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ON STRAIN LOCALIZATION ANALYSIS OF ELASTOPLASTIC MATERIALS AT FINITE STRAINS

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Abstract — The problem of strain localization into planar bands of rate-independent elastoplastic solids with smooth yield surface and plastic potential is analyzed reconsidering the work of Rice and Rudnicki in 1980. It is shown that strain localization with elastic unloading on one side of the band first becomes possible either at localization in the comparison solid corresponding to the loading branch of the constitutive equation or at the snap-back threshold. The elastic unloading is shown to start from the condition of neutral loading, occurring in fact at the onset of localization. The case of localization with elastic unloading into the band and plastic loading outside that was not considered by Rice and Rudnicki is taken into account.

I. INTRODUCTION

The condition of strain localization, which corresponds to a loss of ellipticity of the constitutive equations (RICE [1976]), represents an extreme type of loss of local stability (BIGONI & HUECKEL [1991], BIGONI & ZACCARIA [1992]). Considering a homogeneously deformed space, strain localization is attained when a particular form of bifurcation solution becomes admissible that consists in a discontinuous velocity of deformation inside a planar band (RICE [1976]). The theory of localization of deformation is considered in this paper for elastoplastic solids with smooth yield function and plastic potential.

The nonlinearity (piecewise-linearity) of the tangent constitutive elastoplastic operator makes the direct bifurcation analysis of an elastoplastic solid difficult. The difficulties are reduced by performing the bifurcation analysis on a linear comparison solid, defined, for the associative flow rule, by the loading branch of the constitutive operator ("Hill comparison solid," HILL [1958,1959,1961]). Bifurcation in the elastoplastic solid cannot precede the first bifurcation point detected in the comparison solid (HILL [1958]). In the case of nonassociative flow rules, the comparison theorem of Hill is no longer valid, and, in general, the entire framework of the bifurcation analysis is not well established. RANIECKI [1979] and RANIECKI and BRUHNS [1981] introduced a family of comparison solids to set lower bounds to possible bifurcation points of the elastoplastic solid. Moreover, RANIECKI and BRUHNS [1981] proposed to use, even for the nonassociative flow rule, the Hill comparison solid defined, in the above-mentioned manner, by the loading branch of the constitutive elastoplastic operator. The use of the latter comparison solid in a bifurcation analysis allows one to determine an upper bound to possible bifurcation points of the elastoplastic solid. Strictly speaking, this interpretation is valid if the hypothesis of HUTCHINSON [1973] is accepted that a neutral loading surface is not present in the body at bifurcation (as in the case, for instance, studied by NEEDLEMAN [1979]). In general, however, a bifurcation analysis carried out on the Hill comparison solid for nonassociative plasticity can only set lower bounds to a possible bifurcation pattern for which elastic unloading is excluded in the plastic zone. In other words, if a bifurcation point does exist between the two thresholds corresponding to the bifurcations in the comparison solids of Raniecki and Bruhns and Hill, the bifurcation mode must necessarily involve elastic unloading in some portions of the plastic zone.

In the case of the strain localization analysis, the mode of bifurcation is a priori prescribed. When normality holds, the Hill comparison theorem implies that the localization in the elastoplastic solid cannot occur before the localization in the comparison solid. Analogously, when normality fails to hold, the localization in the elastoplastic solid is excluded before localization in all the comparison solids of Raniecki and Bruhns. From RANIECKI and BRUHNS [1981], it is concluded that if localization in the elastoplastic solid occurs between the localization thresholds set by the two comparison solids, then the localization pattern must involve elastic unloading. Finally, localization corresponding to elastic unloading outside the band and plastic loading inside cannot precede localization in the Hill comparison solid (RICE & RUDNICKI [1980] and BORRÈ & MAIER [1989]).

In this paper, the problem is reconsidered, including the further case of elastic unloading inside the band and plastic loading outside. It is shown that the above-mentioned result of RICE and RUDNICKI [1980] holds under the assumption that localization corresponding to unloading on both sides of the band does not precede localization in the Hill comparison solid. This condition is obviously always verified in the infinitesimal theory and in the finite incremental theory, assuming the normality rule. Excluding the localization with elastic unloading on both sides of the band, it is shown that localization with elastic unloading inside the band and plastic loading outside becomes possible starting from the localization in the Hill comparison solid. In conclusion, at the localization in the Hill comparison solid, two types of localization, corresponding to neutral loading on one side of the band and plastic loading on the other side, are possible. On the other hand, without any assumption on the direction of the plastic flow, it is shown that the localization corresponding to elastic unloading on both sides of the band cannot be a priori excluded to occur between the localization in the Raniecki and Hill comparison solids.

In the proposed analysis, to retain complete generality, the choice of the incremental constitutive equation, involving an objective rate of a symmetric stress measure is left free.

II. NOTATION

Throughout the paper the standard notation of the linear algebra is used (BOWEN & WANG [1976], GURTIN [1981]).

