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# A boundary element technique for incremental, non-linear elasticity Part I: Formulation

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## Abstract

Incremental elastic deformations superimposed upon a given homogeneous strain are analyzed with a boundary element technique. This is based on a recently-developed Green's function for non-linear incremental elastic deformations. Plane strain perturbations are considered of a broad class of incompressible material behaviours (including hyper-, hypoelastic and Navier–Stokes constitutive equations) within the elliptic range. Numerical treatment of the problem is detailed. A possibility of employing the method in the fully non-linear range is outlined, which yields a boundary element approach where the use of domain integrals is avoided, at least in a conventional sense. The methods for bifurcation and shear band analyses will be reported in Part II.

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## 1. Introduction

The analysis of the response to perturbations of a pre-stressed, non-linear elastic solid is important in a broad range of technological circumstances. For instance, pre-stress affects the design of Microelectromechanical Systems [11], it is a concern in the behaviour of geological formations [38,39], biological tissues [10,14], and various structural elements, including seismic insulators and rubber bearings [9,18].

Referring to plane strain deformations of incompressible materials, Biot [2] has shown that the incremental elastic response is governed by two incremental moduli, functions of the current stretch. Biot's constitutive framework was assumed by Bigoni and Capuani [1] to obtain a Green's function and a boundary integral formulation for incremental deformations superimposed upon a given, homogeneous strain. Both Green's function and integral formulation provide the basis to build a boundary element

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technique for solving incremental problems of non-linear elasticity. This is the purpose of the article. In particular, we formulate a general numerical scheme to handle generic boundary value problems with prescribed nominal tractions and/or displacements. When restricted to perturbations of homogeneously deformed, incompressible solids, our boundary elements technique retains all well-known advantages of the small strain formulation. These are:

- discretization only of the boundary of the body;
- automatic satisfaction of the incompressibility constraint;
- possibility of describing singularities arising near corner points of the boundary;
- possibility of employing meshes thoroughly varying in size throughout the body.

Several attempts can be found in the literature to analyze non-linear problems using boundary elements techniques. In some cases the non-linearities were related to the material [4,5,20,34,41], in other cases to large elastic [24,31,35] or elastoplastic [6–8,12,17,28–30] strains. In all cases, in addition to the usual boundary integrals, a domain integral is introduced, leading to the so-called ‘field-boundary element method’. The introduction of this term nullifies a main advantage of BEM and originates from the discrepancy between the non-linear character of the equations governing the problem and the employed fundamental solution (usually referring to linear, isotropic elasticity). In the present paper, the focus is on *incrementally-linear* problems, so that domain integrals do not appear in the formulation that will be presented. However, we believe that the solution of incrementally linear problems should be regarded as the first step toward the analysis of fully non-linear situations and, in particular, we anticipate with a simple example that when employed as a tool to analyze large (thus non-linear) deformations, our incremental method naturally leads to a volume discretization different—in essence—from all those already known.

The paper is organized as follows: the incremental constitutive framework—which includes hyper- and hypoelasticity and the equations governing Stokes flow of fluids—is presented in Section 2 for plane strain, whereas the fundamental solution and boundary integral equations are summarized in Section 3. The boundary element formulation is presented in Section 4 and the discretization detailed in subsequent Section 5. A numerical example is given in Section 6, which offers the simplest context to illustrate the capability of the method to computationally follow a non-linear path of deformation. Systematic investigations of bifurcation and shear band phenomena in two-dimensional elastic materials will be reported in Part II.

## 2. Incremental constitutive equations

The general expression given by Biot [2] for rate constitutive equations of an incompressible material incrementally deformed in plane strain are obtained here with reference to a broad class of material behaviours.

### 2.1. Elasticity

The relation between the Cauchy stress  $\boldsymbol{\sigma}$  and the left Cauchy–Green strain tensor  $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ , with  $\mathbf{F}$  denoting the deformation gradient, for a Cauchy-elastic, incompressible and isotropic solid can be written as [40]

$$\boldsymbol{\sigma} = -q\mathbf{I} + \beta_0\mathbf{B} + \beta_1\mathbf{B}^{-1}, \quad (1)$$

where  $q$  is a parameter connected to the hydrostatic pressure  $\hat{p} = \text{tr } \boldsymbol{\sigma}/3$  through

$$q = -\hat{p} + \beta_0 I_1 + \frac{1}{2} \beta_1 (I_1^2 - I_2), \quad I_1 = \text{tr} \mathbf{B}, \quad I_2 = \text{tr} \mathbf{B}^2, \tag{2}$$

and  $\beta_0$  and  $\beta_1$  are generic functions of two invariants of  $\mathbf{B}$

$$\beta_0 = \beta_0(I_1, I_2), \quad \beta_1 = \beta_1(I_1, I_2). \tag{3}$$

The two particular cases of Mooney–Rivlin and neo-Hookean materials are recovered when  $\beta_0$  and  $\beta_1$  are taken constant and, in addition, when  $\beta_1 = 0$  in the latter case.

Constitutive equation (1) corresponds to Cauchy elasticity, so that it describes a class of behaviours broader than hyperelasticity. The requirement of existence of an elastic potential influences the dependence of coefficients  $\beta_0$  and  $\beta_1$  on the invariants of  $\mathbf{B}$ . To develop this point, let us consider that the constitutive equation (1) implies coaxiality of tensors  $\mathbf{B}$  and  $\boldsymbol{\sigma}$ , so that these share (at least) one principal reference system—the Eulerian principal axes—where

$$\text{diag} \mathbf{B} = (\lambda_1^2, \lambda_2^2, \lambda_3^2), \quad \text{diag} \boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3), \tag{4}$$

in which  $\lambda_i > 0$ ,  $i = 1, 2, 3$  are the principal stretches, satisfying the incompressibility constraint

$$\lambda_1 \lambda_2 \lambda_3 = 1. \tag{5}$$

Expressing Eq. (1) in the Eulerian principal reference system and solving for  $\beta_0$  and  $\beta_1$  yields

$$\beta_0 = \frac{1}{\lambda_1^2 - \lambda_2^2} \left[ \frac{(\sigma_1 - \sigma_3) \lambda_1^2}{\lambda_1^2 - \lambda_3^2} - \frac{(\sigma_2 - \sigma_3) \lambda_2^2}{\lambda_2^2 - \lambda_3^2} \right], \tag{6}$$

$$\beta_1 = \frac{1}{\lambda_1^2 - \lambda_2^2} \left( \frac{\sigma_1 - \sigma_3}{\lambda_1^2 - \lambda_3^2} - \frac{\sigma_2 - \sigma_3}{\lambda_2^2 - \lambda_3^2} \right),$$

two equations that can be alternatively expressed employing every permutation of 1, 2 and 3 as indices. It is clear from the expression (6) that existence of an elastic potential restricts the functional dependence of coefficients  $\beta_i$  on the stretch. In particular, the standard definition of elastic potential  $W = W(\lambda_1, \lambda_2, \lambda_3)$  for incompressible materials is [27]

$$\sigma_i - \sigma_j = \lambda_i \frac{\partial W}{\partial \lambda_i} - \lambda_j \frac{\partial W}{\partial \lambda_j}, \quad i \neq j = 1, 2, 3, \quad \text{not summed}, \tag{7}$$

which can be immediately employed into Eq. (6).

Taking the material derivative of (1) yields

$$\dot{\boldsymbol{\sigma}} = -\dot{q} \mathbf{I} + \beta_0 \dot{\mathbf{B}} + \beta_1 (\mathbf{B}^{-1})^\cdot + \dot{\beta}_0 \mathbf{B} + \dot{\beta}_1 \mathbf{B}^{-1}, \tag{8}$$

where

$$\dot{\mathbf{B}} = \mathbf{D} \mathbf{B} + \mathbf{B} \mathbf{D} + \mathbf{W} \mathbf{B} - \mathbf{B} \mathbf{W}, \tag{9}$$

and

$$(\mathbf{B}^{-1})^\cdot = -\mathbf{B}^{-1} \mathbf{D} - \mathbf{D} \mathbf{B}^{-1} - \mathbf{B}^{-1} \mathbf{W} + \mathbf{W} \mathbf{B}^{-1}, \tag{10}$$

in which  $\mathbf{D}$  is the Eulerian strain rate and  $\mathbf{W}$  the spin tensor. Keeping into account now Eq. (1) and the definition of Jaumann derivative

$$\overset{\nabla}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \mathbf{W} \boldsymbol{\sigma} + \boldsymbol{\sigma} \mathbf{W}, \tag{11}$$

the constitutive equation (1) becomes

$$\overset{\nabla}{\boldsymbol{\sigma}} + \dot{q}\mathbf{I} = \beta_0(\mathbf{DB} + \mathbf{BD}) - \beta_1(\mathbf{B}^{-1}\mathbf{D} + \mathbf{DB}^{-1}) + \dot{\beta}_0\mathbf{B} + \dot{\beta}_1\mathbf{B}^{-1}. \tag{12}$$

