \qquad \forall \boldsymbol{\chi} \neq \mathbf{0} \ \text{ with } \ \chi_{ijk} = \chi_{jik}. \tag{4}$$

Since the discrepancy tensor has the minor symmetries, $\tilde{\mathbf{C}}_{ijhk} = \tilde{\mathbf{C}}_{jihk} = \tilde{\mathbf{C}}_{ijkh}$, the condition (4) can be written as

$$\tilde{\mathbf{C}}_{ijhk}(\chi_{lij} + \chi_{lji})(\chi_{lhk} + \chi_{lkh}) < 0 \qquad \forall \boldsymbol{\chi} \neq \mathbf{0} \ \text{ with } \ \chi_{ijk} = \chi_{jik}, \tag{5}$$

which corresponds to the negative definite condition for the fourth-order constitutive tensor $\tilde{\mathbf{C}}$, because $\chi_{lij} + \chi_{lji} = \mathbf{0}$ if and only if $\boldsymbol{\chi} = \mathbf{0}$.[1] $\square$

[1]The last statement can be proven as follows. With reference to a third-order tensor ς_{ijk}, symmetric with respect to the first two indices ($\varsigma_{ijk} = \varsigma_{jik}$), we define the tensor γ_{ijk} as

$$\gamma_{ijk} = \varsigma_{ijk} + \varsigma_{ikj}, \tag{6}$$

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The fact that the equivalent nonlocal material is positive definite only for 'sufficiently compliant' inclusions was already noted by Bigoni and Drugan (2007) for Cosserat constrained rotation material and is related to the fact that higher-order continua are stiffer than Cauchy elastic materials (imposing boundary conditions on displacement and on its normal derivative). This effect has also an experimental counterpart provided by Gauthier (1982), who showed micropolar effects for porous material, but 'anti-micropolar' behaviour for a soft matrix containing stiff inclusions.

2.2 Higher-order material symmetries for the equivalent SGE solid

Statement. The higher-order material symmetries of the equivalent SGE solid coincide with the material symmetries of the first-order discrepancy tensor $\tilde{\mathsf{C}}$.

Proof. A class of material symmetry corresponds to indifference of a constitutive equation with respect to application of a class of orthogonal transformations represented through an orthogonal tensor $\boldsymbol{Q}$, so that an higher-order material symmetry for the equivalent SGE material occurs when

$$\mathsf{A}^{eq}_{ijhlmn} = Q_{ip}Q_{jq}Q_{hr}Q_{ls}Q_{mt}Q_{nu}\mathsf{A}^{eq}_{pqrstu}, \tag{8}$$

while for the first-order discrepancy tensor when

$$\tilde{\mathsf{C}}_{ijhk} = Q_{ip}Q_{jq}Q_{hr}Q_{ks}\tilde{\mathsf{C}}_{pqrs}. \tag{9}$$

Considering the property of orthogonal transformations ($\boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I}$), the solution (1) for A^{eq} and that this can be inverted as

$$\begin{aligned}\tilde{\mathsf{C}}_{ihln}\delta_{jm} = & -\frac{1}{f}\Big[\mathsf{A}^{eq}_{ijhlmn} + \mathsf{A}^{eq}_{jhimnl} + \mathsf{A}^{eq}_{hijnlm} - \mathsf{A}^{eq}_{ijhnlm} - \mathsf{A}^{eq}_{hijlmn} \\ & +\mathsf{A}^{eq}_{ijhmnl} + \mathsf{A}^{eq}_{jhilmn} - \mathsf{A}^{eq}_{jhinlm} - \mathsf{A}^{eq}_{hijmnl}\Big],\end{aligned} \tag{10}$$

it follows that the symmetry condition for the effective higher-order tensor A^{eq}, eqn (8), is equivalent to that for the first-order discrepancy tensor $\tilde{\mathsf{C}}$, eqn (9).[2]$\square$

2.3 Influence of the volume and shape of the RVE on the higher-order constitutive response

In addition to the dependence on the shape of the inclusion, typical of first-order homogenization, the representation (1) of A^{eq} shows that the higher-order constitutive response in the dilute case depends on the volume and the shape of the RVE through its radius of inertia ρ. This feature distinguishes second-order homogenization from first-order, since in the latter case C^{eq} in the dilute case is independent of the volume and shape of the RVE. Therefore, two composite

resulting symmetric with respect to the last two indices ($\gamma_{ijk} = \gamma_{ikj}$). Relation (6) is invertible, so that

$$\varsigma_{ijk} = \frac{\gamma_{ijk} + \gamma_{jki} - \gamma_{kij}}{2}, \tag{7}$$

and therefore $\boldsymbol{\gamma} = \mathbf{0}$ if and only if $\boldsymbol{\varsigma} = \mathbf{0}$.

[2]Note that isotropic discrepancy at first-order (namely isotropic $\tilde{\mathsf{C}}$) implies isotropy of the strain-gradient equivalent material A^{eq}. On the other hand, it is known from a numerical example by Auffray et al. (2010) that a Cauchy composite material with an hexagonal symmetry can yield a nonlocal anisotropic response. Their example, not referred to a dilute suspension, is not in direct contrast with the results presented here.

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materials $\mathcal{M}$ and $\mathcal{N}$ differing only in the geometrical distribution of the inclusions correspond to the same equivalent local tensor $\mathbf{C}^{eq}(\mathcal{M}) = \mathbf{C}^{eq}(\mathcal{N})$, but lead to a different higher-order equivalent tensor $\mathbf{A}^{eq}(\mathcal{M}) \neq \mathbf{A}^{eq}(\mathcal{N})$.

An example in 2D is reported in Fig. 1 where the hexagonal RVE ($\mathcal{N}$) compared to the squared RVE ($\mathcal{M}$) yields

$$\mathbf{A}^{eq}(\mathcal{M}) = \frac{3\sqrt{3}}{5}\mathbf{A}^{eq}(\mathcal{N}) \sim 1.039\mathbf{A}^{eq}(\mathcal{N}), \tag{11}$$

while in the 3D example reported in Fig. 2 a truncated-octahedral RVE ($\mathcal{N}$) is compared to a cubic RVE ($\mathcal{M}$) yielding

$$\mathbf{A}^{eq}(\mathcal{M}) = \frac{16\sqrt[3]{2}}{19}\mathbf{A}^{eq}(\mathcal{N}) \sim 1.061\mathbf{A}^{eq}(\mathcal{N}). \tag{12}$$

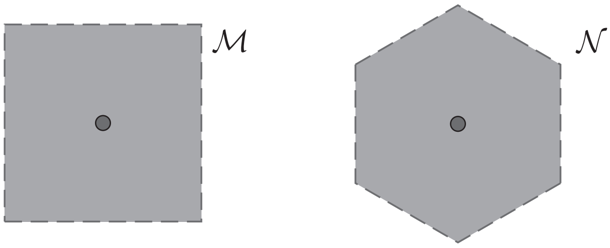

Figure 1: Two-phase RVEs differing only in the shape of the boundary, namely $\mathcal{M}$ and $\mathcal{N}$. In the dilute limit, both composites are characterized by the same equivalent local tensor, $\mathbf{C}^{eq}(\mathcal{M}) = \mathbf{C}^{eq}(\mathcal{N})$, but by different higher-order equivalent tensors, $\mathbf{A}^{eq}(\mathcal{M}) \neq \mathbf{A}^{eq}(\mathcal{N})$, see eqn (11).

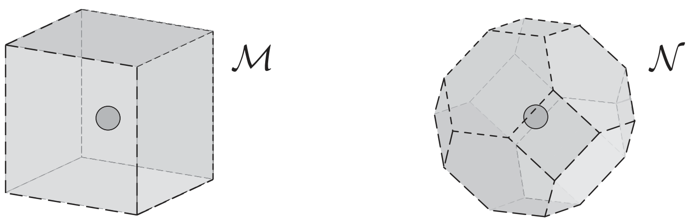

Figure 2: Similarly to Fig. 1, two RVEs $\mathcal{M}$ (cubic RVE) and $\mathcal{N}$ (truncated-octahedral RVE) leading to the same equivalent local tensor, $\mathbf{C}^{eq}(\mathcal{M}) = \mathbf{C}^{eq}(\mathcal{N})$, but to different higher-order equivalent tensors, $\mathbf{A}^{eq}(\mathcal{M}) \neq \mathbf{A}^{eq}(\mathcal{N})$, see eqn (12).

The fact that different shapes of the RVE yield, through their radii of inertia, different nonlocal properties is inherent to the proposed identification procedure. However, this effect is small –as shown by the estimates (11) and (12)– and has to be understood under the light of the dilute assumption for a random distribution of inclusions, so that the choice of the shape of the RVE is to a certain extent limited.

3 Application cases

Several applications of eqn (1) are presented in this Section for composites of different geometries and constitutive properties. Situations in which the homogenized material results isotropic are first considered and finally some cases of anisotropic behaviour are presented.