A tensor $\mathbf{A}$ is a linear transformation over a (three-dimensional) inner product space $\mathcal{V}$. Lin denotes the set of all tensors and Sym the set of all symmetric tensors.

An operator ("fourth order tensor") $\mathbb{A}$ is a (linear) transformation over Lin or Sym.

The symbol $\otimes$ denotes the tensor product over an inner product space (including Lin).

The symbol $\boxtimes$ denotes the tensor product of transformations defined, for the specific case of $\mathbf{A}, \mathbf{B} \in$ Lin, by (HALMOS [1958]):

$$\mathbf{A} \boxtimes \mathbf{B}[\mathbf{a} \otimes \mathbf{b}] = \mathbf{A}\mathbf{a} \otimes \mathbf{B}\mathbf{b}, \quad \forall \mathbf{a}, \mathbf{b} \in \mathcal{V}. \tag{1}$$

The fourth-order tensor $\mathbb{S}$ denotes the symmetrizing operator over Lin, and the second order tensor $\mathbf{S}$ indicates the first Piola–Kirchhoff stress tensor (transpose of the nominal stress tensor, cf. OGDEN [1984], §3.4.1) defined by:

$$\mathbf{t} = \mathbf{S}\mathbf{n}, \tag{2}$$

where $\mathbf{t}$ is the nominal traction and $\mathbf{n}$ the normal to the material area element. Finally, the symbol $\overset{\triangle}{\boldsymbol{\Sigma}}$ denotes an objective rate of a generic symmetric stress measure.

III. CONSTITUTIVE RELATIONS

Elastoplastic behavior is assumed to be characterized by a piecewise-linear constitutive operator $\mathbb{C}$ relating an objective rate of a symmetric stress measure $\overset{\triangle}{\boldsymbol{\Sigma}}$ to the velocity of deformation $\mathbf{D}$:

$$\overset{\triangle}{\boldsymbol{\Sigma}} = \mathbb{C}[\mathbf{D}]. \tag{3}$$

The operator $\mathbb{C}$ is equal to the fourth-order tensor $\mathbb{E}$ of instantaneous elastic moduli, in the case of *elastic unloading*, i.e. when the velocity of deformation $\mathbf{D}$ satisfies:

$$\mathbf{D}\cdot\mathbb{E}[\mathbf{Q}] \le 0. \tag{4}$$

The operator $\mathbb{C}$ is equal to the fourth-order tensor

$$\mathbb{C}^{\mathrm{h}} = \mathbb{E} - \frac{1}{g}\,\mathbb{E}[\mathbf{P}] \otimes \mathbb{E}[\mathbf{Q}] \tag{5}$$

in the case of *plastic loading*, i.e. when the velocity of deformation $\mathbf{D}$ satisfies:

$$\mathbf{D}\cdot\mathbb{E}[\mathbf{Q}] \ge 0. \tag{6}$$

In eqns (4–6) the symmetric tensor $\mathbf{Q}$ is the *yield surface gradient*, the symmetric tensor $\mathbf{P}$ gives the mode of the plastic flow and g is the *plastic modulus*, related to the hardening modulus H by:

$$g = \mathrm{H} + \mathbf{P}\cdot\mathbb{E}[\mathbf{Q}]. \tag{7}$$

Strain hardening occurs when H is positive, whereas the softening regime is characterized by negative values of H. Moreover, perfect plasticity corresponds to $\mathrm{H} = 0$ and the snap-back (or critical) modulus to $\mathrm{H} = -\mathbf{P}\cdot\mathbb{E}[\mathbf{Q}]$. Finally, the subcritical behavior (CASEY & LIN [1986], MAIER & HUECKEL [1979]) corresponds to the values of the hardening modulus inferior to the snap-back threshold.

The elastic fourth-order tensor $\mathbb{E}$ is assumed to be a positive definite linear mapping Sym $\rightarrow$ Sym. The relative Lagrangian description is adopted:

$$\mathbf{F} = \mathbf{I}, \tag{8}$$

where $\mathbf{F}$ is the deformation gradient. The constitutive relationship (3) may be expressed in terms of the material derivative $\dot{\mathbf{S}}$ of the first Piola–Kirchhoff stress tensor and the Eulerian velocity gradient $\mathbf{L}$, as:

$$\dot{\mathbf{S}} = \bar{\mathbb{C}}[\mathbf{L}], \tag{9}$$

where, in the case of the Zaremba–Jaumann derivative of the Kirchhoff stress, with $\mathbf{T}$ denoting the Cauchy stress, $\bar{\mathbb{C}}$ becomes:

$$\bar{\mathbb{C}} = (\mathbb{C} - \mathbf{I} \boxtimes \mathbf{T} - \mathbf{T} \boxtimes \mathbf{I})\mathbb{S} + \mathbf{I} \boxtimes \mathbf{T} \tag{10}$$

in the case of the Lie derivative (or Oldroyd derivative) of the Kirchhoff stress:

$$\bar{\mathbb{C}} = \mathbb{C}\mathbb{S} + \mathbf{I} \boxtimes \mathbf{T} \tag{11}$$

or, in the case of the Zaremba–Jaumann derivative of the Cauchy stress:

$$\bar{\mathbb{C}} = (\mathbb{C} - \mathbf{I} \boxtimes \mathbf{T} - \mathbf{T} \boxtimes \mathbf{I})\mathbb{S} + \mathbf{I} \boxtimes \mathbf{T} + \mathbf{T} \otimes \mathbf{I}. \tag{12}$$