Noting that

$$\dot{\beta}_i = \frac{\partial\beta_i}{\partial I_1} \text{tr}\dot{\mathbf{B}} + 2\frac{\partial\beta_i}{\partial I_2} \mathbf{B} \cdot \dot{\mathbf{B}}, \tag{13}$$

where  $\beta_i = \beta_i(I_1, I_2)$ , with  $i = 0, 1$  and

$$\text{tr}\dot{\mathbf{B}} = 2\mathbf{B} \cdot \mathbf{D}, \quad \mathbf{B} \cdot \dot{\mathbf{B}} = 2\mathbf{B}^2 \cdot \mathbf{D}, \tag{14}$$

or

$$\dot{\hat{\beta}}_i = \frac{\partial\hat{\beta}_i}{\partial\lambda_1} \dot{\lambda}_1 + \frac{\partial\hat{\beta}_i}{\partial\lambda_2} \dot{\lambda}_2, \tag{15}$$

where  $\hat{\beta}_i = \hat{\beta}_i(\lambda_1, \lambda_2)$  with  $i = 0, 1$  are the coefficients  $\beta_0$  and  $\beta_1$  expressed as functions of the principal stretches, we arrive at

$$\begin{aligned} \overset{\nabla}{\boldsymbol{\sigma}} + \dot{q}\mathbf{I} = & \beta_0(\mathbf{DB} + \mathbf{BD}) - \beta_1(\mathbf{B}^{-1}\mathbf{D} + \mathbf{DB}^{-1}) + 2\left(\mathbf{B} \cdot \mathbf{D} \frac{\partial\beta_0}{\partial I_1} + 2\mathbf{B}^2 \cdot \mathbf{D} \frac{\partial\beta_0}{\partial I_2}\right)\mathbf{B} \\ & + 2\left(\mathbf{B} \cdot \mathbf{D} \frac{\partial\beta_1}{\partial I_1} + 2\mathbf{B}^2 \cdot \mathbf{D} \frac{\partial\beta_1}{\partial I_2}\right)\mathbf{B}^{-1}, \end{aligned} \tag{16}$$

which is equivalent to

$$\overset{\nabla}{\boldsymbol{\sigma}} + \dot{q}\mathbf{I} = \beta_0(\mathbf{DB} + \mathbf{BD}) - \beta_1(\mathbf{B}^{-1}\mathbf{D} + \mathbf{DB}^{-1}) + \left(\frac{\partial\hat{\beta}_0}{\partial\lambda_1} \dot{\lambda}_1 + \frac{\partial\hat{\beta}_0}{\partial\lambda_2} \dot{\lambda}_2\right)\mathbf{B} + \left(\frac{\partial\hat{\beta}_1}{\partial\lambda_1} \dot{\lambda}_1 + \frac{\partial\hat{\beta}_1}{\partial\lambda_2} \dot{\lambda}_2\right)\mathbf{B}^{-1}. \tag{17}$$

The incremental constitutive equation in the form (16) or (17) is valid for three-dimensional, incompressible Cauchy elasticity.

We are interested here in the particularization of (16) to *incremental plane strain deformations* superimposed on a generic state of *homogeneous* deformation. In the Eulerian principal reference system

$$\text{diag}\mathbf{B} = \left(\lambda_1^2, \lambda_2^2, \frac{1}{\lambda_1^2\lambda_2^2}\right), \quad D_{i3} = D_{3i} = 0, \quad i = 1, 2, 3, \tag{18}$$

so that the out-of-plane stress rate components can be determined as

$$\overset{\nabla}{\sigma}_{3i} = \overset{\nabla}{\sigma}_{i3} = 0, \quad i = 1, 2 \tag{19}$$

and

$$\overset{\nabla}{\sigma}_{33} = -\dot{q} + (\lambda_1^2 - \lambda_2^2) \left\{ \frac{1}{\lambda_1^2\lambda_2^2} \left[ \frac{\partial\beta_0}{\partial I_1} + 2(\lambda_1^2 + \lambda_2^2) \frac{\partial\beta_0}{\partial I_2} \right] + \lambda_1^2\lambda_2^2 \left[ \frac{\partial\beta_1}{\partial I_1} + 2(\lambda_1^2 + \lambda_2^2) \frac{\partial\beta_1}{\partial I_2} \right] \right\} (D_{11} - D_{22}), \tag{20}$$

or

$$\overset{\nabla}{\sigma}_{33} = -\dot{q} + \left[ \frac{1}{\lambda_1^2\lambda_2^2} \left( \lambda_1 \frac{\partial\hat{\beta}_0}{\partial\lambda_1} - \lambda_2 \frac{\partial\hat{\beta}_0}{\partial\lambda_2} \right) + \lambda_1^2\lambda_2^2 \left( \lambda_1 \frac{\partial\hat{\beta}_1}{\partial\lambda_1} - \lambda_2 \frac{\partial\hat{\beta}_1}{\partial\lambda_2} \right) \right] \frac{(D_{11} - D_{22})}{2}. \tag{21}$$

Finally, the in-plane rate components can be expressed in the Biot [2] form as

$$\begin{cases} \overset{\nabla}{\sigma}_{12} = 2\mu D_{12}, \\ \overset{\nabla}{\sigma}_{11} - \overset{\nabla}{\sigma}_{22} = 2\mu_*(D_{11} - D_{22}), \\ D_{11} + D_{22} = 0, \end{cases} \tag{22}$$

where  $\mu$  and  $\mu_*$  are two incremental moduli corresponding respectively to shearing parallel to, and at 45° to, the Eulerian principal axes. These can be expressed as functions of the invariants of **B**

$$\begin{aligned} \mu &= \frac{\lambda_1^2 + \lambda_2^2}{2} \left( \beta_0 - \frac{\beta_1}{\lambda_1^2 \lambda_2^2} \right), \\ \mu_* &= \frac{\lambda_1^2 + \lambda_2^2}{2} \beta_0 + \frac{(\lambda_1^2 - \lambda_2^2)^2}{2} \left[ \frac{\partial \beta_0}{\partial I_1} + 2(\lambda_1^2 + \lambda_2^2) \frac{\partial \beta_0}{\partial I_2} \right] \\ &\quad - \frac{1}{\lambda_1^2 \lambda_2^2} \left\{ \frac{\lambda_1^2 + \lambda_2^2}{2} \beta_1 + \frac{(\lambda_1^2 - \lambda_2^2)^2}{2} \left[ \frac{\partial \beta_1}{\partial I_1} + 2(\lambda_1^2 + \lambda_2^2) \frac{\partial \beta_1}{\partial I_2} \right] \right\}, \end{aligned} \tag{23}$$

or as functions of the principal stretches

$$\begin{aligned} \mu &= \frac{\lambda_1^2 + \lambda_2^2}{2} \left( \hat{\beta}_0 - \frac{\hat{\beta}_1}{\lambda_1^2 \lambda_2^2} \right), \\ \mu_* &= \frac{\lambda_1^2 + \lambda_2^2}{2} \hat{\beta}_0 + \frac{\lambda_1^2 - \lambda_2^2}{4} \left( \lambda_1 \frac{\partial \hat{\beta}_0}{\partial \lambda_1} - \lambda_2 \frac{\partial \hat{\beta}_0}{\partial \lambda_2} \right) \\ &\quad - \frac{1}{\lambda_1^2 \lambda_2^2} \left[ \frac{\lambda_1^2 + \lambda_2^2}{2} \hat{\beta}_1 + \frac{\lambda_1^2 - \lambda_2^2}{4} \left( \lambda_1 \frac{\partial \hat{\beta}_1}{\partial \lambda_1} - \lambda_2 \frac{\partial \hat{\beta}_1}{\partial \lambda_2} \right) \right]. \end{aligned} \tag{24}$$

An alternative expression for the two incremental moduli  $\mu$  and  $\mu_*$ , related to the existence of a strain-energy function, was given by Biot [2] (see Appendix A) in the form

$$\begin{aligned} \mu &= \frac{1}{2} \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 - \lambda_2^2} (\sigma_1 - \sigma_2), \\ \mu_* &= \frac{1}{4} \left( \lambda_1 \frac{\partial W}{\partial \lambda_1} + \lambda_2 \frac{\partial W}{\partial \lambda_2} + \lambda_1^2 \frac{\partial^2 W}{\partial \lambda_1^2} + \lambda_2^2 \frac{\partial^2 W}{\partial \lambda_2^2} - 2\lambda_1 \lambda_2 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} \right). \end{aligned} \tag{25}$$

As it will become clear later, constitutive relations of the form (22) with generic coefficients  $\mu$  and  $\mu_*$  embrace a much broader class of material behaviours than Cauchy, isotropic elasticity.

### 2.1.1. Mooney–Rivlin material

A simple explicit form of strain-energy functions for isotropic rubber-like elastic media was proposed by Mooney [22] in the form

$$W(I_1, I_2) = \frac{\mu_1}{2} (I_1 - 3) - \frac{\mu_2}{4} (I_1^2 - I_2 - 6), \tag{26}$$

or

$$\hat{W}(\lambda_1, \lambda_2) = \frac{\mu_1}{2} (\lambda_1^2 + \lambda_2^2 + \lambda_1^{-2} \lambda_2^{-2} - 3) - \frac{\mu_2}{2} (\lambda_1^{-2} + \lambda_2^{-2} + \lambda_1^2 \lambda_2^2 - 3), \tag{27}$$

where  $\mu_1$  and  $\mu_2$  are material parameters and  $\mu_0 = \mu_1 - \mu_2$  represents the shear modulus in the original unstressed state.