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3.1 Equivalent isotropic SGE

For an isotropic composite, the first-order discrepancy tensor $\tilde{\mathbf{C}}$ is

$$\tilde{\mathbf{C}}^{iso}_{ijhk} = \tilde{\lambda}\delta_{ij}\delta_{hk} + \tilde{\mu}(\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}), \tag{13}$$

so that the equivalent sixth-order tensor $\mathbf{A}^{eq}$, eqn (1), is given by

$$\begin{aligned} \mathbf{A}^{eq}_{ijhlmn} = & -f\frac{\rho^2}{4}\left\{\tilde{\lambda}\left[\delta_{ih}\left(\delta_{jl}\delta_{mn} + \delta_{jm}\delta_{ln}\right) + \delta_{jh}\left(\delta_{il}\delta_{mn} + \delta_{im}\delta_{ln}\right)\right]\right. \\ & \left. +\tilde{\mu}\left[2\left(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}\right)\delta_{hn} + \delta_{in}\left(\delta_{jl}\delta_{hm} + \delta_{jm}\delta_{hl}\right) + \delta_{jn}\left(\delta_{il}\delta_{hm} + \delta_{im}\delta_{hl}\right)\right]\right\}, \end{aligned} \tag{14}$$

which is a special case of isotropic sixth-order tensor

$$\begin{aligned} \mathbf{A}^{iso}_{ijhlmn} = & \frac{a_1}{2}\left[\delta_{ij}\left(\delta_{hl}\delta_{mn} + \delta_{hm}\delta_{ln}\right) + \delta_{lm}\left(\delta_{in}\delta_{jh} + \delta_{ih}\delta_{jn}\right)\right] \\ & +\frac{a_2}{2}\left[\delta_{ih}\left(\delta_{jl}\delta_{mn} + \delta_{jm}\delta_{ln}\right) + \delta_{jh}\left(\delta_{il}\delta_{mn} + \delta_{im}\delta_{ln}\right)\right] \\ & +2\,a_3\left(\delta_{ij}\delta_{hn}\delta_{lm}\right) + a_4\left(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}\right)\delta_{hn} \\ & +\frac{a_5}{2}\left[\delta_{in}\left(\delta_{jl}\delta_{hm} + \delta_{jm}\delta_{hl}\right) + \delta_{jn}\left(\delta_{il}\delta_{hm} + \delta_{im}\delta_{hl}\right)\right], \end{aligned} \tag{15}$$

with the following constants

$$a_1 = a_3 = 0, \qquad a_2 = -f\frac{\rho^2}{2}\tilde{\lambda}, \qquad a_4 = a_5 = -f\frac{\rho^2}{2}\tilde{\mu}. \tag{16}$$

The related strain energy is positive definite when parameters a_i $(i = 1, ..., 5)$ satisfy eqn (18) of Part I, which for the values (16) implies

$$\tilde{K} < 0, \quad \tilde{\mu} < 0, \tag{17}$$

where $\tilde{K}$ is the bulk modulus, equal to $\tilde{\lambda}+2\tilde{\mu}/3$ in 3D and $\tilde{\lambda}+\tilde{\mu}$ in plane strain, and corresponding to the negative definiteness condition for $\tilde{\mathbf{C}}$, according to our previous results (Section 2.1).

An explicit evaluation of the constants (a_2, $a_4 = a_5$) is given now, in the case when an isotropic fourth-order tensor $\tilde{\mathbf{C}}$ is obtained from homogenization of a RVE with both isotropic phases, matrix denoted by '1' (with Lamé constants λ_1 and μ_1) and inclusion denoted by '2' (with Lamé constants λ_2 and μ_2), having a shape leading to an isotropic equivalent constitutive tensor

$$\mathbf{C}^{eq}_{ijhk} = \lambda_{eq}\delta_{ij}\delta_{hk} + \mu_{eq}(\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}), \tag{18}$$

where

$$\lambda_{eq} = \lambda_1 + f\tilde{\lambda}, \qquad \mu_{eq} = \mu_1 + f\tilde{\mu}, \qquad K_{eq} = K_1 + f\tilde{K}. \tag{19}$$

In particular, the following forms of inclusions are considered within an isotropic matrix.

- For 3D deformation:
  - spherical elastic inclusions.
- For plane strain:
  - circular elastic inclusions;

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– regular n-polygonal holes with $n \neq 4$ (the case $n = 4$ leads to an orthotropic material and is treated in the next subsection).

For all of the above cases it is shown that a positive definite equivalent SGE material, eqn (17), is obtained only when the inclusion phase is 'softer' than the matrix in terms of *both* shear and bulk moduli,

$$\mu_2 < \mu_1, \quad K_2 < K_1, \tag{20}$$

which is always satisfied when the inclusions are voids. The positive definiteness condition (20) can be written in terms of the ratio μ_2/μ_1 and the Poisson's ratio of the phases ν_1 and ν_2 [where $\nu_i = \lambda_i/(2(\lambda_i + \mu_i))$] as

$$\frac{\mu_2}{\mu_1} < \min\left\{1;\ \frac{1-2\nu_2}{1-2\nu_1}\right\}, \tag{21}$$

for the case of plane strain, and

$$\frac{\mu_2}{\mu_1} < \min\left\{1;\ \frac{(1+\nu_1)(1-2\nu_2)}{(1+\nu_2)(1-2\nu_1)}\right\}, \tag{22}$$

for three-dimensional case. The regions where a positive definite SGE material is obtained, eqns (21) - (22), are mapped in the plane $\mu_2/\mu_1 - \nu_1$ for different values of the inclusion Poisson's ratio ν_2 (Fig. 3, plane strain on the left and 3D-deformation on the right).

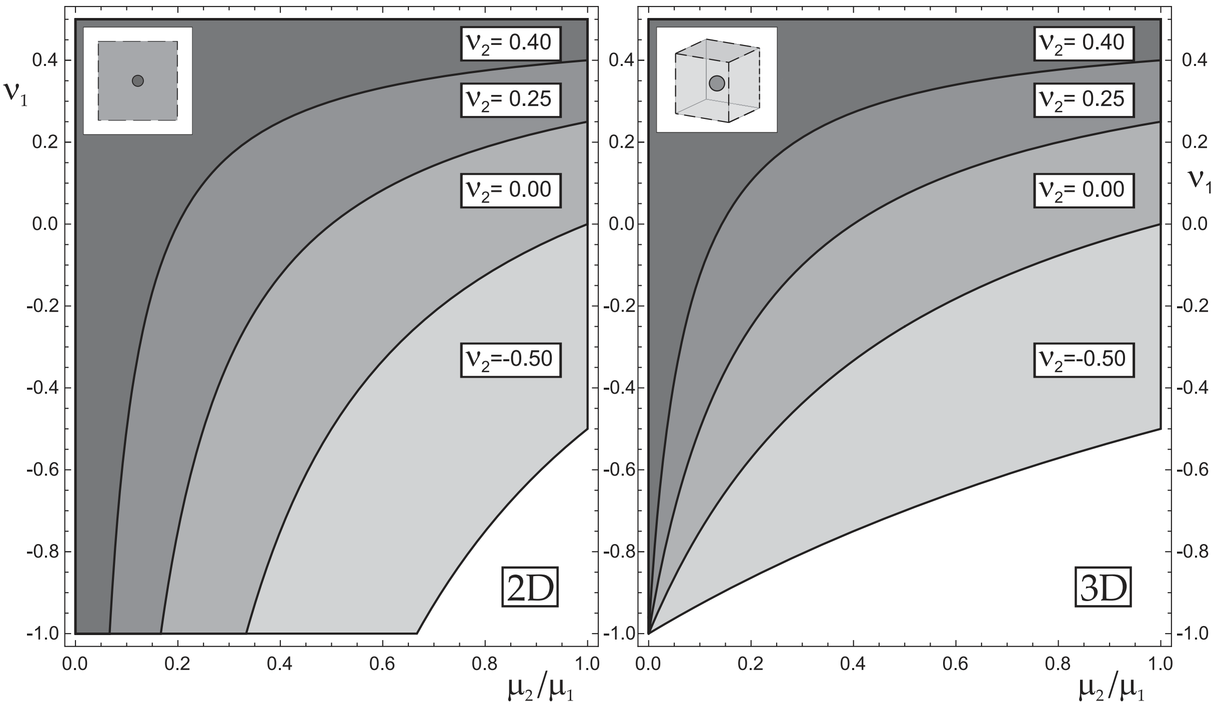

Figure 3: Regions in the plane $\mu_2/\mu_1 - \nu_1$ where the higher-order effective constitutive tensor $\mathbf{A}^{eq}$ is positive definite (for different values of ν_2). The regions for the plane strain case, eqn (21), are reported on the left, while the case of three-dimensional deformations, eqn (22), is reported on the right.