The constitutive operator (10) incorporates, as a particular case, the "classical" associative rule at finite strain (HILL [1958,1959,1961]; HUTCHINSON [1973]). The constitutive operator (12) was proposed by RUDNICKI and RICE [1975] (and later employed in NEEDLEMAN and RICE [1978], RICE [1976], and RICE and RUDNICKI [1980]). The difference between the constitutive operators (10) and (12) vanishes when the term (tr $\mathbf{D}$)$\mathbf{T}$ is negligible, as in the cases analyzed by HILL and HUTCHINSON [1975] and NEEDLEMAN [1979]. Finally, the constitutive operator (11) was used by HUTCHINSON and MILES [1974] and DUSZEK and PERZYNA [1991].

IV. LOCALIZATION ANALYSIS

Strain localization corresponding to the velocity gradient $\mathbf{L}_o$ outside the band and $\mathbf{L}_i$ inside, is possible if:

$$\bar{\mathbb{C}}[\mathbf{L}_i]\mathbf{n} - \bar{\mathbb{C}}[\mathbf{L}_o]\mathbf{n} = \mathbf{0}, \tag{13}$$

and

$$\mathbf{L}_i - \mathbf{L}_o = \mathbf{g} \otimes \mathbf{n}, \tag{14}$$

where $\mathbf{n}$ is the unit vector normal to the band and $\mathbf{g}$ is the vector giving the direction of the relative velocity into the band (Fig. 1). Condition (13) can be obtained (RICE [1976]) by using the constitutive equations into the *stress compatibility* condition:

$$(\dot{\mathbf{S}}_i - \dot{\mathbf{S}}_o)\mathbf{n} = \mathbf{0}. \tag{15}$$

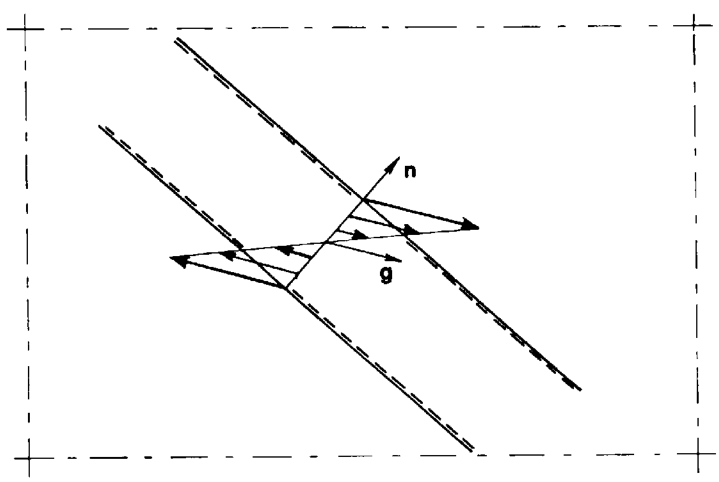

Fig. 1. Kinematics of localization of deformation.

Condition (14) expresses the strain compatibility across the two planes of discontinuity that identify the band (THOMAS [1961], HILL [1962]). It is worth noting that, in the case of the strain localization analysis, where reference is made to a homogeneous situation before bifurcation, *only the jump is prescribed* between the two fields $\mathbf{L}_i$ and $\mathbf{L}_o$. Therefore, unless the analysis is applied to a specific boundary value problem, either $\mathbf{L}_i$ or $\mathbf{L}_o$ is *free*. Equation (13) is valid regardless of the linearity of the operator $\bar{\mathbb{C}}$. Having assumed the incremental piecewise linear relationships (3), four cases may arise, i.e.:

a. plastic loading inside and outside the band
b. elastic unloading inside and outside the band
c. plastic loading inside the band and elastic unloading outside
d. elastic unloading inside the band and plastic loading outside

Only cases a and c were considered by RICE and RUDNICKI [1980] who showed that, for a generic continuous loading path and excluding the postcritical behavior, localization corresponding to plastic loading in the band and elastic unloading outside first becomes possible at localization in the Hill comparison solid. In the following, all the cases a–d are examined in detail.