In this case, with reference to the representation (1), we simply obtain:

$$\begin{aligned} \beta_0 &= \mu_1, & \beta_1 &= \mu_2, \\ \mu &= \mu_* = \frac{1}{2}(\lambda_1^2 + \lambda_2^2) \left( \mu_1 - \frac{\mu_2}{\lambda_1^2 \lambda_2^2} \right) \end{aligned} \quad (28)$$

and

$$\overset{\nabla}{\sigma}_{33} = -\dot{q}. \quad (29)$$

### 2.1.2. Ogden material

The following class of strain-energy functions was proposed by [25] to fit experimental results on rubber:

$$\widehat{W}(\lambda_1, \lambda_2) = \sum_{i=1}^N \frac{\mu_i}{\alpha_i} [\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + (\lambda_1 \lambda_2)^{-\alpha_i} - 3], \quad (30)$$

where  $\mu_i$  and  $\alpha_i$  are material parameters, subjected to the constraints:

$$2\mu_0 = \sum_{i=1}^N \mu_i \alpha_i, \quad \text{with } \mu_i \alpha_i > 0, \quad i = 1, \dots, N, \quad (31)$$

in which,  $N$  is a positive integer determining the number of terms in the strain-energy function,  $\mu_i$  are constant shear moduli and  $\alpha_i$  are dimensionless parameters, with  $i = 1, \dots, N$ . In particular, Mooney–Rivlin material can be recovered as a particular case taking  $N = 2$ ,  $\alpha_1 = 2$  and  $\alpha_2 = -2$ .

An excellent correlation with experimental data relative to simple-tension, equibiaxial tension and pure shear of vulcanised rubber is obtained employing the values [25,36,37]:

$$\begin{cases} \alpha_1 = 1.3 & \mu_1 = 6.3 \times 10^5 \text{ N/m}^2, \\ \alpha_2 = 5.0 & \mu_2 = 0.012 \times 10^5 \text{ N/m}^2, \\ \alpha_3 = -2.0 & \mu_3 = 0.1 \times 10^5 \text{ N/m}^2, \end{cases} \quad (32)$$

yielding  $\mu_0 = 4.225 \times 10^5 \text{ N/m}^2$ .

Coefficients in the representations (16) and (22) can be obtained from Eq. (30) in the form:

$$\begin{aligned} \hat{\beta}_0 &= \frac{1}{\lambda_1^2 - \lambda_2^2} \sum_{i=1}^N \mu_i \left[ \frac{\lambda_1^{\alpha_i} - (\lambda_1 \lambda_2)^{-\alpha_i}}{\lambda_1^2 - (\lambda_1 \lambda_2)^{-2}} \lambda_1^2 - \frac{\lambda_2^{\alpha_i} - (\lambda_1 \lambda_2)^{-\alpha_i}}{\lambda_2^2 - (\lambda_1 \lambda_2)^{-2}} \lambda_2^2 \right], \\ \hat{\beta}_1 &= \frac{1}{\lambda_1^2 - \lambda_2^2} \sum_{i=1}^N \mu_i \left[ \frac{\lambda_1^{\alpha_i} - (\lambda_1 \lambda_2)^{-\alpha_i}}{\lambda_1^2 - (\lambda_1 \lambda_2)^{-2}} - \frac{\lambda_2^{\alpha_i} - (\lambda_1 \lambda_2)^{-\alpha_i}}{\lambda_2^2 - (\lambda_1 \lambda_2)^{-2}} \right] \end{aligned} \quad (33)$$

and

$$\begin{aligned} \mu &= \frac{1}{2} \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 - \lambda_2^2} \sum_{i=1}^N \mu_i (\lambda_1^{\alpha_i} - \lambda_2^{\alpha_i}), \\ \mu_* &= \frac{1}{4} \sum_{i=1}^N \alpha_i \mu_i (\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i}). \end{aligned} \quad (34)$$

The component  $\overset{\nabla}{\sigma}_{33}$  is given in Appendix B.

2.2. Hypoelasticity and the loading branch of elastoplastic constitutive laws

We consider a general incremental constitutive equation relating the Jaumann derivative of the Cauchy stress tensor to the Eulerian strain rate  $\mathbf{D}$  through a generic tensor  $\mathbf{T} \in \text{Sym}$  in the form

$$\overset{\nabla}{\boldsymbol{\sigma}} = -\dot{q}\mathbf{I} + \gamma_1\mathbf{D} + (\gamma_2\mathbf{T} \cdot \mathbf{D} + \gamma_3\mathbf{T}^2 \cdot \mathbf{D})\mathbf{I} + (\gamma_4\mathbf{T} \cdot \mathbf{D} + \gamma_5\mathbf{T}^2 \cdot \mathbf{D})\mathbf{T} + (\gamma_6\mathbf{T} \cdot \mathbf{D} + \gamma_7\mathbf{T}^2 \cdot \mathbf{D})\mathbf{T}^2 + \gamma_8(\mathbf{DT} + \mathbf{TD}) + \gamma_9(\mathbf{DT}^2 + \mathbf{T}^2\mathbf{D}), \tag{35}$$

where coefficients  $\gamma_i, i = 1, \dots, 9$  are polynomial functions of the invariants of  $\mathbf{T}$ . In the particular case in which  $\mathbf{T}$  is identified with the Cauchy stress  $\boldsymbol{\sigma}$ , the constitutive equation (35) describes an incompressible, *hypoelastic* material [40]. However, even if  $\mathbf{T}$  does not represent the Cauchy stress and the coefficients  $\gamma_i, i = 1, \dots, 9$ , remain completely unspecified (but independent of  $\mathbf{D}$ ), in a principal reference system of  $\mathbf{T}$  and for plane, incremental deformation, we get

$$\overset{\nabla}{\sigma}_{i3} = \overset{\nabla}{\sigma}_{3i} = 0, \quad i = 1, 2, \tag{36}$$

$$\overset{\nabla}{\sigma}_{33} = -\dot{q} + (T_1 - T_2)[\gamma_2 + \gamma_4 T_3 + \gamma_6 T_3^2 + (T_1 + T_2)(\gamma_3 + \gamma_5 T_3 + \gamma_7 T_3^2)]D_{11},$$

where  $T_i$  with  $i = 1, 2, 3$  denote the principal values of  $\mathbf{T}$  and  $\overset{\nabla}{\sigma}_{12}, \overset{\nabla}{\sigma}_{21}$  and  $\overset{\nabla}{\sigma}_{11} - \overset{\nabla}{\sigma}_{22}$  can be expressed in the form (22), with

$$\mu = \frac{\gamma_1 + \gamma_8(T_1 + T_2) + \gamma_9(T_1^2 + T_2^2)}{2}, \tag{37}$$

$$\mu_* = \frac{1}{2}\{\gamma_1 + (T_1 - T_2)^2[\gamma_4 + (T_1 + T_2)(\gamma_5 + \gamma_6) + (T_1 + T_2)^2\gamma_7] + (T_1 + T_2)\gamma_8 + (T_1^2 + T_2^2)\gamma_9\}.$$

2.2.1.  $J_2$ -deformation theory of plasticity: hyperelastic and hypoelastic approaches

A  $J_2$ -deformation theory of plasticity was proposed by Hutchinson and Neale [15] (see also [16,23]) in the framework of hyperelasticity, to model metals subject to proportional loading. With reference to the representation (1), the constitutive-law in the  $J_2$ -deformation theory of plasticity can be expressed as

$$\sigma_i = \frac{2}{3}E_s\epsilon_i + \hat{p}, \quad i = 1, 2, 3, \quad \epsilon_1 + \epsilon_2 + \epsilon_3 = 0, \tag{38}$$

where  $\epsilon_i = \log \lambda_i$  are the logarithmic strains and  $\hat{p} = \text{tr } \boldsymbol{\sigma} / 3$ . In the same equation  $E_s$  is the secant modulus to the curve representing the effective stress  $\sigma_E$  versus effective strain  $\epsilon_E$

$$\epsilon_E = \sqrt{\frac{2}{3}(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2)}, \quad \sigma_E = \sqrt{\frac{3}{2}(S_1^2 + S_2^2 + S_3^2)}, \tag{39}$$

where  $S_i$  are the principal components of deviatoric stress. The curve is assumed to be determined by

$$E_s = K\epsilon_E^{N-1}, \tag{40}$$

where  $N \in ]0, 1[$  is an hardening exponent,  $K$  is a positive constitutive parameter. The strain-energy function results therefore to be

$$W^\epsilon = \frac{K}{N+1}\epsilon_E^{N+1}. \tag{41}$$

The above framework corresponds to the following choice of parameters  $q$ ,  $\beta_0$  and  $\beta_1$  (see (1) and (38)):