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Cylindrical elastic inclusions The elastic constants K_{eq} and μ_{eq} of the isotropic material equivalent to a dilute suspension of parallel isotropic cylindrical inclusions embedded in an isotropic matrix have been obtained by Hashin and Rosen (1964), in our notation

$$\tilde{K} = \frac{(K_2 - K_1)(K_1 + \mu_1)}{K_2 + \mu_1}, \qquad \tilde{\mu} = \frac{2\mu_1(\mu_2 - \mu_1)(K_1 + \mu_1)}{2\mu_1\mu_2 + K_1(\mu_1 + \mu_2)}. \tag{23}$$

Exploiting equation (16), the equivalent higher-order constants a_i $(i = 1, ..., 5)$ can be obtained from the first-order discrepancy quantities, eqn (23), so that the non-null constants are evaluated as

$$\begin{aligned} a_2 &= f\frac{\rho^2}{2}\left[\frac{(K_1 - K_2)(K_1 + \mu_1)}{K_2 + \mu_1} - \frac{2\mu_1(\mu_1 - \mu_2)(K_1 + \mu_1)}{2\mu_1\mu_2 + K_1(\mu_1 + \mu_2)}\right], \\ a_4 &= a_5 = f\rho^2\frac{\mu_1(\mu_1 - \mu_2)(K_1 + \mu_1)}{2\mu_1\mu_2 + K_1(\mu_1 + \mu_2)}. \end{aligned} \tag{24}$$

The higher-order equivalent constants a_2 and a_4 given by eqn (24) are reported in Figs. 4 and 5 as functions of the ratio μ_2/μ_1 and for different Poisson's ratios of matrix and inclusion. In all the figures, a red spot denotes the threshold for which the strain energy of the equivalent material looses positive definiteness. The dashed curves refer to regions where this positive definiteness is lost.

With reference to Fig. 4, we may note that $a_2 \to \infty$ in the limit $\nu_1 \to 1/2$. Furthermore, a_4 is not affected by the Poisson's ratio of the inclusion ν_2, except that the threshold for positive definiteness condition for the equivalent material strain energy changes, eqn (21).

Spherical elastic inclusions The equivalent elastic constants K_{eq} and μ_{eq} of the isotropic material equivalent to a dilute suspension of isotropic spherical inclusions within an isotropic matrix have been obtained by Eshelby (1957) and independently by Hashin (1959), in our notation

$$\tilde{K} = \frac{(3K_1 + 4\mu_1)(K_2 - K_1)}{3K_2 + 4\mu_1}, \qquad \tilde{\mu} = \frac{5\mu_1(\mu_2 - \mu_1)(3K_1 + 4\mu_1)}{\mu_1(3K_1 + 4\mu_2) + 2(3K_1 + 4\mu_1)(\mu_2 + \mu_1)}, \tag{25}$$

so that, through equation (16), the non-null equivalent higher-order constants are given by

$$\begin{aligned} a_2 &= f\frac{\rho^2}{2}\left[\frac{(3K_1 + 4\mu_1)(K_2 - K_1)}{3K_2 + 4\mu_1} - \frac{2}{3}\frac{5\mu_1(\mu_2 - \mu_1)(3K_1 + 4\mu_1)}{\mu_1(3K_1 + 4\mu_2) + 2(3K_1 + 4\mu_1)(\mu_2 + \mu_1)}\right], \\ a_4 &= a_5 = f\frac{\rho^2}{2}\frac{5\mu_1(\mu_2 - \mu_1)(3K_1 + 4\mu_1)}{\mu_1(3K_1 + 4\mu_2) + 2(3K_1 + 4\mu_1)(\mu_2 + \mu_1)}, \end{aligned} \tag{26}$$

which are reported in Fig. 6 and Fig. 7 as a function of the shear stiffness ratio μ_2/μ_1 and for different Poisson's ratios of the phases. In these figures the curves become dashed when the strain energy of the equivalent material looses positive definiteness. Moreover, the higher-order constants are reported in Fig. 8 as functions of the matrix Poisson's ratio ν_1 in the particular case of spherical voids.

Similar to the case of cylindrical elastic inclusions, $a_2 \to \infty$ in the limit $\nu_1 \to 1/2$ and a_4 is not affected by the Poisson's ratio of the inclusion ν_2, except for the threshold of strain energy's positive definiteness, eqn. (22).

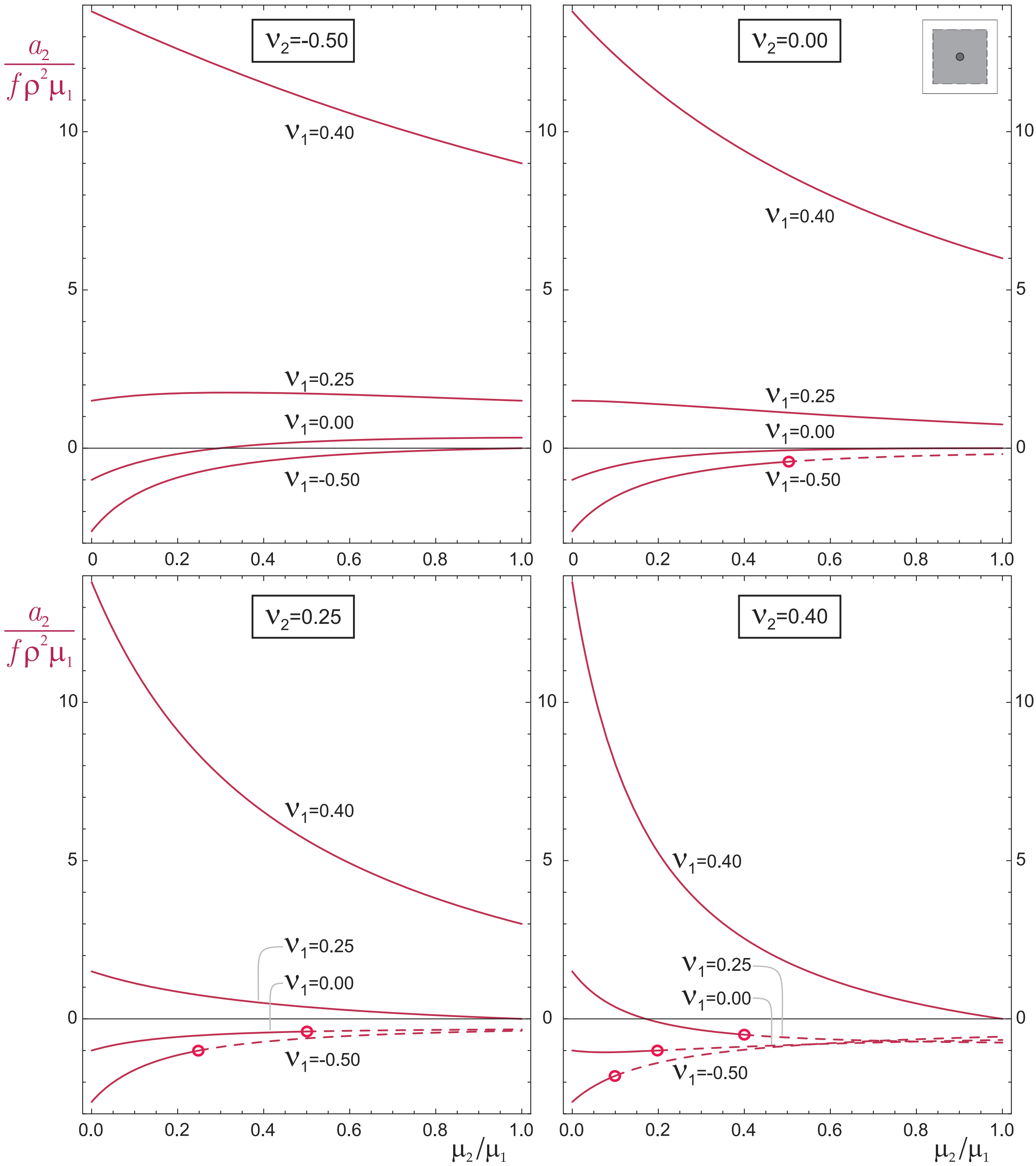

Figure 4: Higher-order equivalent constant a_2, eqn $(24)_1$, of the SGE solid equivalent to a composite made up of an isotropic matrix containing a dilute suspension of cylindrical elastic inclusions, as a function of the ratio μ_2/μ_1, for different values of the Poisson's ratio of the phases $\{\nu_1, \nu_2\}$ ={-0,5;-0.25;0;0.4}. The constant a_2 is made dimensionless through division by parameter $f\rho^2\mu_1$. The curves are dashed where the strain energy of the equivalent material is not positive definite, a red spot marks where the loss of positive definiteness occurs.