IV.1. *Plastic loading inside and outside the band*

$$\mathbb{S}[\mathbf{L}_i]\cdot\mathbb{E}[\mathbf{Q}] \geq 0 \quad \text{and} \quad \mathbb{S}[\mathbf{L}_o]\cdot\mathbb{E}[\mathbf{Q}) \geq 0. \tag{16}$$

In this case $\bar{\mathbb{C}}$ becomes linear and is denoted by $\bar{\mathbb{C}}^{\mathrm{h}}$ ($\mathbb{C}$ must be replaced by $\mathbb{C}^{\mathrm{h}}$ in (10)–(12)). Equation (13) becomes the condition of the loss of ellipticity of $\bar{\mathbb{C}}^{\mathrm{h}}$, i.e.:

$$\bar{\mathbf{A}}_{\mathrm{h}}(\mathbf{n})\mathbf{g} = \mathbf{0}, \tag{17}$$

where $\bar{\mathbf{A}}_{\mathrm{h}}(\mathbf{n})$ is the acoustic tensor associated with $\bar{\mathbb{C}}^{\mathrm{h}}$:

$$\bar{\mathbf{A}}_{\mathrm{h}}(\mathbf{n})\mathbf{g} = \bar{\mathbb{C}}^{\mathrm{h}}[\mathbf{g} \otimes \mathbf{n}]\mathbf{n}, \quad \forall \mathbf{g} \in \mathcal{V}. \tag{18}$$

The acoustic tensor $\bar{\mathbf{A}}_{\mathrm{h}}(\mathbf{n})$ may be expressed as the sum of two parts:

$$\bar{\mathbf{A}}_{\mathrm{h}}(\mathbf{n}) = \mathbf{A}_{\mathrm{h}}(\mathbf{n}) + \mathbf{A}_{\mathrm{g}}(\mathbf{n}), \tag{19}$$

where:

$$\mathbf{A}_{\mathrm{h}}(\mathbf{n})\mathbf{g} = \mathbb{C}^{\mathrm{h}}\mathbb{S}[\mathbf{g} \otimes \mathbf{n}], \quad \forall \mathbf{g} \in \mathcal{V}, \tag{20}$$

and, in the case of the constitutive operator (10):

$$\mathbf{A}_{\mathrm{g}}(\mathbf{n}) = \tfrac{1}{2}[(\mathbf{n}\cdot\mathbf{T}\mathbf{n})\mathbf{I} - \mathbf{T}\mathbf{n} \otimes \mathbf{n} - \mathbf{n} \otimes \mathbf{n}\mathbf{T} - \mathbf{T}], \tag{21}$$

in the case of (11):

$$\mathbf{A}_{\mathrm{g}}(\mathbf{n}) = (\mathbf{n}\cdot\mathbf{T}\mathbf{n})\mathbf{I} \tag{22}$$

or, finally, in the case of (12):

$$\mathbf{A}_g(\mathbf{n}) = \tfrac{1}{2}[(\mathbf{n}\cdot\mathbf{Tn})\mathbf{I} + \mathbf{Tn}\otimes\mathbf{n} - \mathbf{n}\otimes\mathbf{nT} - \mathbf{T}]. \tag{23}$$

The "geometrical part" $\mathbf{A}_g(\mathbf{n})$ of the acoustic tensor is symmetric when the Jaumann derivative or the Oldroyd derivative of the Kirchhoff stress are adopted in (3). If the Jaumann derivative of the Cauchy stress is adopted, $\mathbf{A}_g(\mathbf{n})$ results to be nonsymmetric (as in the cases of RUDNICKI & RICE [1975] and AN & SCHAEFFER [1992]). Clearly, loss of ellipticity may originate from the tensor $\mathbf{A}_g(\mathbf{n})$, even if $\mathbf{A}_h(\mathbf{n})$ is not singular. On the other hand, even in the absence of $\mathbf{A}_g(\mathbf{n})$, as in the case of the infinitesimal theory, loss of ellipticity may be caused by the plastic deformation only. As shown by RICE [1976], a solution of (17) is possible when the plastic modulus $\mathcal{g}$ becomes equal to the critical plastic modulus:

$$\mathcal{g}^h(\mathbf{n}) = \mathbb{E}[\mathbf{Q}]\mathbf{n}\cdot\bar{\mathbf{A}}_e^{-1}(\mathbf{n})\mathbb{E}[\mathbf{P}]\mathbf{n}. \tag{24}$$

When $\mathcal{g} = \mathcal{g}^h(\mathbf{n})$, the vector $\mathbf{g}^h(\mathbf{n})$ that satisfies (17), normalized through the condition $\mathbf{g}^h(\mathbf{n})\otimes\mathbf{n}\cdot\mathbb{E}[\mathbf{Q}] = \mathcal{g}^h(\mathbf{n})$, is given by:

$$\mathbf{g}^h(\mathbf{n}) = \bar{\mathbf{A}}_e^{-1}(\mathbf{n})\mathbb{E}[\mathbf{P}]\mathbf{n}. \tag{25}$$

The tensor $\bar{\mathbf{A}}_e(\mathbf{n})$ in (25) is the elastic acoustic tensor defined as:

$$\bar{\mathbf{A}}_e(\mathbf{n})\mathbf{g} = \bar{\mathbb{E}}[\mathbf{g}\otimes\mathbf{n}]\mathbf{n}, \tag{26}$$

where $\bar{\mathbb{E}}$ is the fourth-order tensor of instantaneous elastic moduli associated with the conjugate pair $(\mathbf{S},\mathbf{F})$, defined analogously to $\bar{\mathbb{C}}$ as in (10)–(12).