$$\begin{aligned}
 q &= -\hat{\mathbf{p}} + \frac{2}{3} E_s \frac{(\lambda_2^4 - \lambda_2^{-2})(\lambda_1^2 + \lambda_3^2)\epsilon_1 - (\lambda_1^4 - \lambda_1^{-2})(\lambda_2^2 + \lambda_3^2)\epsilon_2}{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)(\lambda_2^2 - \lambda_3^2)}, \\
 \beta_0 &= \frac{2}{3} E_s \frac{1}{\lambda_1^2 - \lambda_2^2} \left[ \frac{(\epsilon_1 - \epsilon_3)\lambda_1^2}{\lambda_1^2 - \lambda_3^2} - \frac{(\epsilon_2 - \epsilon_3)\lambda_2^2}{\lambda_2^2 - \lambda_3^2} \right], \\
 \beta_1 &= \frac{2}{3} E_s \frac{1}{\lambda_1^2 - \lambda_2^2} \left( \frac{\epsilon_1 - \epsilon_3}{\lambda_1^2 - \lambda_3^2} - \frac{\epsilon_2 - \epsilon_3}{\lambda_2^2 - \lambda_3^2} \right)
 \end{aligned}
 \tag{42}$$

and to the incremental shear moduli:

$$\begin{aligned}
 \mu &= \frac{1}{3} E_s (\epsilon_1 - \epsilon_2) \coth(\epsilon_1 - \epsilon_2), \\
 \mu_* &= \frac{1}{9} \frac{E_s}{\epsilon_E^2} [3(\epsilon_1 + \epsilon_2)^2 + N(\epsilon_1 - \epsilon_2)^2].
 \end{aligned}
 \tag{43}$$

The out-of-plane stress increment is given by

$$\overset{\nabla}{\sigma}_{33} = \dot{\mathbf{p}}.
 \tag{44}$$

Differently from Hutchinson and Neale [15], Stören and Rice [33] present the following hypoelastic law (their Eq. (26)) to model elastoplastic materials subject to proportional loading:

$$\overset{\nabla}{\mathbf{S}} = 2h_1 \mathbf{D} - \frac{1 - N}{N} \frac{\mathbf{S} \cdot \overset{\nabla}{\boldsymbol{\sigma}}}{\mathbf{S} \cdot \mathbf{S}} \mathbf{S},
 \tag{45}$$

where  $\mathbf{S} = \boldsymbol{\sigma} - \text{Itr} \boldsymbol{\sigma}/3$ , is the deviatoric stress,  $N$  is a work-hardening exponent and  $h_1$  is the secant modulus on the shear stress–strain curve. Eq. (45) can be written as

$$\overset{\nabla}{\boldsymbol{\sigma}} = \dot{\mathbf{p}}\mathbf{I} + 2h_1 \left[ \mathbf{D} - (1 - N) \frac{\mathbf{S} \cdot \mathbf{D}}{\mathbf{S} \cdot \mathbf{S}} \mathbf{S} \right],
 \tag{46}$$

where  $\dot{\mathbf{p}} = \text{tr} \dot{\boldsymbol{\sigma}}/3$ , which is, evidently, a particular case of (35) and can therefore be cast in the form (22).

### 2.2.2. The loading branch of non-associative, elastoplastic law

A generic constitutive equation for an incompressible isotropic-elastic, plastic material depending on a generic collection of state variables  $\mathcal{H}$  can be written in the form

$$\overset{\nabla}{\boldsymbol{\sigma}} - \dot{\mathbf{p}}\mathbf{I} = \begin{cases} 2\mu\mathbf{D} - \frac{4\mu^2}{H} \langle \mathbf{Q} \cdot \mathbf{D} \rangle \mathbf{P} & \text{if } f(\boldsymbol{\sigma}, \mathcal{H}) = 0, \\ 2\mu\mathbf{D} & \text{if } f(\boldsymbol{\sigma}, \mathcal{H}) < 0, \end{cases}
 \tag{47}$$

where  $f$  is the yield function,  $\mathbf{Q} \in \text{Sym}$  is the yield function gradient and  $\mathbf{P} \in \text{Sym}$  the plastic potential gradient. The Macaulay brackets  $\langle \cdot \rangle$  apply to every scalar  $\alpha$  in such a way that  $\langle \alpha \rangle = (\alpha + |\alpha|)/2$  and introduce the incremental non-linearity, typical of plasticity. The scalar  $H > 0$  is the plastic modulus, related to the hardening modulus  $h$  through

$$H = h + h_e \text{ with } h_e = 2\mu\mathbf{Q} \cdot \mathbf{P}.
 \tag{48}$$

Finally, tensors  $\mathbf{D}$ ,  $\mathbf{P}$  and  $\mathbf{Q}$  are subject to the incompressibility constraints

$$\text{tr} \mathbf{D} = 0, \quad \text{tr} \mathbf{P} = 0, \quad \text{tr} \mathbf{Q} = 0,
 \tag{49}$$

so that  $\dot{\mathbf{p}} = \text{tr} \overset{\nabla}{\boldsymbol{\sigma}}/3$ .

Restricting the constitutive equation (47) to its loading branch is synonymous of assuming  $f(\boldsymbol{\sigma}, \mathcal{H}) = 0$  and eliminating the incremental non-linearity, thus obtaining



$$\overset{\nabla}{\boldsymbol{\sigma}} = \dot{\hat{p}}\mathbf{I} + 2\mu\mathbf{D} - \frac{4\mu^2}{H}(\mathbf{Q} \cdot \mathbf{D})\mathbf{P}. \tag{50}$$

Let us assume now that  $\mathbf{P}$  and  $\mathbf{Q}$  are coaxial (and not necessarily coaxial with the Cauchy stress). In the principal reference system of  $\mathbf{P}$  and  $\mathbf{Q}$  and for plane incremental deformations, Eq. (50) can be cast in the form (22) with

$$\mu_* = \mu - \frac{\mu^2}{H}(Q_1 - Q_2)(P_1 - P_2). \tag{51}$$

The constitutive fourth-order tensor in (50) contains the non-symmetric term  $\mathbf{P} \otimes \mathbf{Q}$ , where  $\mathbf{Q} \neq \mathbf{P}$  corresponds to the so-called non-associative elastoplasticity. However, due to the incompressibility and plane strain constraints and without altering the material response, we can add a term

$$\mathbf{P} \otimes \mathbf{Q} + \mathbf{A} \otimes (\mathbf{I} - 3\mathbf{e}_3 \otimes \mathbf{e}_3), \tag{52}$$

where  $\mathbf{A}$  is a symmetric tensor coaxial to  $\mathbf{P}$  and  $\mathbf{Q}$  and  $\mathbf{e}_3$  is the unit vector singling the out-of-plane direction. Now, the choice

$$\text{diag} \mathbf{A} = \{P_2Q_1 - P_1Q_2 + \alpha, \alpha, -P_2Q_1 + P_1Q_2 - 2\alpha\}, \tag{53}$$

taken in the principal reference system of  $\mathbf{P}$  and  $\mathbf{Q}$ , symmetrizes the constitutive operator for every  $\alpha \in \mathbb{R}$ , so that non-associativity does not yield an unsymmetric tangent constitutive operator for incompressible, isotropic-elastic, plane-strain plasticity with coaxial yield function and plastic potential gradients.

### 2.3. Newtonian fluids

The constitutive equation describing a Newtonian, incompressible fluid takes the form

$$\boldsymbol{\sigma} = \hat{p}\mathbf{I} + 2\mu\mathbf{D}, \tag{54}$$

where the Cauchy stress is related to the Eulerian strain rate through the viscous coefficient  $\mu$  and the pressure in the fluid  $\hat{p} = \text{tr} \boldsymbol{\sigma} / 3$ . Clearly Eq. (54) has a structure identical to a very particular case of (35), so that, for plane strain

$$\begin{aligned} \sigma_{i3} &= 0, \quad i = 1, 2, \quad \sigma_{33} = \frac{\sigma_{11} + \sigma_{22}}{2}, \\ \sigma_{11} - \sigma_{22} &= 2\mu(D_{11} - D_{22}), \quad \sigma_{12} = \mu D_{12}, \end{aligned} \tag{55}$$

a form akin to (22). Therefore, keeping into account that we will address the problem of quasi-static deformations, our framework can be immediately employed to describe—*mutatis mutandis*—two-dimensional Stokes flow. As a consequence, several results presented in the following reduce to findings by Ladyzhenskaya [19] as particular cases.

### 2.4. The general form of constitutive equations for plane, incompressible, incremental deformations

We have shown in the section above that a very broad class of incremental material behaviours—including Cauchy elasticity, hyperelasticity, hypoelasticity, the loading branch of coaxial elastoplasticity, and Newtonian fluids—can be described by constitutive equation (22).