Regular n-polygonal holes ($n \neq 4$) The elastic constants μ_{eq} and K_{eq} of the isotropic material equivalent to a dilute suspension of n-polygonal holes ($n \neq 4$) in an isotropic matrix have been obtained by Jasiuk et al. (1994) and Thorpe et al. (1995), from which the first-order

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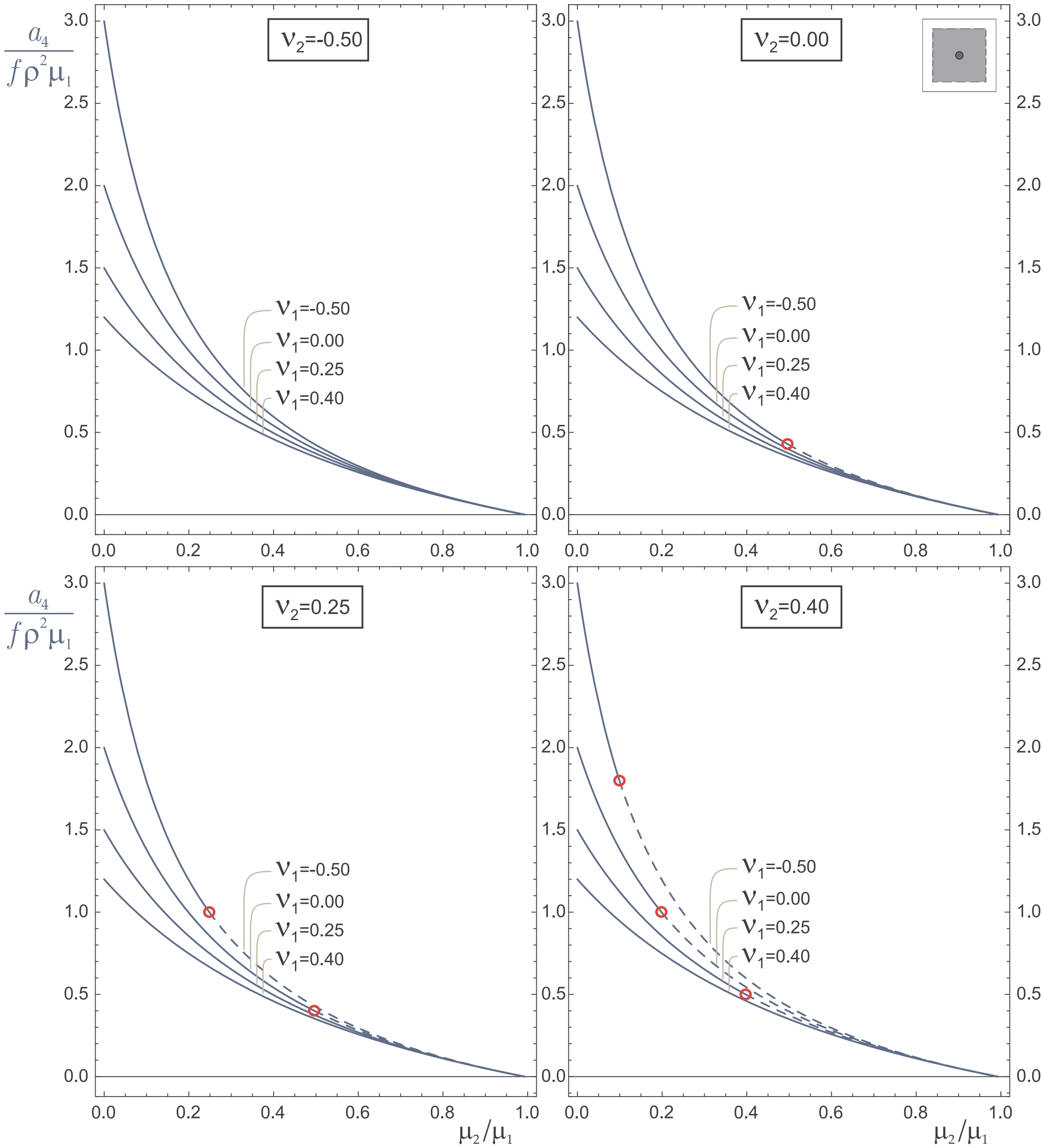

Figure 5: Higher-order equivalent constant $a_4 = a_5$, eqn $(24)_2$, of the SGE solid equivalent to a composite made up of an isotropic matrix containing a dilute suspension of cylindrical elastic inclusions, as a function of the ratio μ_2/μ_1, for different values of Poisson's ratio of the phases $\{\nu_1, \nu_2\}$ ={-0,5;-0.25;0;0.4}. The constant a_4 is made dimensionless through division by parameter $f\rho^2\mu_1$. Note that the curves are not affected by the Poisson's ratio of the inclusion ν_2, except that the threshold (red spot) for positive definiteness of the equivalent material strain energy changes, eqn (21). Dashed curve represents values for which the strain energy of the equivalent material is not positive definite.

discrepancy stiffness can be written in our notation as

$$\tilde{K}(n) = -\mathcal{A}(n)[1 - \mathcal{B}(n)]\frac{K_1 + \mu_1}{\mu_1}K_1, \qquad \tilde{\mu}(n) = -\mathcal{A}(n)[1 + \mathcal{B}(n)]\frac{K_1 + \mu_1}{K_1}\mu_1, \tag{27}$$

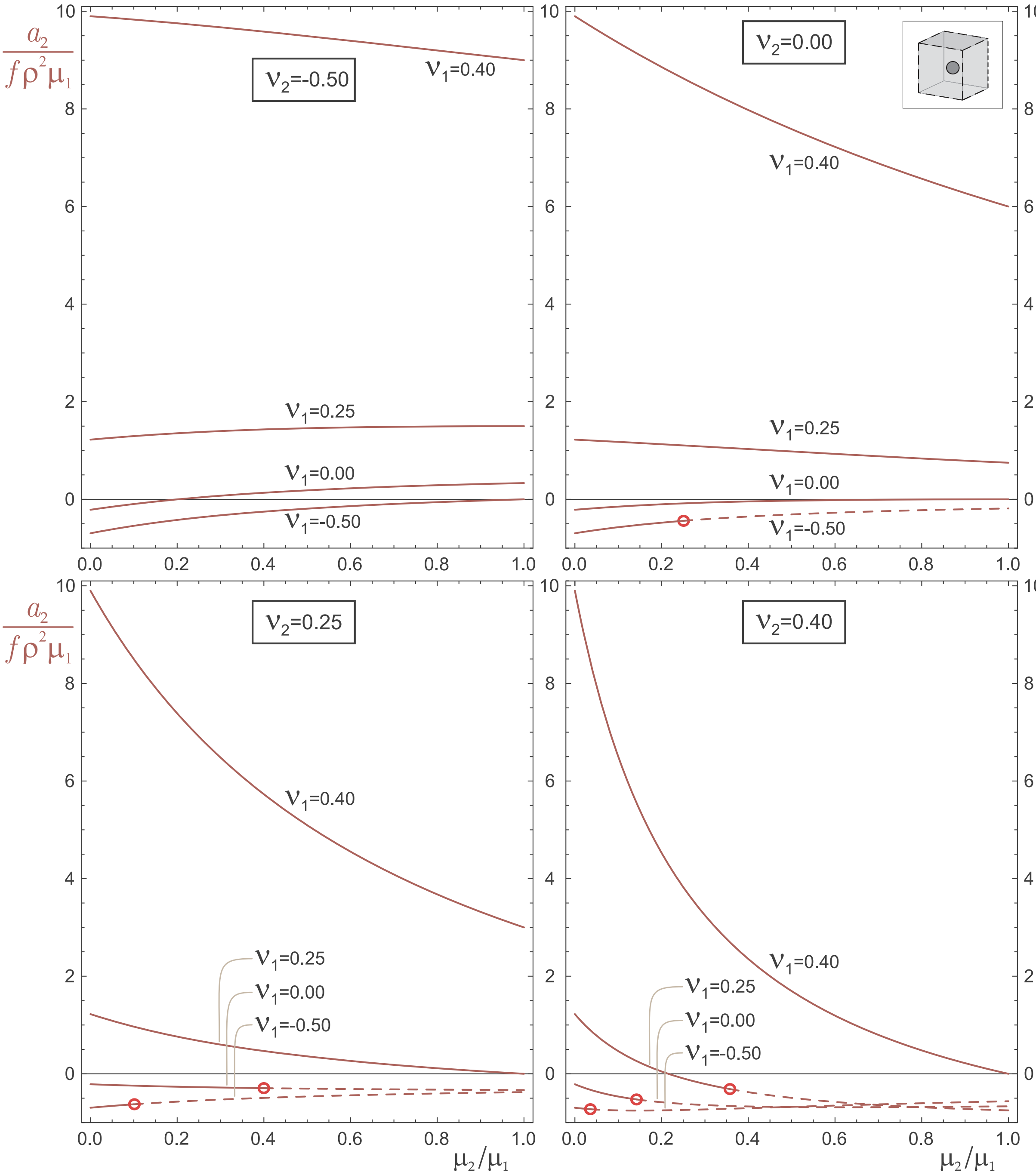

Figure 6: Higher-order equivalent constant a_2, eqn $(26)_1$, of the SGE solid equivalent to a composite made up of an isotropic matrix containing a dilute suspension of spherical elastic inclusions as a function of the ratio μ_2/μ_1, for different values of Poisson's ratio of the phases $\{\nu_1,\nu_2\}$ ={-0,5;-0.25;0;0.4}. The constant a_2 is made dimensionless through division by parameter $f\rho^2\mu_1$. The curves are dashed where the strain energy of the equivalent material is not positive definite, a red spot marks where the loss of positive definiteness occurs.