It is worth noting that the solution (24–25) is valid only if $\bar{\mathbf{A}}_e(\mathbf{n})$ is not singular. In other words, the solution (24–25) is meaningful under the assumption that elastic strain localization cannot occur.

IV.2. *Elastic unloading inside and outside the band*

$$\mathbb{S}[\mathbf{L}_i]\cdot\mathbb{E}[\mathbf{Q}] \le 0 \quad \text{and} \quad \mathbb{S}[\mathbf{L}_o]\cdot\mathbb{E}[\mathbf{Q}] \le 0. \tag{27}$$

The constitutive operator $\bar{\mathbb{C}}$, as in the preceding case, becomes linear and equal to $\bar{\mathbb{E}}$. From condition (13), localization is attained when:

$$\bar{\mathbf{A}}_e(\mathbf{n})\mathbf{g} = 0. \tag{28}$$

IV.3. *Plastic loading inside the band and elastic unloading outside*

$$\mathbb{S}[\mathbf{L}_i]\cdot\mathbb{E}[\mathbf{Q}] \ge 0 \quad \text{and} \quad \mathbb{S}[\mathbf{L}_o]\cdot\mathbb{E}[\mathbf{Q}] \le 0. \tag{29}$$

In this case the operator $\bar{\mathbb{C}}$ in the first term of (13) becomes equal to $\bar{\mathbb{C}}^h$, whereas $\bar{\mathbb{C}}$ becomes equal to $\bar{\mathbb{E}}$ in the second term of (13). Therefore, the condition (13) implies:

$$\bar{\mathbb{C}}^h[\mathbf{L}_i]\mathbf{n} - \bar{\mathbb{E}}[\mathbf{L}_o]\mathbf{n} = \mathbf{0}. \tag{30}$$

Condition (30), using the linearity of $\mathbb{C}^h$, may be expressed in two equivalent forms:

$$\bar{\mathbf{A}}_e(\mathbf{n})\mathbf{g} = \frac{1}{\mathcal{g}}(\mathbf{D}_i \cdot \mathbb{E}[\mathbf{Q}])\mathbb{E}[\mathbf{P}]\mathbf{n}, \tag{31}$$

or

$$\bar{\mathbf{A}}_h(\mathbf{n})\mathbf{g} = \frac{1}{\mathcal{g}}(\mathbf{D}_o \cdot \mathbb{E}[\mathbf{Q}])\mathbb{E}[\mathbf{P}]\mathbf{n}. \tag{32}$$

IV.4. *Elastic unloading inside the band and plastic loading outside*

$$\mathbb{S}[\mathbf{L}_i] \cdot \mathbb{E}[\mathbf{Q}] \leq 0 \quad \text{and} \quad \mathbb{S}[\mathbf{L}_o] \cdot \mathbb{E}[\mathbf{Q}] \geq 0. \tag{33}$$

In this case eqn (13) yields:

$$\bar{\mathbb{E}}[\mathbf{L}_i]\mathbf{n} - \bar{\mathbb{C}}^h[\mathbf{L}_o]\mathbf{n} = \mathbf{0}, \tag{34}$$

giving the following two equivalent localization conditions:

$$\bar{\mathbf{A}}_e(\mathbf{n})\mathbf{g} = -\frac{1}{\mathcal{g}}(\mathbf{D}_o \cdot \mathbb{E}[\mathbf{Q}])\mathbb{E}[\mathbf{P}]\mathbf{n}, \tag{35}$$

or

$$\bar{\mathbf{A}}_h(\mathbf{n})\mathbf{g} = -\frac{1}{\mathcal{g}}(\mathbf{D}_i \cdot \mathbb{E}[\mathbf{Q}])\mathbb{E}[\mathbf{P}]\mathbf{n}. \tag{36}$$

V. STRAIN LOCALIZATION WITH ELASTIC UNLOADING

Assuming a continuous loading path, at a given instant the condition of strain localization in the Hill comparison solid is given by eqn (24) in terms of the critical value $\mathcal{g}^h(\mathbf{n})$ of the plastic modulus. The condition is valid under the hypothesis that the localization with elastic unloading on both the sides of the band does not precede the elastoplastic localization. In this section, it is shown that localization becomes possible with elastic unloading on one side of the band for values of the plastic modulus internal to the closed interval defined by $\mathcal{g}^h(\mathbf{n})$ and the snap-back threshold $\mathcal{g} = 0$. Finally the case of elastic unloading on both sides of the band is discussed, and it is shown that it cannot be a priori excluded to occur before localization in the Hill comparison solid.

Proposition 1. Localization with elastic unloading inside the band and plastic loading outside is possible if the plastic modulus belongs to the closed interval defined by the two extreme values 0 and $\mathcal{g}^h(\mathbf{n})$.