It is expedient now to transform the constitutive equation (22) in terms of material derivative of the nominal stress tensor  $\mathbf{t}$ . In particular, a Lagrangean formulation of field equations is employed in the following, with the current state taken as reference. As a consequence, constitutive equations (22) become

$$\dot{\mathbf{t}}_{ij} = \mathbb{K}_{ijkl}v_{l,k} + \hat{p}\delta_{ij} = \tilde{\mathbb{K}}_{ijkl}v_{l,k} + \hat{\pi}\delta_{ij}, \quad v_{i,i} = 0, \tag{56}$$

where  $v_i$  is the velocity,  $\delta_{ij}$  the Kronecker delta and

$$\dot{p} = \frac{\dot{\sigma}_1 + \dot{\sigma}_2}{2}, \quad \dot{\pi} = \frac{\dot{t}_{11} + \dot{t}_{22}}{2} = \dot{p} - \frac{\sigma_1 - \sigma_2}{2} v_{1,1}. \quad (57)$$

Both  $\dot{p}$  and  $\dot{\pi}$  measure the in plane hydrostatic stress rate (positive in tension) as related respectively to Cauchy and to nominal stresses. Tensor  $\mathbb{K}_{ijkl}$  represents the instantaneous moduli, possessing the major symmetry  $\mathbb{K}_{ijkl} = \mathbb{K}_{klij}$  and having the form [13]

$$\begin{aligned} \mathbb{K}_{1111} &= \mu_* - \frac{\sigma}{2} - p, & \mathbb{K}_{1122} &= -\mu_*, & \mathbb{K}_{1112} &= \mathbb{K}_{1121} = 0, \\ \mathbb{K}_{2211} &= -\mu_*, & \mathbb{K}_{2222} &= \mu_* + \frac{\sigma}{2} - p, & \mathbb{K}_{2212} &= \mathbb{K}_{2221} = 0, \\ \mathbb{K}_{1212} &= \mu + \frac{\sigma}{2}, & \mathbb{K}_{1221} &= \mathbb{K}_{2112} = \mu - p, & \mathbb{K}_{2121} &= \mu - \frac{\sigma}{2}, \end{aligned} \quad (58)$$

with

$$\sigma = \sigma_1 - \sigma_2, \quad p = \frac{\sigma_1 + \sigma_2}{2}. \quad (59)$$

Note that in Eq. (56)  $\tilde{\mathbb{K}}_{ijkl}$  differs from  $\mathbb{K}_{ijkl}$  only in the following components:

$$\tilde{\mathbb{K}}_{1111} = \tilde{\mathbb{K}}_{2222} = \mu_* - p, \quad (60)$$

so that  $\tilde{\mathbb{K}}_{ijkl}$  also possesses the major symmetry. Only the forms (58) and (60) of the constitutive equation (22) will be needed in the following.

The formulation in the present paper is restricted to the elliptic range (E), where the characteristic equation associated to the equilibrium equations governing perturbations superimposed upon a given homogeneous strain does not admit real solutions. In particular, the elliptic range corresponds to negative or complex coefficients  $\gamma_1$  and  $\gamma_2$ , functions of the material properties and state of pre-stress as follows:

$$\left. \begin{array}{l} \gamma_1 \\ \gamma_2 \end{array} \right\} = \frac{1 - 2\frac{\mu_*}{\mu} \pm \sqrt{\Delta}}{1 + k}, \quad (61)$$

where

$$\Delta = k^2 - 4\frac{\mu_*}{\mu} + 4\left(\frac{\mu_*}{\mu}\right)^2. \quad (62)$$

Employing Eq. (25)<sub>1</sub>, parameter  $k$  appearing in Eqs. (61) and (62) can be written as

$$k = \frac{\sigma}{2\mu} = \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 + \lambda_2^2}, \quad (63)$$

and it represents a normalized measure of the pre-stress or—more precisely—a measure of the current state of stretch.

The elliptic range may be further sub-divided into elliptic-imaginary (EI) and elliptic-complex (EC) regimes. These are defined as follows:

- $\Delta > 0$ , so that  $\gamma_1$  and  $\gamma_2$  are both negative in (EI),
- $\Delta < 0$ , so that  $\gamma_1$  and  $\gamma_2$  are a conjugate pair in (EC).

It may be important to remark that shear bands, corresponding to the appearance of discontinuous strain rates, are formally excluded within the elliptic range, i.e. in the context analyzed here. However, as shown

by Bigoni and Capuani [1] and Radi et al. [32], formation of zones of concentrated strains in response to a perturbation becomes possible when the boundary of the elliptic regime is approached. This will be also demonstrated with numerical examples referring to van Hove conditions in Part II.

The boundary of the elliptic range is characterized by the following conditions:

- elliptic imaginary/parabolic (EI/P) boundary is attained when  $k = 1$ , corresponding to  $\gamma_1 = 0$ ;
- elliptic complex/hyperbolic (EC/H) boundary is attained when  $\Delta = 0$ , corresponding to  $\gamma_1 = \gamma_2$ .

### 3. The fundamental solution

With reference to the constitutive equation (56), the Green’s function set  $\{v_i^g; \pi^g\}$  for an infinite and uniformly pre-strained medium can be written in the form [1]

$$v_i^g = \frac{1}{2\pi^2\mu(1+k)} \left\{ \frac{\pi\delta_{ig} \log r}{[(2-i)\gamma_2 + 1 - i]\sqrt{-\gamma_1} + [(2-i)\gamma_1 + 1 - i]\sqrt{-\gamma_2}} - \int_0^{\frac{\pi}{2}} [K_i^g(\alpha + \theta) + (3 - 2i)(3 - 2g)K_i^g(\alpha - \theta)] \log(\cos \alpha) d\alpha \right\}, \tag{64}$$

$$\pi^g = -\frac{1}{2\pi r} \left\{ \cos \left[ \theta - (g - 1)\frac{\pi}{2} \right] + \frac{1}{\pi(1+k)} \int_0^\pi \frac{\tilde{K}_g(\alpha + \theta)}{\cos \alpha} d\alpha \right\},$$

where  $r$  and  $\theta$  are the polar coordinates singling out the generic point with respect to the position  $\mathbf{y}$  of the concentrated force and

$$K_i^g(\alpha) = \frac{\sin \left[ \alpha + (i - 1)\frac{\pi}{2} \right] \sin \left[ \alpha + (g - 1)\frac{\pi}{2} \right]}{A(\alpha)}, \tag{65}$$

$$A(\alpha) = \sin^4 \alpha (\cot^2 \alpha - \gamma_1)(\cot^2 \alpha - \gamma_2) > 0,$$

$$\tilde{K}_g(\alpha) = K_g^g(\alpha) \cos \left[ \alpha + (g - 1)\frac{\pi}{2} \right] \Gamma(\alpha), \tag{66}$$

$$\Gamma(\alpha) = 2 \left( \frac{\mu_*}{\mu} - 1 \right) (2 \cos^2 \alpha - 1) - k.$$

Note that in expression (64)<sub>2</sub> of  $\pi^g$  we have introduced the usual symbol to denote the Cauchy principal value integral.

The gradient of the Green’s velocity may be obtained directly from (64) and can be written as

$$v_{i,g}^j = \frac{1}{2\pi^2\mu(1+k)r} \left\{ \frac{(3 - 2i)\pi \cos \left[ \theta + (1 - g)\frac{\pi}{2} \right]}{[(2-i)\gamma_2 + 1 - i]\sqrt{-\gamma_1} + [(2-i)\gamma_1 + 1 - i]\sqrt{-\gamma_2}} + \sin \left[ \theta + (1 - g)\frac{\pi}{2} \right] \int_0^{\frac{\pi}{2}} \Sigma \left( \alpha + \theta + (i - 1)\frac{\pi}{2}, \alpha + \theta \right) \log(\cos \alpha) d\alpha - \sin \left[ \theta + (1 - g)\frac{\pi}{2} \right] \int_0^{\frac{\pi}{2}} \Sigma \left( \alpha - \theta - (i - 1)\frac{\pi}{2}, \alpha + \theta \right) \log(\cos \alpha) d\alpha \right\}, \tag{67}$$

where index  $i$  is not summed and

$$\Sigma(\alpha, \beta) = \frac{\sin \alpha [2 \cos \alpha A(\beta) - \sin \alpha A'(\beta)]}{A^2(\beta)}, \quad A'(\beta) = \frac{\partial A(\beta)}{\partial \beta}. \tag{68}$$

The four components of the velocity gradient not included in Eq. (67) can be obtained using incompressibility and symmetry of Green’s tensor:

$$v_{2,2}^1 = v_{1,2}^2 = -v_{1,1}^1, \quad v_{2,1}^1 = v_{1,1}^2 = -v_{2,2}^2. \tag{69}$$

According to constitutive equations (56)–(58), from Green’s velocity gradient and pressure rate  $\dot{\pi}^g$  we obtain the in-plane hydrostatic stress rate

$$\dot{p}^g = \dot{\pi}^g - \frac{\sigma}{2} v_{1,1}^g, \tag{70}$$

and the associated incremental nominal stress

$$\begin{aligned} \dot{t}_{11}^g &= (2\mu_* - p)v_{1,1}^g + \dot{\pi}^g, & \dot{t}_{12}^g &= (\mu - p)v_{1,2}^g + (\mu + \mu k)v_{2,1}^g, \\ \dot{t}_{21}^g &= (\mu - p)v_{2,1}^g + (\mu - \mu k)v_{1,2}^g, & \dot{t}_{22}^g &= -(2\mu_* - p)v_{1,1}^g + \dot{\pi}^g. \end{aligned} \tag{71}$$

It is important to note that the Green’s velocity  $v_i^g$  can be given the form

$$v_i^g(r, \theta) = \zeta_1^{ig} \log r + \zeta_2^{ig}(\theta), \tag{72}$$

where the dependence on the current state is condensed in the coefficients  $\zeta_1^{ig}$  and  $\zeta_2^{ig}$ . The former coefficient satisfies  $\zeta_1^{ig} = 0$  if  $i \neq g$ , while the latter contains the directional dependence on  $\theta$ . It follows from representation (72) that the gradient of the Green’s velocity, the in-plane hydrostatic stress rate and thus the incremental stress rate can be expressed as explicit functions of  $1/r$ .