where $\mathcal{A}(n)$ and $\mathcal{B}(n)$ are constants depending on the number of edges n of the regular polygonal hole, which can be approximated through numerical computations, and are reported in Tab. 1 for n={3; 5; 6}. In the case of a regular polygon with infinite number of edges, in other words a

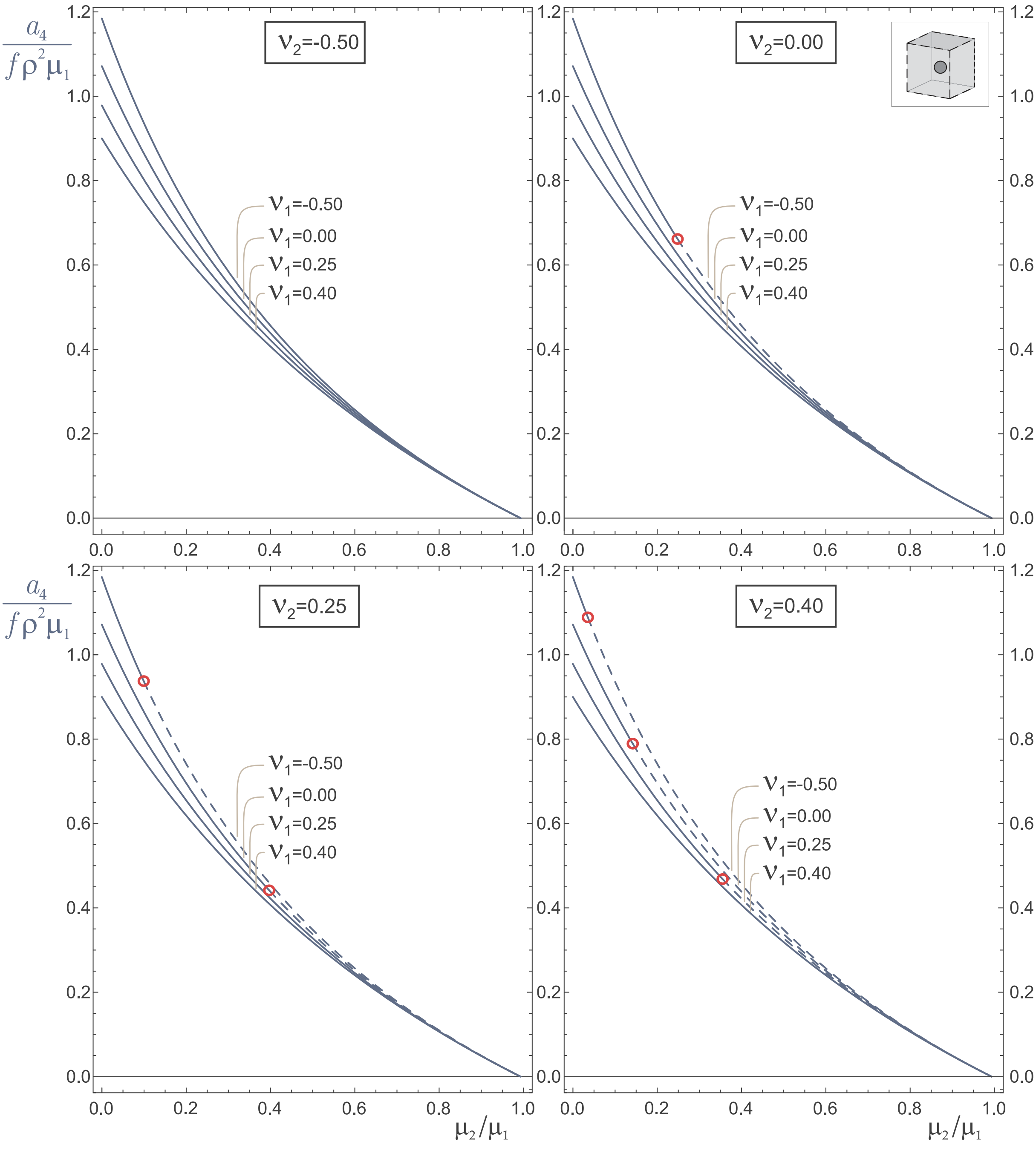

Figure 7: Higher-order equivalent constant $a_4 = a_5$, eqn $(26)_2$, of the SGE solid equivalent to a composite made up of an isotropic matrix containing a dilute suspension of cylindrical elastic inclusions as a function of the ratio μ_2/μ_1, for different values of Poisson's ratio of the phases $\{\nu_1, \nu_2\}$ ={-0,5;-0.25;0;0.4}. The constant a_4 is made dimensionless through division by parameter $f\rho^2\mu_1$. Note that the curves are not affected by the Poisson's ratio of the inclusion ν_2, except that the threshold (red spot) for positive definiteness of the equivalent material strain energy changes, eqn (22). Dashed curve represents values for which the strain energy of the equivalent material is not positive definite.

circle, the value of the constants is $\mathcal{A}(n \to \infty) = 3/2$ and $\mathcal{B}(n \to \infty) = 1/3$, so that the case of a cylindrical void inclusion is recovered, eqn (23) with $\mu_2 = K_2 = 0$. The equivalent higher-order

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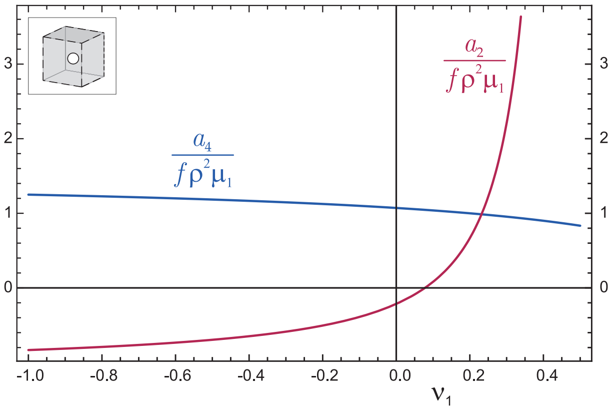

Figure 8: Higher-order equivalent constants a_2 and $a_4 = a_5$ of the equivalent SGE material for a composite made up of an isotropic matrix containing a dilute suspension of spherical voids as a function of the matrix Poisson's ratio ν_1, eqn (26) with $\mu_2 = K_2 = 0$. The constants are made dimensionless through division by parameter $f\rho^2\mu_1$.

constants can be obtained from eqn (16) by using the first-order discrepancy quantities, eqn (27), from which the non-null constants follow

$$\begin{aligned} a_2 &= f\frac{\rho^2}{2}\mathcal{A}(n)\left\{[1-\mathcal{B}(n)]K_1^2 - [1+\mathcal{B}(n)]\mu_1^2\right\}\frac{K_1+\mu_1}{\mu_1 K_1}, \\ a_4 = a_5 &= f\frac{\rho^2}{2}\mathcal{A}(n)[1+\mathcal{B}(n)]\frac{K_1+\mu_1}{K_1}\mu_1, \end{aligned} \tag{28}$$

and are shown in Fig. 9 as functions of the matrix Poisson's ratio ν_1.

Polygonal hole	n	Approximated values $\mathcal{A}(n)$	$\mathcal{B}(n)$
Triangle	3	2.1065	0.2295
Pentagon	5	1.6198	0.3233
Hexagon	6	1.5688	0.3288
Circle	∞	3/2	1/3

Tab. 1: Values of the constants $\mathcal{A}(n)$ and $\mathcal{B}(n)$ for triangular ($n = 3$), pentagonal ($n = 5$), hexagonal ($n = 6$), and circular ($n \to \infty$) holes in an isotropic elastic matrix (Thorpe et al., 1995). These values are instrumental to obtain the equivalent properties $\tilde{K}(n)$ and $\tilde{\mu}(n)$, eqn (27), of the higher-order material.