Proof: Let $\bar{\mathbf{A}}_e(\mathbf{n})$ be nonsingular. From eqn (35) one obtains:

$$\mathbf{g}(\mathbf{n}) = -\frac{1}{\mathcal{g}}(\mathbf{D}_o \cdot \mathbb{E}[\mathbf{Q}])\bar{\mathbf{A}}_e^{-1}(\mathbf{n})\mathbb{E}[\mathbf{P}]\mathbf{n}. \tag{37}$$

By substituting (37) through (14) into the unloading condition (33_1) and keeping into account (33_2), the following inequality is obtained:

$$1 - \frac{1}{\mathscr{g}} \mathbb{E}[\mathbf{Q}]\mathbf{n}\cdot\bar{\mathbf{A}}_c^{-1}(\mathbf{n})\mathbb{E}[\mathbf{P}]\mathbf{n} \leq 0. \tag{38}$$

Condition (38), together with the definition (24) of $\mathscr{g}^h(\mathbf{n})$, yields:

$$1 - \frac{\mathscr{g}^h(\mathbf{n})}{\mathscr{g}} \leq 0, \tag{39}$$

therefore:

$$\mathscr{g}^h(\mathbf{n}) > 0 \Rightarrow \mathscr{g} \in [0,\mathscr{g}^h(\mathbf{n})], \tag{40}$$

$$\mathscr{g}^h(\mathbf{n}) \leq 0 \Rightarrow \mathscr{g} \in [\mathscr{g}^h(\mathbf{n}),0]. \tag{41}$$

Thus, if $\mathbf{D}_o$ satisfies the plastic loading condition and $\mathscr{g}$ is internal to the closed interval defined by $\mathscr{g}^h(\mathbf{n})$ and 0, the velocity of deformation in the band $\mathbf{D}_i = \mathbf{D}_o + \mathbf{g} \otimes \mathbf{n}$, where $\mathbf{g}$ is given by (37), necessarily corresponds to elastic unloading.

Proposition 2. Localization with plastic loading inside the band and elastic unloading outside is possible if the plastic modulus belongs to the closed interval defined by the two extreme values 0 and $\mathscr{g}^h(\mathbf{n})$.

Proof: Let $\bar{\mathbf{A}}_e(\mathbf{n})$ be nonsingular. From eqn (31) one obtains:

$$\mathbf{g}(\mathbf{n}) = \frac{1}{\mathscr{g}} (\mathbf{D}_i\cdot\mathbb{E}[\mathbf{Q}])\bar{\mathbf{A}}_e^{-1}(\mathbf{n})\mathbb{E}[\mathbf{P}]\mathbf{n}. \tag{42}$$

By substituting (42) through (14) into the unloading condition (29_2) and keeping into account (29_1), the following inequality is obtained:

$$1 - \frac{1}{\mathscr{g}} \mathbb{E}[\mathbf{Q}]\mathbf{n}\cdot\bar{\mathbf{A}}_e^{-1}(\mathbf{n})\mathbb{E}[\mathbf{P}]\mathbf{n} \leq 0. \tag{43}$$

Condition (43) is identical to (38) and thus the inequalities (40–41) are deduced.

Remark 1: If in a continuous loading path $\bar{\mathbf{A}}_e(\mathbf{n})$ is singular at isolated points only, continuity implies that the results of Propositions 1 and 2 remain valid. In other words, at the point where $\bar{\mathbf{A}}_e(\mathbf{n})$ is singular, vector $\mathbf{g}^h(\mathbf{n})$ can be obtained as a limit on the loading path.

Remark 2: RICE and RUDNICKI [1980] arrived at the same conclusion of Proposition 2, with the only difference that the existence of $\bar{\mathbf{A}}_h^{-1}(\mathbf{n})$ is required in their proof. In order to make clear the analogies with that work, we use condition (32):

$$\mathbf{g}(\mathbf{n}) = \frac{1}{\mathscr{g}} (\mathbf{D}_o\cdot\mathbb{E}[\mathbf{Q}])\bar{\mathbf{A}}_h^{-1}(\mathbf{n})\mathbb{E}[\mathbf{P}]\mathbf{n} \tag{44}$$

into the loading condition (29_1), to obtain

$$\mathbf{D}_o \cdot \mathbb{E}[\mathbf{Q}] \left\{ 1 + \frac{1}{\mathscr{g}} \mathbb{E}[\mathbf{Q}] \cdot \mathbb{S}[\mathbf{n} \otimes \bar{\mathbf{A}}_h^{-1}(\mathbf{n}) \mathbb{E}[\mathbf{P}]\mathbf{n}] \right\} \geq 0. \tag{45}$$

By using the unloading condition (29_2) and the symmetries of $\mathbb{E}$, eqn (45) becomes:

$$1 + \frac{1}{\mathscr{g}} \mathbb{E}[\mathbf{Q}]\mathbf{n} \cdot \bar{\mathbf{A}}_h^{-1}(\mathbf{n}) \mathbb{E}[\mathbf{P}]\mathbf{n} \leq 0. \tag{46}$$