#### 4. Boundary element formulation

We restrict the presentation to mixed boundary value problems in which velocities and incremental nominal tractions  $\dot{t}$  are prescribed functions defined on separate portions  $\partial B_v$  and  $\partial B_t$  of the boundary  $\partial B = \partial B_v \cup \partial B_t$

$$v_i = \bar{v}_i, \text{ on } \partial B_v, \quad \dot{t}_{ij}n_i = \bar{t}_j \text{ on } \partial B_t, \tag{73}$$

of a solid  $B$ , currently in a state of homogeneous, finite deformation. In this context, two integral representations exist relating the velocity and the pressure rate in interior points of the body to the boundary values of nominal traction rates  $\dot{t}_j = \dot{t}_{ij}n_i$  and velocities [1]. These are

$$v_g(\mathbf{y}) = \int_{\partial B} [\dot{t}_j(\mathbf{x})v_j^g(\mathbf{x}, \mathbf{y}) - \dot{t}_j^g(\mathbf{x}, \mathbf{y})v_j(\mathbf{x})] dl_x, \tag{74}$$

and

$$\begin{aligned} \dot{p}(\mathbf{y}) &= - \int_{\partial B} \dot{t}_g(\mathbf{x})\dot{p}^g(\mathbf{x}, \mathbf{y}) dl_x + \int_{\partial B} n_i(\mathbf{x})v_j(\mathbf{x}) \llbracket \kappa_{ijk} \dot{p}_{,k}^g(\mathbf{x}, \mathbf{y}) \rrbracket dl_x \\ &\quad - \left( 4\mu\mu_* - 4\mu_*^2 + \mu\sigma - 2\mu_*\sigma - \frac{\sigma^2}{2} \right) \int_{\partial B} n_i(\mathbf{x})v_i(\mathbf{x})v_{1,11}^1(\mathbf{x}, \mathbf{y}) dl_x \\ &\quad - \sigma \left( \mu + \frac{\sigma}{2} \right) \int_{\partial B} n_i(\mathbf{x})v_i(\mathbf{x})v_{2,11}^2(\mathbf{x}, \mathbf{y}) dl_x. \end{aligned} \tag{75}$$

If the point  $\mathbf{y}$  is on the boundary, Eq. (74) becomes [1]

$$C_j^g v_j(\mathbf{y}) = \int_{\partial B} \dot{t}_j(\mathbf{x})v_j^g(\mathbf{x}, \mathbf{y}) dl_x - \int_{\partial B} \dot{t}_j^g(\mathbf{x}, \mathbf{y})v_j(\mathbf{x}) dl_x, \tag{76}$$

where

$$C_i^g = \lim_{\varepsilon \rightarrow 0} \int_{\partial C_\varepsilon} \dot{\tau}_i^g(\mathbf{x}, \mathbf{y}) dL_x \tag{77}$$

is the so-called **C**-matrix [3], depending on the material parameters, the state of pre-stress and the geometry of the boundary (in the case of a smooth boundary,  $C_i^g = (1/2)\delta_{gi}$ ). Note that symbol  $\partial C_\varepsilon$  introduced in (77) denotes the intersection between a circle of radius  $\varepsilon$  centred at  $\mathbf{y}$  and the domain  $B$ .

### 5. Boundary discretization

The boundary equation (76) is the starting point to derive the *collocation boundary element method*. To this purpose, the boundary  $\partial B$  is divided into  $m$  elements  $\Gamma^e$  ( $e = 1, \dots, m$ ), with subsets  $m_v$  and  $m_\tau$  belonging respectively to  $\partial B_v$  and  $\partial B_\tau$  (clearly  $m = m_v + m_\tau$ ).

For simplicity the same discretization is assumed for velocity and traction rates at the boundary. In particular, inside each boundary element  $\Gamma^e$  we use

$$\begin{aligned} v_i(\mathbf{x}) &= \varphi_\alpha(\mathbf{x}) \bar{v}_{ix}, \\ \dot{\tau}_i(\mathbf{x}) &= \varphi_\alpha(\mathbf{x}) \dot{\bar{\tau}}_{ix}, \quad \alpha = 0, \dots, \Theta, \end{aligned} \tag{78}$$

where  $\bar{v}_{ix}$ ,  $\dot{\bar{\tau}}_{ix}$  are the nodal values of velocities and nominal traction rates, respectively, and  $\varphi_\alpha$  are the relevant shape functions, selected as polynomials of degree  $\Theta$ .

The discretized form of Eq. (76), *collocating* the point  $\mathbf{y}$  at  $\mathbf{y}^{(\bar{e}, \bar{\alpha})}$ , corresponding to the node  $\bar{\alpha}$  of the element  $\bar{e}$ , is

$$C_i^g \bar{v}_{i\bar{\alpha}} + \sum_{e=1}^m \bar{v}_{ix}^e \int_{\Gamma^e} \varphi_\alpha(\mathbf{x}) \dot{\bar{\tau}}_i^g(\mathbf{x}, \mathbf{y}^{(\bar{e}, \bar{\alpha})}) dL_x = \sum_{e=1}^m \dot{\bar{\tau}}_{ix}^e \int_{\Gamma^e} \varphi_\alpha(\mathbf{x}) v_i^g(\mathbf{x}, \mathbf{y}^{(\bar{e}, \bar{\alpha})}) dL_x, \tag{79}$$

where indices  $\alpha$  and  $i$  are summed and range between 0 and  $\Theta$  and 1 and 2, respectively.

An analysis of Eq. (79) reveals that the number of unknowns is  $n_{\text{unk}} = 2(m_u + m_\tau)\Theta = 2m\Theta$ , being  $\bar{v}_{ix}^e$  prescribed on  $\partial B_v$  and  $\dot{\bar{\tau}}_{ix}^e$  on  $\partial B_\tau$ .

Collocating now Eq. (79) at  $m\Theta$  nodes along the two directions  $x_1$  and  $x_2$ , yields the following algebraic system:

$$\mathbf{H}\hat{\mathbf{v}} = \mathbf{G}\hat{\mathbf{t}}, \tag{80}$$

where  $\hat{\mathbf{v}}$  and  $\hat{\mathbf{t}}$  are the vectors collecting  $\bar{v}_{ix}^e$  and  $\dot{\bar{\tau}}_{ix}^e$

$$\hat{v}_{2\Theta(e-1)+2\alpha+i} = \bar{v}_{ix}^e, \quad \hat{t}_{2\Theta(e-1)+2\alpha+i} = \dot{\bar{\tau}}_{ix}^e. \tag{81}$$

Solution of system (80) gives the nodal velocities  $\bar{v}_{ix}^e$  on  $\partial B_\tau$  and the nominal traction rates  $\dot{\bar{\tau}}_{ix}^e$  on  $\partial B_v$ .

Once system (80) has been solved, fields  $\mathbf{v}(\mathbf{y})$  and  $\dot{\mathbf{p}}(\mathbf{y})$  can be evaluated at internal points  $\mathbf{y}$  by applying the discretized forms of Eqs. (74) and (75). In particular, Eq. (74) reduces to Eq. (79) with  $C_i^g = \delta_{gi}$  and the integration is straightforward, as  $r$  is always different from zero.

We limit the presentation to discretization of the boundary into rectilinear elements and linear shape functions, so that  $\Theta = 1$  and  $n_{\text{unk}} = 2m$ .