3.2 Equivalent cubic SGE

When the first-order discrepancy tensor $\tilde{\mathbf{C}}$ has a cubic symmetry, it can be represented in a cartesian system aligned parallel to the symmetry axes as (see Thomas, 1966)

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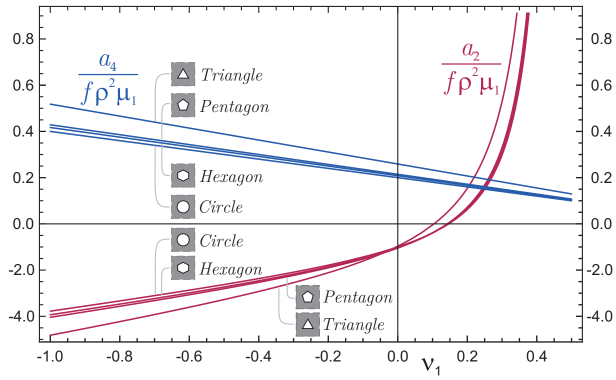

Figure 9: Higher-order equivalent constants a_2 and $a_4 = a_5$ of the equivalent SGE material for a dilute suspension of triangular ($n = 3$), pentagonal ($n = 5$), hexagonal ($n = 6$), and circular ($n \to \infty$) holes in an isotropic matrix, as functions of the matrix Poisson's ratio ν_1, eqn (28). The constants are made dimensionless through division by parameter $f\rho^2\mu_1$.

$$\begin{aligned}\tilde{\mathbf{C}}^{cub}_{ijhk} = \; & \tilde{\mathbf{C}}^{iso}_{ijhk} + \tilde{\xi}\left[\left(\delta_{i2}\delta_{j3}+\delta_{i3}\delta_{j2}\right)\left(\delta_{h2}\delta_{k3}+\delta_{h3}\delta_{k2}\right)+\left(\delta_{i1}\delta_{j3}+\delta_{i3}\delta_{j1}\right)\left(\delta_{h1}\delta_{k3}+\delta_{h3}\delta_{k1}\right)\right. \\ & \left.+\left(\delta_{i1}\delta_{j2}+\delta_{i2}\delta_{j1}\right)\left(\delta_{h1}\delta_{k2}+\delta_{h2}\delta_{k1}\right)\right],\end{aligned} \tag{29}$$

where $\tilde{\mathbf{C}}^{iso}$ is given by eqn (13). The sixth-order tensor $\mathbf{A}^{eq}$ for the equivalent material is obtained using eqn (1) in the form

$$\begin{aligned}\mathbf{A}^{eq}_{ijhlmn} = \; & \mathbf{A}^{iso}_{ijhlmn} + \frac{a_6}{2}\left\{\left(\delta_{i1}\delta_{h2}+\delta_{i2}\delta_{h1}\right)\left[\left(\delta_{l1}\delta_{n2}+\delta_{l2}\delta_{n1}\right)\delta_{jm}+\left(\delta_{m1}\delta_{n2}+\delta_{m2}\delta_{n1}\right)\delta_{jl}\right]\right. \\ & +\left(\delta_{j1}\delta_{h2}+\delta_{j2}\delta_{h1}\right)\left[\left(\delta_{l1}\delta_{n2}+\delta_{l2}\delta_{n1}\right)\delta_{im}+\left(\delta_{m1}\delta_{n2}+\delta_{m2}\delta_{n1}\right)\delta_{il}\right]\} \\ & +\left(\delta_{i1}\delta_{h3}+\delta_{i3}\delta_{h1}\right)\left[\left(\delta_{l1}\delta_{n3}+\delta_{l3}\delta_{n1}\right)\delta_{jm}+\left(\delta_{m1}\delta_{n3}+\delta_{m3}\delta_{n1}\right)\delta_{jl}\right] \\ & +\left(\delta_{j1}\delta_{h3}+\delta_{j3}\delta_{h1}\right)\left[\left(\delta_{l1}\delta_{n3}+\delta_{l3}\delta_{n1}\right)\delta_{im}+\left(\delta_{m1}\delta_{n3}+\delta_{m3}\delta_{n1}\right)\delta_{il}\right] \\ & +\left(\delta_{i2}\delta_{h3}+\delta_{i3}\delta_{h2}\right)\left[\left(\delta_{l2}\delta_{n3}+\delta_{l3}\delta_{n2}\right)\delta_{jm}+\left(\delta_{m2}\delta_{n3}+\delta_{m3}\delta_{n2}\right)\delta_{jl}\right] \\ & \left.+\left(\delta_{j2}\delta_{h3}+\delta_{j3}\delta_{h2}\right)\left[\left(\delta_{l2}\delta_{n3}+\delta_{l3}\delta_{n2}\right)\delta_{im}+\left(\delta_{m2}\delta_{n3}+\delta_{m3}\delta_{n2}\right)\delta_{il}\right]\right\},\end{aligned} \tag{30}$$

with $\mathbf{A}^{iso}$ given by eqn (15), parameters a_i ($i = 1, ..., 5$) by eqn (16), and

$$a_6 = -f\frac{\rho^2}{2}\tilde{\xi}. \tag{31}$$

According to results presented in subsections 2.1 and 2.2, the effective higher-order tensor $\mathbf{A}^{eq}$ results to be a cubic sixth-order tensor and is positive definite when $\tilde{\mathbf{C}}$, eqn (29), is negative definite, namely, eqn (17) together with

$$\tilde{\xi} + \tilde{\mu} < 0. \tag{32}$$

Aligned square holes within an isotropic matrix There are no results available for the plane strain homogenization of a dilute suspension of square holes distributed (with parallel edges) within an isotropic matrix. Therefore, we have compared with a conformal mapping technique (Misseroni et al. 2013) stress and strain averages, and found the following discrepancy at first-order in the constitutive quantities[3]

$$\tilde{\lambda} = -(1.198K_1^2 - 1.864\mu_1^2)\frac{K_1+\mu_1}{K_1\mu_1}, \qquad \tilde{\mu} = -1.864\frac{K_1+\mu_1}{K_1}\mu_1, \qquad \tilde{\xi} = -0.796\frac{K_1+\mu_1}{K_1}\mu_1, \tag{33}$$

showing that $\tilde{\mathbf{C}}$ is negative definite, eqn (32), and therefore the corresponding effective higher-order tensor $\mathbf{A}^{eq}$, eqn (30), is positive definite.

The equivalent higher-order constants a_i $(i = 1, ..., 6)$ can be obtained from the first-order discrepancy quantities, eqn (33), so that the non-null constants are evaluated by exploiting eqns (16) and (31) as

$$\begin{aligned} a_2 &= f\rho^2\,(0.599K_1^2 - 0.932\mu_1^2)\frac{K_1+\mu_1}{K_1\mu_1}, \\ a_4 &= a_5 = 0.932f\rho^2\frac{K_1+\mu_1}{K_1}\mu_1, \\ a_6 &= 0.398f\rho^2\frac{K_1+\mu_1}{K_1}\mu_1. \end{aligned} \tag{34}$$

These three independent constants are reported in Fig. 10 as functions of the matrix Poisson's ratio ν_1.

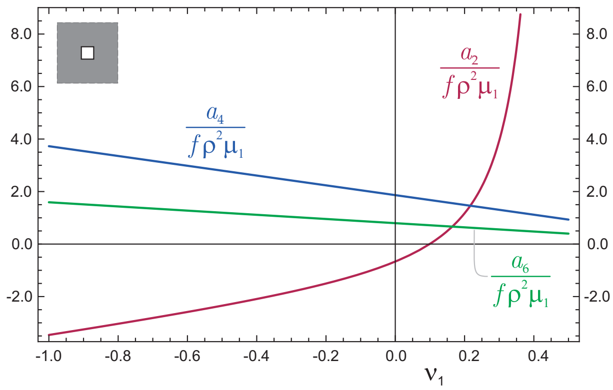

Figure 10: Higher-order equivalent constants a_2, $a_4 = a_5$, and a_6 of the equivalent SGE material for the plane strain case of a dilute suspension of square holes (with parallel edges) in an isotropic matrix, as a function of the matrix Poisson's ratio ν_1, eqn (34). The constants are made dimensionless through division by parameter $f\rho^2\mu_1$.

[3]Thorpe et al. (1995) give results for composites with a random orientation of square holes, so that the effective behaviour is isotropic and given by eqn (27) with $\mathcal{A}(n=4) = 1.738$ and $\mathcal{B}(n=4) = 0.306$. This isotropic effective response can be independently obtained by averaging the cubic effective response given by eqn (33) over two orientations of the square hole differing by an angle $\pi/4$.