When $\mathscr{g}^h(\mathbf{n}) \neq \mathscr{g}$, the inverse of the acoustic tensor of the Hill comparison solid was obtained by RICE and RUDNICKI [1980], in the form (see the Appendix):

$$\bar{\mathbf{A}}_h^{-1}(\mathbf{n}) = \left[\mathbf{I} + \frac{\bar{\mathbf{A}}_e^{-1}(\mathbf{n}) \mathbb{E}[\mathbf{P}]\mathbf{n} \otimes \mathbb{E}[\mathbf{Q}]\mathbf{n}}{\mathscr{g} - \mathscr{g}^h(\mathbf{n})} \right] \bar{\mathbf{A}}_e^{-1}(\mathbf{n}). \tag{47}$$

Note that $\bar{\mathbf{A}}_e^{-1}(\mathbf{n})$ and therefore $\bar{\mathbf{A}}_h^{-1}(\mathbf{n})$ are supposed to exist. A substitution of (47) into (46) yields:

$$\frac{\mathscr{g}}{\mathscr{g} - \mathscr{g}^h(\mathbf{n})} \leq 0, \tag{48}$$

that is identical to (39).

Remark 3: Inequality (41) allows the conclusion that localization with elastic unloading is always possible at least for values of the plastic modulus inferior to the snap-back threshold. In other words, it is not possible to reach the snap-back modulus without passing the possibility of a strain localization.

Proposition 3. If $\mathscr{g}(\mathbf{n}) = \mathscr{g}^h(\mathbf{n})$, elastoplastic localization corresponds to neutral loading on the side of unloading.

Proof: In the case of elastic unloading into the band, a substitution of the jump condition (14), together with (37), into the unloading condition (33_1) yields:

$$\mathbb{S}[\mathbf{L}_i] \cdot \mathbb{E}[\mathbf{Q}] = \mathbf{D}_o \cdot \mathbb{E}[\mathbf{Q}] - \frac{1}{\mathscr{g}} (\mathbf{D}_o \cdot \mathbb{E}[\mathbf{Q}]) \cdot \bar{\mathbf{A}}_e^{-1}(\mathbf{n}) \mathbb{E}[\mathbf{P}]\mathbf{n}, \tag{49}$$

from which the condition of neutral loading is readily obtained using the expression (24) of the critical plastic modulus. In the case of elastic unloading outside the band, a substitution of the jump condition (14), together with (42), into the unloading condition (29_2) yields:

$$\mathbb{S}[\mathbf{L}_o] \cdot \mathbb{E}[\mathbf{Q}] = \mathbf{D}_i \cdot \mathbb{E}[\mathbf{Q}] - \frac{1}{\mathscr{g}} (\mathbf{D}_i \cdot \mathbb{E}[\mathbf{Q}]) \cdot \bar{\mathbf{A}}_e^{-1}(\mathbf{n}) \mathbb{E}[\mathbf{P}]\mathbf{n}, \tag{50}$$

from which the condition of neutral loading is readily obtained using the expression (24) of the critical plastic modulus.

Remark 4: In the case of nonassociative flow rules, localization corresponding to elastic unloading inside and outside the band can in principle precede (or occur at) the localization in the Hill comparison solid.

In fact, let $\mathbf{g} \otimes \mathbf{n}$ be the dyad for which $\bar{\mathbb{E}}$ loses ellipticity and for which elastic unloading occurs, i.e. $\mathbb{S}[\mathbf{g} \otimes \mathbf{n}] \cdot \mathbb{E}[\mathbf{Q}] < 0$. Due to the nonassociativity of the flow rule, the dyad can verify the following inequality:

$$-\frac{1}{\mathcal{g}}\left(\mathbf{g} \otimes \mathbf{n} \cdot \mathbb{E}[\mathbf{Q}]\right)\left(\mathbf{g} \otimes \mathbf{n} \cdot \mathbb{E}[\mathbf{P}]\right) > 0. \tag{51}$$

Therefore the Hill comparison solid can satisfy, in correspondence of the critical dyad $\mathbf{g} \otimes \mathbf{n}$, the strong ellipticity condition and, thus, the ellipticity. This possibility becomes evident if no restrictions are placed on the direction of plastic flow $\mathbf{P}$. Trivially, if $\mathbf{P} = -\mathbf{Q}$ is assumed, the localization corresponding to elastic unloading inside and outside the band always precedes the localization in the Hill comparison solid. It is a well-known effect that, in the presence of nonassociative flow rules, the second order work along certain strain rate directions in elastic–plastic deformation may be larger than the second order work corresponding to elastic deformation (MRÓZ [1963,1966], RUNESSON & MRÓZ [1989]. This effect explains the possibility of strain localization corresponding to elastic unloading inside and outside the band before strain localization in the Hill comparison solid.