The singular integrals in (79), computed over the elements adjacent to the node  $e$  in which the equation is collocated, are evaluated analytically. In particular, the strongly singular integral  $I_{\text{strong}}^{(ig,e)}$  in the left hand side of Eq. (79) is equal to

$$I_{\text{strong}}^{(ig,e)} = \int_{\Gamma_{e-1}} \varphi_1(\mathbf{x}) \dot{\bar{\tau}}_i^g(\mathbf{x}, \mathbf{y}^{(e-1,1)}) dL_x + \int_{\Gamma_e} \varphi_0(\mathbf{x}) \dot{\bar{\tau}}_i^g(\mathbf{x}, \mathbf{y}^{(e,0)}) dL_x, \tag{82}$$

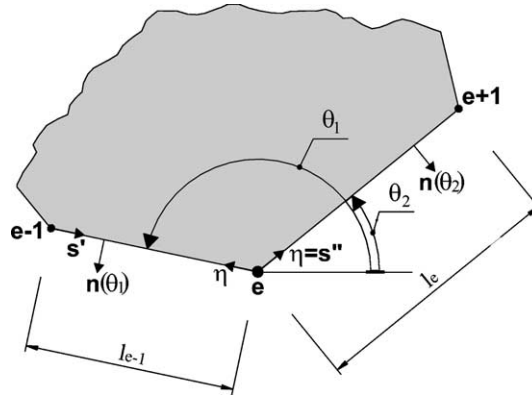


Fig. 1. Geometry at node e.

which, with reference to the geometry sketched in Fig. 1, can be transformed into

$$I_{\text{strong}}^{(ig,e)} = l_{e-1} \int_0^1 s' \dot{\tau}_i^g(r, \theta_1) ds' + l_e \int_0^1 (1 - s'') \dot{\tau}_i^g(r, \theta_2) ds'', \tag{83}$$

so that the change of variable

$$\eta = l_{e-1}(1 - s') = l_e s'' \tag{84}$$

yields

$$I_{\text{strong}}^{(ig,e)} = \int_0^{l_{e-1}} \left(1 - \frac{\eta}{l_{e-1}}\right) \dot{\tau}_i^g(\eta, \theta_1) d\eta + \int_0^{l_e} \left(1 - \frac{\eta}{l_e}\right) \dot{\tau}_i^g(\eta, \theta_2) d\eta. \tag{85}$$

As far as the elements  $e - 1$  and  $e$  are concerned, the incremental Green's tractions can be computed as

$$\dot{\tau}_i^g(r) = (-1)^e \frac{\chi_{ig}(k, \frac{\mu_*}{\mu})}{r}, \tag{86}$$

which are independent of  $\theta$ , so that  $\chi_{ig}$  can be calculated as  $\tau_i^g(1)$ . Introducing Eq. (86) into Eq. (85), we obtain an explicit formula for the singular integrals (82) in the form

$$I_{\text{strong}}^{(ig,e)} = (-1)^e \chi_{ig} \log \left( \frac{l_e}{l_{e-1}} \right). \tag{87}$$

In all our examples we have always found numerically that  $\chi_{11} = \chi_{22} = 0$ , a result that can be analytically checked in the special case of Newtonian fluid, i.e. when  $k = 0$  and  $\mu_* = \mu$ , but still requires a proof in the general case.

The weakly singular integrals  $I_{\text{weak}}^{(ig,e)}$  in the right hand side of Eq. (79) is equal to

$$I_{\text{weak}}^{(ig,e)} = \int_{\Gamma_{e-1}} \varphi_1(\mathbf{x}) v_i^g(\mathbf{x}, \mathbf{y}^{(e-1,1)}) dI_x + \int_{\Gamma_e} \varphi_0(\mathbf{x}) v_i^g(\mathbf{x}, \mathbf{y}^{(e,0)}) dI_x, \tag{88}$$

which becomes (Fig. 1)

$$I_{\text{weak}}^{(ig,e)} = \int_0^{l_{e-1}} \left(1 - \frac{\eta}{l_{e-1}}\right) v_i^g(\eta, \theta_1) d\eta + \int_0^{l_e} \left(1 - \frac{\eta}{l_e}\right) v_i^g(\eta, \theta_2) d\eta. \tag{89}$$

Table 1  
Analytic expression for the weakly singular integrals (88) at the right hand side of Eq. (79)

$i$	$g$	Integral $I_{\text{weak}}^{(ig,e)}$
1	1	$\frac{l_{e-1}}{2} \left[ \frac{2 \log(l_{e-1}) - 3}{A_{11}} + v_1^1(1, \theta_1) \right] + \frac{l_e}{2} \left[ \frac{2 \log(l_e) - 3}{A_{11}} + v_1^1(1, \theta_2) \right]$
1	2	$\frac{l_{e-1}}{2} v_1^2(\theta_1) + \frac{l_e}{2} v_1^2(\theta_2)$
2	1	$\frac{l_{e-1}}{2} v_2^1(\theta_1) + \frac{l_e}{2} v_2^1(\theta_2)$
2	2	$\frac{l_{e-1}}{2} \left[ -\frac{2 \log(l_{e-1}) - 3}{A_{22}} + v_2^2(1, \theta_1) \right] + \frac{l_e}{2} \left[ -\frac{2 \log(l_e) - 3}{A_{22}} + v_2^2(1, \theta_2) \right]$

$$A_{11} = 4\mu\pi(1+k)(\gamma_2\sqrt{-\gamma_1} + \gamma_1\sqrt{-\gamma_2}), A_{22} = 4\mu\pi(1+k)(\sqrt{-\gamma_1} + \sqrt{-\gamma_2}).$$

Taking Eq. (72) into account, the integral (89) can be analytically evaluated and the results are listed in Table 1.

We note, in passing, that the use of higher-order polynomials as shape functions is straightforward, since the singularities in the above integrals arise from the constant part of the interpolating functions.

### 6. A numerical example: non-linear elasticity without domain integrals

An elastic block subject to homogeneous, increasing deformation is considered, with the purpose of illustrating with a simple example that our formulation does not involve domain integrals.

In particular, the elastic block is constrained to plane deformations starting from an unstressed square configuration and is subjected to tension or compression in one direction. We refer to Mooney–Rivlin (26) and Ogden (30) and (31) non-linear elastic laws. The trivial, analytical solution to this problem is reported by Ogden [26]

$$\sigma_1 - \sigma_2 = \lambda_1 \frac{\partial \widehat{W}}{\partial \lambda_1} - \lambda_2 \frac{\partial \widehat{W}}{\partial \lambda_2}, \tag{90}$$

so that for uniaxial stress ( $\sigma_2 = 0$ ) and plane strain deformations ( $\lambda_1 = 1/\lambda_2 = \lambda, \lambda_3 = 1$ ), from (27) and (30) we get

$$\sigma_1 = (\mu_1 - \mu_2)(\lambda^2 - \lambda^{-2}) \text{ and } \sigma_1 = \sum_{i=1}^N \mu_i (\lambda^{2\alpha_i} - \lambda^{-2\alpha_i}), \tag{91}$$

for Mooney–Rivlin and Ogden materials, respectively. Within our framework, the solution may be obtained by integrating the incremental equations, using a forward Euler scheme. Since the basic step of the method is a linear increment superimposed upon a homogeneous configuration *domain integrals or volume discretizations are completely avoided*. It is worth mentioning that in the special case of homogeneous deformations, the domain integrals can be brought to the boundary even within the usual framework, see for instance [21], with reference to elastoplasticity. In that case, however, specific manipulations of the domain integrals are requested to transform these to boundary integrals, while in our method domain integrals are simply absent.

Results are presented in Fig. 2, where for the Mooney–Rivlin material we have taken  $\mu_0 = 0.35412$  MPa, whereas for the Ogden material we have referred to the values listed in (32). The analytical solution is compared to the results given by the numerical procedure with a uniform mesh of 16 boundary elements.

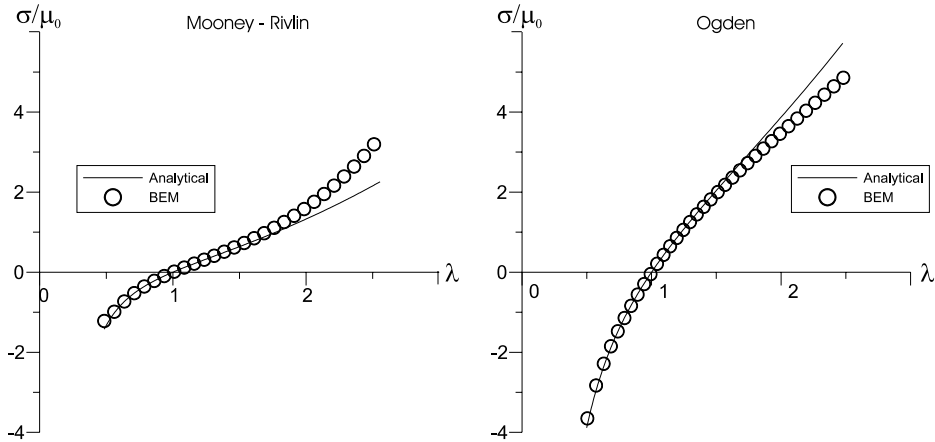


Fig. 2. Uniaxial deformation of Mooney–Rivlin and Ogden elastic blocks. Cauchy stress  $\sigma$  versus stretch  $\lambda$ .

Gaussian quadrature formulae have been employed with 12 integration points for Green's functions (64)–(71) and 18 points for integrals in the discretized equation (79).

We are in a position now to speculate on a more general procedure allowing for increments superimposed upon a generic, non-homogeneous deformation. This would necessarily lead to some volume discretization to describe the material inhomogeneity, but this should be regarded as different—in essence—from the usual domain discretization (see for instance [31]). The demonstration is provided by the above example, showing that our method does not need any domain discretization when the material properties are homogeneous.

## 7. Conclusions

A boundary element technique has been proposed for plane strain and incompressible, incremental deformations of a broad class of materials (Cauchy-elastic, hyper- and hypoelastic, and Newtonian fluids). Difficulties inherent to the treatment of the incompressibility constraint are simply avoided by the presented method. When used as the basis to analyze non-linear deformations, our approach yields a formulation avoiding domain integrals. It also allows for the analysis of different bifurcation situations, that will be investigated in Part II of this study.