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3.3 Equivalent orthotropic SGE

When the first-order discrepancy tensor $\tilde{\mathbf{C}}$ is orthotropic, it can be represented in a cartesian system aligned parallel to the symmetry axes as (see Spencer, 1982)

$$\begin{aligned}\tilde{\mathbf{C}}^{orth}_{ijhk} = \; & \tilde{\mathbf{C}}^{iso}_{ijhk} + \tilde{\xi}^{I}\left(\delta_{i2}\delta_{j3}+\delta_{i3}\delta_{j2}\right)\left(\delta_{h2}\delta_{k3}+\delta_{h3}\delta_{k2}\right) + \tilde{\xi}^{II}\left(\delta_{i1}\delta_{j3}+\delta_{i3}\delta_{j1}\right)\left(\delta_{h1}\delta_{k3}+\delta_{h3}\delta_{k1}\right)\\ & + \tilde{\xi}^{III}\left(\delta_{i1}\delta_{j2}+\delta_{i2}\delta_{j1}\right)\left(\delta_{h1}\delta_{k2}+\delta_{h2}\delta_{k1}\right) + \tilde{\omega}^{I}\delta_{i1}\delta_{j1}\delta_{h1}\delta_{k1} + \tilde{\omega}^{II}\delta_{i3}\delta_{j3}\delta_{h3}\delta_{k3}\\ & + \tilde{\omega}^{III}\left(\delta_{ij}\delta_{h3}\delta_{k3}+\delta_{hk}\delta_{i3}\delta_{j3}\right) + \tilde{\omega}^{IV}\left(\delta_{i1}\delta_{j1}\delta_{h3}\delta_{k3}+\delta_{i3}\delta_{j3}\delta_{h1}\delta_{k1}\right),\end{aligned} \tag{35}$$

where $\tilde{\xi}^{III}$, $\tilde{\omega}^{I}$, $\tilde{\xi}^{I}$, $\tilde{\xi}^{II}$, $\tilde{\omega}^{II}$, $\tilde{\omega}^{III}$ and $\tilde{\omega}^{IV}$ are seven independent constants (in addition to $\tilde{\lambda}$ and $\tilde{\mu}$) defining the orthotropic behaviour in 3D.[4] The in-plane behaviour is defined by groups of four independent constants, which for the x_1–x_2 plane are $\{\tilde{\lambda};\tilde{\mu};\tilde{\xi}^{III};\tilde{\omega}^{I}\}$.

In the case of orthotropic $\tilde{\mathbf{C}}$, eqn (1) defining the sixth-order nonlocal tensor $\mathbf{A}^{eq}$ leads to

$$\begin{aligned}\mathbf{A}^{eq}_{ijhlmn} = \; & \mathbf{A}^{iso}_{ijhlmn} + \frac{a_6}{2}\{\left(\delta_{i1}\delta_{h2}+\delta_{i2}\delta_{h1}\right)\left[\left(\delta_{l1}\delta_{n2}+\delta_{l2}\delta_{n1}\right)\delta_{jm}+\left(\delta_{m1}\delta_{n2}+\delta_{m2}\delta_{n1}\right)\delta_{jl}\right]\\ & +\left(\delta_{j1}\delta_{h2}+\delta_{j2}\delta_{h1}\right)\left[\left(\delta_{l1}\delta_{n2}+\delta_{l2}\delta_{n1}\right)\delta_{im}+\left(\delta_{m1}\delta_{n2}+\delta_{m2}\delta_{n1}\right)\delta_{il}\right]\}\\ & +\frac{a_7}{2}\{\left(\delta_{i1}\delta_{h3}+\delta_{i3}\delta_{h1}\right)\left[\left(\delta_{l1}\delta_{n3}+\delta_{l3}\delta_{n1}\right)\delta_{jm}+\left(\delta_{m1}\delta_{n3}+\delta_{m3}\delta_{n1}\right)\delta_{jl}\right]\\ & +\left(\delta_{j1}\delta_{h3}+\delta_{j3}\delta_{h1}\right)\left[\left(\delta_{l1}\delta_{n3}+\delta_{l3}\delta_{n1}\right)\delta_{im}+\left(\delta_{m1}\delta_{n3}+\delta_{m3}\delta_{n1}\right)\delta_{il}\right]\}\\ & +\frac{a_8}{2}\{\left(\delta_{i2}\delta_{h3}+\delta_{i3}\delta_{h2}\right)\left[\left(\delta_{l2}\delta_{n3}+\delta_{l3}\delta_{n2}\right)\delta_{jm}+\left(\delta_{m2}\delta_{n3}+\delta_{m3}\delta_{n2}\right)\delta_{jl}\right]\\ & +\left(\delta_{j2}\delta_{h3}+\delta_{j3}\delta_{h2}\right)\left[\left(\delta_{l2}\delta_{n3}+\delta_{l3}\delta_{n2}\right)\delta_{im}+\left(\delta_{m2}\delta_{n3}+\delta_{m3}\delta_{n2}\right)\delta_{il}\right]\}\\ & +\frac{a_9}{2}\left[\delta_{i1}\left(\delta_{l1}\delta_{jm}+\delta_{m1}\delta_{jl}\right)+\delta_{j1}\left(\delta_{l1}\delta_{im}+\delta_{m1}\delta_{il}\right)\right]\delta_{h1}\delta_{n1}\\ & +\frac{a_{10}}{2}\left[\delta_{i3}\left(\delta_{l3}\delta_{jm}+\delta_{m3}\delta_{jl}\right)+\delta_{j3}\left(\delta_{l3}\delta_{im}+\delta_{m3}\delta_{il}\right)\right]\delta_{h3}\delta_{n3}\\ & +\frac{a_{11}}{2}\{\delta_{h3}\left[\delta_{ln}\left(\delta_{jm}\delta_{i3}+\delta_{im}\delta_{j3}\right)+\delta_{mn}\left(\delta_{jl}\delta_{i3}+\delta_{il}\delta_{j3}\right)\right]\\ & +\delta_{n3}\left[\delta_{ih}\left(\delta_{jm}\delta_{l3}+\delta_{jl}\delta_{m3}\right)+\delta_{jh}\left(\delta_{im}\delta_{l3}+\delta_{il}\delta_{m3}\right)\right]\}\\ & +\frac{a_{12}}{2}\{\delta_{h1}\delta_{n3}\left[\delta_{i1}\left(\delta_{jm}\delta_{l3}+\delta_{jl}\delta_{m3}\right)+\delta_{j1}\left(\delta_{im}\delta_{l3}+\delta_{il}\delta_{m3}\right)\right]\\ & \delta_{h3}\delta_{n1}\left[\delta_{i3}\left(\delta_{jm}\delta_{l1}+\delta_{jl}\delta_{m1}\right)+\delta_{j3}\left(\delta_{im}\delta_{l1}+\delta_{il}\delta_{m1}\right)\right]\},\end{aligned} \tag{36}$$

with $\mathbf{A}^{iso}$ given by eqn (15), parameters a_i $(i = 1, ..., 5)$ by eqn (16), and

$$\begin{gathered} a_6 = -f\frac{\rho^2}{2}\tilde{\xi}^{III}, \qquad a_7 = -f\frac{\rho^2}{2}\tilde{\xi}^{II}, \qquad a_8 = -f\frac{\rho^2}{2}\tilde{\xi}^{I},\\ a_9 = -f\frac{\rho^2}{2}\tilde{\omega}^{I}, \qquad a_{10} = -f\frac{\rho^2}{2}\tilde{\omega}^{II}, \qquad a_{11} = -f\frac{\rho^2}{2}\tilde{\omega}^{III}, \qquad a_{12} = -f\frac{\rho^2}{2}\tilde{\omega}^{IV}, \end{gathered} \tag{37}$$

According to the results presented in subsections 2.1 and 2.2, the effective higher-order tensor $\mathbf{A}^{eq}$ results to be an orthotropic sixth-order tensor, positive definite when $\tilde{\mathbf{C}}$, eqn (35),

[4] Note that the cubic representation (29) is obtained as a particular case by setting $\tilde{\xi}^{I} = \tilde{\xi}^{II} = \tilde{\xi}^{III} = \tilde{\xi}$ and $\tilde{\omega}^{I} = \tilde{\omega}^{II} = \tilde{\omega}^{III} = \tilde{\omega}^{IV} = 0$.

is negative definite, namely

$$
\begin{cases}
\tilde{\mu} + \tilde{\xi}^{III} < 0, \\
\tilde{\mu} + \tilde{\xi}^{II} < 0, \\
\tilde{\mu} + \tilde{\xi}^{I} < 0, \\
\tilde{\lambda} + 2\tilde{\mu} + \tilde{\omega}^{I} < 0, \\
4\tilde{\mu}(\tilde{\lambda} + \tilde{\mu}) + (\tilde{\lambda} + 2\tilde{\mu})\tilde{\omega}^{I} < 0, \\
8\tilde{\mu}^3 - \tilde{\omega}^{I}\tilde{\omega}^{III\,2} + 4\tilde{\mu}^2(\tilde{\omega}^{I} + \tilde{\omega}^{II} + 2\tilde{\omega}^{III}) + \tilde{\lambda}\left(12\tilde{\mu}^2 + \tilde{\omega}^{I}\tilde{\omega}^{II} + 4\tilde{\mu}(\tilde{\omega}^{I} + \tilde{\omega}^{II} - \tilde{\omega}^{IV}) - \tilde{\omega}^{IV\,2}\right) \\
\qquad -2\tilde{\mu}\left(2\tilde{\omega}^{III\,2} - \tilde{\omega}^{I}(\tilde{\omega}^{II} + 2\tilde{\omega}^{III}) + 2\tilde{\omega}^{III}\tilde{\omega}^{IV} + \tilde{\omega}^{IV\,2}\right) < 0,
\end{cases}
\tag{38}
$$

while in the case of plane strain, conditions (38) become, in the x_1–x_2 plane

$$
\begin{cases}
\tilde{\mu} + \tilde{\xi}^{III} < 0, \\
\tilde{\lambda} + 2\tilde{\mu} + \tilde{\omega}^{I} < 0, \\
4\tilde{\mu}(\tilde{\lambda} + \tilde{\mu}) + (\tilde{\lambda} + 2\tilde{\mu})\tilde{\omega}^{I} < 0.
\end{cases}
\tag{39}
$$

Orthotropic matrix with cylindrical holes We consider the plane strain of an orthotropic matrix containing a dilute suspension of circular holes. In particular, assuming x_3 as the out-of-plane direction and x_1 and x_2 as the orthotropy axes, the discrepancy tensor has the form (35) and is characterized by the following constants [5] (Tsukrov and Kachanov, 2000)

$$
\begin{aligned}
\tilde{\lambda} &= \frac{\gamma(\lambda_1 + 2\mu_1)\left\{\left[(-1+\gamma)^2 - (1+\gamma)\delta\right]\lambda_1^2 + 2\left[2(-1+\gamma)\gamma - (1+\gamma)\delta\right]\lambda_1\mu_1 + 4\gamma^2\mu_1^2\right\}}{\left[(-1+\gamma)\lambda_1 + 2\gamma\mu_1\right](\lambda_1 + \gamma\lambda_1 + 2\gamma\mu_1)}, \\
\tilde{\mu} &= -(\lambda_1 + 2\mu_1) \times \\
&\quad \frac{\left(-1+\gamma^2\right)(-1+\gamma-\delta)\lambda_1^2 + 2(-1+\gamma)\gamma(2+2\gamma-\delta)\lambda_1\mu_1 + 4\gamma\left(\gamma+\gamma^2+\delta\right)\mu_1^2}{2\left[(-1+\gamma)\lambda_1 + 2\gamma\mu_1\right](\lambda_1 + \gamma\lambda_1 + 2\gamma\mu_1)}, \\
\tilde{\xi} &= -\tilde{\mu} - \frac{\delta(1+\gamma+\delta)(\lambda_1 + 2\mu_1)\left[(-1+\gamma)\lambda_1 + 2\gamma\mu_1\right](\lambda_1 + \gamma\lambda_1 + 2\gamma\mu_1)}{\left[\left(-2+2\gamma-\delta^2\right)\lambda_1 + 4\gamma\mu_1 - 2\delta^2\mu_1\right]^2}. \\
\tilde{\omega} &= -\tilde{\mu} - \gamma(\lambda_1 + 2\mu_1) \times \\
&\quad \frac{\left(-1+\gamma^2\right)(-1+\gamma+\gamma\delta)\lambda_1^2 + 2(-1+\gamma)\left[\delta + 2\gamma(1+\gamma)(1+\delta)\right]\lambda_1\mu_1 + 4\gamma^2(1+\gamma+\gamma\delta)\mu_1^2}{2\left[(-1+\gamma)\lambda_1 + 2\gamma\mu_1\right](\lambda_1 + \gamma\lambda_1 + 2\gamma\mu_1)},
\end{aligned}
\tag{40}
$$

[5] For conciseness, in this subsection the in-plane orthotropy parameters ξ^{III} and ω^{I} are denoted by ξ and ω, respectively, in the representation of both matrix and discrepancy quantities.

where

$$\gamma = \sqrt{\Gamma^2 - \Delta}, \qquad \delta = \sqrt{\Gamma + \sqrt{\Delta}} + \sqrt{\Gamma - \sqrt{\Delta}}, \qquad \Gamma = \frac{2\mu_1(\mu_1 + \omega_1) + \lambda_1(\mu_1 - \xi_1 + \omega_1)}{(\lambda_1 + 2\mu_1)(\mu_1 + \xi_1)},$$

$$\Delta = \frac{[-2\xi_1(\lambda_1 + 2\mu_1 + \xi_1) + (\lambda_1 + 2\mu_1)\omega_1][2\mu_1(\mu_1 + \omega_1) + \lambda_1(2\mu_1 + \omega_1)]}{(\lambda_1 + 2\mu_1)^2(\mu_1 + \xi_1)^2}. \tag{41}$$

The non-null constants a_2, $a_4 = a_5$, a_6, and a_9 defining the effective higher-order tensor $\mathbf{A}^{eq}$ can explicitly be evaluated using eqns (16) and (37), when a specific orthotropic matrix is considered. With reference to orthotropic properties of olivine, pine wood, olivinite, marble, and canine femora (which orthotropic constitutive parameters are reported in Tab. 2 for the three possible orientations of orthotropy) used as matrix material, the corresponding non-null higher-order constants are given in Tab. 3 for a dilute suspension of cylindrical holes with centers aligned parallel to the in-plane orthotropy axes. All the three possible orientations (Or1, Or2, Or3) are considered for the axis of the cylindrical inclusion, defining the out-of-plane direction in the plane strain problem considered.

Matrix material	Orientation	λ_1	μ_1	ξ_1	ω_1
Olivine	Or1	66.000	47.000	−17.000	32.000
	Or2	60.000	106.000	−75.000	−80.000
	Or3	56.000	52.000	−27.500	112.000
Pine (softwood)	Or1	0.740	8.180	−7.590	−15.860
	Or2	0.760	0.515	−0.476	−0.550
	Or3	0.940	8.080	−7.625	−15.310
Olivinite	Or1	93.000	58.500	−21.85	22.000
	Or2	92.000	53.500	−18.05	33.000
	Or3	82.000	64.000	−29.7	−11.000
Marble	Or1	51.000	29.500	−14.65	9.000
	Or2	52.000	26.000	−10.65	15.000
	Or3	47.000	31.500	−15.2	−6.000
Canine femora	Or1	9.730	6.235	−2.900	−3.200
	Or2	11.900	8.900	−6.065	−10.700
	Or3	11.900	5.150	−2.815	7.500

Tab. 2: Values of the elastic constants $\lambda_1, \mu_1, \xi_1, \omega_1$ for different orthotropic materials, namely: olivine (Chevrot and Browaeys, 2004), pine wood (Yamai, 1957), olivinite, marble (Aleksandrov, Ryzhove and Belikov, 1968), and canine femora (Cowin and Van Buskirk, 1986). The reported values are in GPa.

4 Conclusions

Assuming Cauchy elastic composites made up of a dilute suspension of inclusions and an RVE with a spherical ellipsoid of inertia, the equivalent higher-order constitutive behaviour (of 'Mindlin type') can be defined in a rigorous way, even for anisotropy of the constituents and complex shape of the inclusions. Through this procedure a perfect match of the elastic

Matrix material	Orientation	$\frac{a_2}{f\rho^2\mu_1}$	$\frac{a_4}{f\rho^2\mu_1}$	$\frac{a_6}{f\rho^2\mu_1}$	$\frac{a_9}{f\rho^2\mu_1}$
Olivine	Or1	2.426	1.661	3.077	−1.198
	Or2	1.133	2.105	−1.014	−1.804
	Or3	3.254	1.497	0.858	−0.780
Pine wood	Or1	0.269	3.789	−3.754	−3.737
	Or2	10.297	3.551	−3.268	−3.497
	Or3	0.142	3.478	−3.455	−3.399
Olivinite	Or1	3.119	1.644	−0.220	−1.045
	Or2	4.398	1.414	0.804	−0.675
	Or3	4.011	1.481	0.487	−0.782
Marble	Or1	4.023	1.629	−0.257	−1.068
	Or2	5.866	1.389	0.823	−0.768
	Or3	5.080	1.532	0.440	−1.015
Canine femora	Or1	8.279	1.219	2.465	−0.801
	Or2	4.401	2.110	−1.875	−1.788
	Or3	4.273	1.660	−0.690	−1.063

Tab. 3: Higher-order equivalent constants a_2, $a_4 = a_5$, a_6, and a_9, eqns (16) and (37), of the orthotropic SGE material equivalent to an orthotropic matrix containing a dilute suspension of cylindrical holes, collinear to three possible orientations of orthotropy. The constants are made dimensionless through division by parameter $f\rho^2\mu_1$ and are reported for different matrices, which orthotropy parameters are given in Tab. 2.

energies of the RVE and of the equivalent higher-order material is obtained, for a general class of displacements prescribed on the two respective boundaries. However, it has been shown that, to achieve a positive definite strain energy of the equivalent higher-order material, the inclusions have to be less stiff (in a way previously detailed) than the matrix, a situation already found by Bigoni and Drugan (2007) for Cosserat equivalent materials, which limits the applicability of the presented results, but explains the interpretation of previous experiments and results showing nonlocal effects for soft inclusions and 'anti-micropolar' behaviour for stiff ones (Gauthier, 1982).

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