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## Appendix A. Biot's expression of incremental moduli (25)

To obtain the expression (25)<sub>1</sub> of the incremental modulus  $\mu$ , let us begin by considering the spectral representations of the left Cauchy–Green strain tensor  $\mathbf{B}$ , and of the Cauchy stress  $\boldsymbol{\sigma}$



$$\begin{aligned} \mathbf{B} &= \lambda_1^2 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2^2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3^2 \mathbf{e}_3 \otimes \mathbf{e}_3, \\ \boldsymbol{\sigma} &= \sigma_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \sigma_3 \mathbf{e}_3 \otimes \mathbf{e}_3, \end{aligned} \tag{A.1}$$

where  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$  denote the Eulerian principal axes. The time derivative of (A.1) is

$$\begin{aligned} \dot{\mathbf{B}} &= 2\lambda_1 \dot{\lambda}_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + 2\lambda_2 \dot{\lambda}_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + 2\lambda_3 \dot{\lambda}_3 \mathbf{e}_3 \otimes \mathbf{e}_3 + \lambda_1^2 (\dot{\mathbf{e}}_1 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \dot{\mathbf{e}}_1) \\ &\quad + \lambda_2^2 (\dot{\mathbf{e}}_2 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \dot{\mathbf{e}}_2) + \lambda_3^2 (\dot{\mathbf{e}}_3 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \dot{\mathbf{e}}_3), \\ \dot{\boldsymbol{\sigma}} &= \dot{\sigma}_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \dot{\sigma}_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \dot{\sigma}_3 \mathbf{e}_3 \otimes \mathbf{e}_3 + \sigma_1 (\dot{\mathbf{e}}_1 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \dot{\mathbf{e}}_1) \\ &\quad + \sigma_2 (\dot{\mathbf{e}}_2 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \dot{\mathbf{e}}_2) + \sigma_3 (\dot{\mathbf{e}}_3 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \dot{\mathbf{e}}_3). \end{aligned} \tag{A.2}$$

Taking into account that for incremental plane strain deformations is  $\dot{\mathbf{e}}_3 = \mathbf{0}$  and that  $\mathbf{e}_1 \cdot \dot{\mathbf{e}}_2 = -\dot{\mathbf{e}}_1 \cdot \mathbf{e}_2$ , the in-plane out-of-diagonal components result to be

$$\begin{aligned} \dot{B}_{12} &= (\lambda_1^2 - \lambda_2^2) \dot{\mathbf{e}}_1 \cdot \mathbf{e}_2, \\ \dot{\sigma}_{12} &= (\sigma_1 - \sigma_2) \dot{\mathbf{e}}_1 \cdot \mathbf{e}_2. \end{aligned} \tag{A.3}$$

On the other hand, Eqs. (9) and (11) give

$$\begin{aligned} \dot{B}_{12} &= (\lambda_1^2 + \lambda_2^2) D_{12} - (\lambda_1^2 - \lambda_2^2) W_{12}, \\ \dot{\sigma}_{12} &= \overset{\nabla}{\sigma}_{12} - (\sigma_1 - \sigma_2) W_{12}. \end{aligned} \tag{A.4}$$

Considering Eqs. (A.4) and (A.3) yields

$$\begin{aligned} D_{12} &= \frac{(\dot{\mathbf{e}}_1 \cdot \mathbf{e}_2 + W_{12})(\lambda_1^2 - \lambda_2^2)}{\lambda_1^2 + \lambda_2^2}, \\ \overset{\nabla}{\sigma}_{12} &= (\dot{\mathbf{e}}_1 \cdot \mathbf{e}_2 + W_{12})(\sigma_1 - \sigma_2), \end{aligned} \tag{A.5}$$

from which, eliminating the term  $\dot{\mathbf{e}}_1 \cdot \mathbf{e}_2 + W_{12}$ , we get

$$\overset{\nabla}{\sigma}_{12} = \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 - \lambda_2^2} (\sigma_1 - \sigma_2) D_{12}. \tag{A.6}$$

A comparison between Eq. (22)<sub>1</sub> and Eq. (A.6) gives the Biot's expression of  $\mu$ , Eq. (25)<sub>1</sub>.

The incremental moduli  $\mu_*$  can be obtained by taking the time derivative of (7) written for the in-plane components and taking into account that  $\dot{\lambda}_3 = 0$  for plane strain incremental deformations

$$\dot{\sigma}_1 - \dot{\sigma}_2 = \dot{\lambda}_1 \frac{\partial W}{\partial \lambda_1} + \lambda_1 \dot{\lambda}_1 \frac{\partial^2 W}{\partial \lambda_1^2} + \lambda_1 \dot{\lambda}_2 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} - \dot{\lambda}_2 \frac{\partial W}{\partial \lambda_2} - \lambda_2 \dot{\lambda}_1 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} - \lambda_2 \dot{\lambda}_2 \frac{\partial^2 W}{\partial \lambda_2^2}. \tag{A.7}$$

If instead of the potential  $W(\lambda_1, \lambda_2, \lambda_3)$  the potential  $\widehat{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, 1/\lambda_1 \lambda_2)$  is used, the same relation Eq. (A.7) is obtained, except that  $\widehat{W}$  replaces  $W$ .

Considering Eqs. (9), (A.2), and (11) for the diagonal components, we get

$$\begin{aligned} \dot{\lambda}_i &= D_{ii} \lambda_i, \quad i = 1, 2, \\ \overset{\nabla}{\sigma}_{22} - \overset{\nabla}{\sigma}_{11} &= \dot{\sigma}_{11} - \dot{\sigma}_{22}, \end{aligned} \tag{A.8}$$

which, used in Eq. (A.7), yield

$$\overset{\nabla}{\sigma}_{22} - \overset{\nabla}{\sigma}_{11} = \left( \lambda_1 \frac{\partial W}{\partial \lambda_1} + \lambda_2 \frac{\partial W}{\partial \lambda_2} + \lambda_1^2 \frac{\partial^2 W}{\partial \lambda_1^2} + \lambda_2^2 \frac{\partial^2 W}{\partial \lambda_2^2} - 2\lambda_1 \lambda_2 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} \right) \frac{D_{11} - D_{22}}{2}. \tag{A.9}$$

A comparison between Eq. (22)<sub>2</sub> and Eq. (A.9) gives the Biot's expression of  $\mu_*$ , Eq. (25)<sub>2</sub>.

## Appendix B. The out-of-plane stress increment for the Ogden hyperelastic material

The only non-zero component of the Jaumann derivative of the Cauchy stress in the out-of-plane direction is

$$\overset{\nabla}{\sigma}_{33} = \sum_{i=1}^N \left[ \mu_i \frac{g_1^i(\lambda_1, \lambda_2) + g_2^i(\lambda_1, \lambda_2) + g_3^i(\lambda_1, \lambda_2)}{(\lambda_1 - \lambda_2)^2 (\lambda_1 + \lambda_2) (\lambda_1^2 - \lambda_3^2) (\lambda_2^2 - \lambda_3^2)^2} \lambda_3^{2\alpha_i+1} \right], \quad (\text{B.1})$$

where

$$\begin{aligned} g_1^i(\lambda_1, \lambda_2) &= \lambda_1^{\alpha_i} \lambda_2^{2(\alpha_i+1)} (\lambda_1^2 - \lambda_3^2)^2 [\lambda_1 (\lambda_1 - \lambda_2) (\lambda_1^2 + \lambda_3^2) (\alpha_i (\lambda_2^2 - \lambda_3^2) - 2(\lambda_2^2 + \lambda_3^2)) \\ &\quad - 2\lambda_3^4 ((\lambda_1^2 + \lambda_2^2) - \lambda_3 (1 + \lambda_3^{-6}))], \\ g_2^i(\lambda_1, \lambda_2) &= (\lambda_1 + \lambda_2) (\lambda_1 - \lambda_2)^3 \lambda_3^6 [\alpha_i (\lambda_1^2 - \lambda_3^2) (\lambda_2^2 - \lambda_3^2) (1 + \lambda_3^6) \\ &\quad + 2\lambda_3^{-1} ((\lambda_3^3 - 1) + \lambda_3^{-9} (\lambda_3^3 + 1) + \lambda_3^{-5} (\lambda_1^2 + \lambda_2^2))], \\ g_3^i(\lambda_1, \lambda_2) &= \lambda_1^{(1+2\alpha_i)} \lambda_2^{\alpha_i} (\lambda_2^2 - \lambda_3^2) [\alpha_i \lambda_2 (\lambda_2 - \lambda_1) ((\lambda_3^2 - \lambda_2^2) + (\lambda_1^{-2} - \lambda_3^{-2}) \lambda_3^{-2}) \\ &\quad + 2\lambda_1^{-4} \lambda_2^{-4} (\lambda_1 - \lambda_2)^2 + 2\lambda_2 (\lambda_2 - \lambda_1) ((\lambda_1^2 + \lambda_3^2) + (\lambda_2^2 + \lambda_3^2)) \\ &\quad + 2\lambda_2^4 (1 - \lambda_1^5 \lambda_2) + 2\lambda_2^{-2} (1 - \lambda_1^5 \lambda_2^7)], \end{aligned} \quad (\text{B.2})$$

in which, due to incompressibility

$$\lambda_3 = \frac{1}{\lambda_1 \lambda_2}. \quad (\text{B.3})$$

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