From the above Propositions 1, 2, and 3 it is concluded:

The critical value of the hardening modulus for localization in the Hill comparison solid represents the transition to the possibility of localization with elastic unloading on one side of the band. On this side, the elastic solution starts to develop from a neutral loading condition.

Clearly, the velocity gradient outside (or inside) the band is free only if a specific boundary value problem is not considered.

VI. ONE-DIMENSIONAL EXAMPLE: SHANLEY ANALOGY

As noted by RICE and RUDNICKI [1980], the problem of strain localization presents analogies with the SHANLEY [1947] model of column buckling. These analogies are now evident from the preceding propositions. First of all, the onset of bifurcation (localization) is determined by the Hill comparison solid. The solution corresponding to elastic unloading on one side of the band initially satisfies the condition of neutral loading (as in one of the Shanley springs). After the first bifurcation threshold, a range of possible bifurcations is observed in the presence of elastic unloading on one side of the band. The condition of elastic unloading inside or outside of the band corresponds to the possibility of elastic unloading in one or in the other of the Shanley springs. This analogy is, however, not strict because of the fact that the Shanley problem is symmetric in respect to the springs, whereas the localization problem is symmetric in respect to the velocity deformation inside and outside the band only in the homogeneous problem when boundary conditions are disregarded. In the case of the associative flow rule, the localization in the elastoplastic solid occurs always before the localization corresponding to elastic unloading on both sides of the band, as in the Shanley problem. This circumstance, however, may be not verified for nonassociative flow rules (cfr. Remark 4). In any case, assuming that the localization of deformation with elastic unloading on both

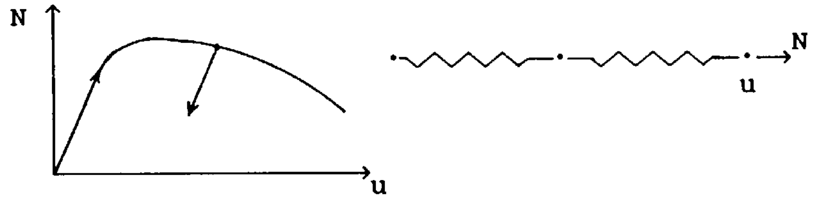

Fig. 2. Nonlinear spring model.

sides of the band occurs after localization in the Hill comparison solid, at the instant of localization with the elastic unloading on both sides of the band, the neutral loading condition on one sides of the band is attained. This circumstance is also verified for the Shanley springs.

In order to stress the analogies with the Shanley column buckling problem let us consider the special case of uniaxial extension of a prismatic block. In this specific case the constitutive parameters (elastic constants and tensors **P** and **Q**) can be chosen in such a way to obtain from the uniaxial state of strain a uniaxial state of stress. Then, if the band is prescribed to occur orthogonal to the direction of stretching, the block behaves essentially as the system shown in Fig. 2 of two nonlinear springs with linear unloading. The two-spring system behaves similarly to the Shanley model. In fact, bifurcation occurs at the peak of the uniaxial curves, where one of the springs is in the condition of neutral loading (see Fig. 3). This spring unloads elastically after the peak, whereas the other one continues to be subject to plastic loading. Finally, if both springs are constrained to pass the peak remaining on the fundamental path, bifurcation starts directly with elastic unloading in one spring.

VII. CONCLUSIONS

The condition of strain localization has been examined in the context of the incremental finite formulation of elastoplasticity with nonassociative flow rule and smooth yield and plastic potential functions. The result obtained by RICE and RUDNICKI [1980] has been reconsidered. The possibility of strain localization corresponding to elastic unload-

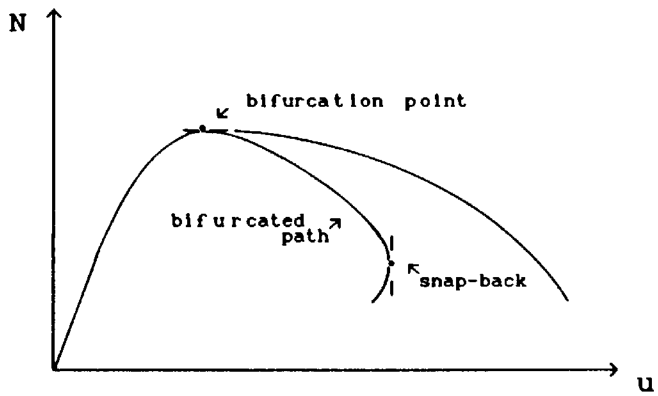

Fig. 3. Behavior of the spring model.

ing inside the band and plastic loading outside has been considered. It is shown that this possibility of localization first occurs, satisfying the condition of neutral loading into the band, at strain localization in the Hill comparison solid (corresponding to plastic loading inside and outside the band). The result, as well as the result due to RICE and RUDNICKI [1980], is discussed, including the critical and subcritical behavior (in the sense defined by MAIER and HUECKEL [1979]). It is shown that localizations with elastic unloading on one side of the band and plastic loading on the other side are possible for values of the plastic modulus internal to the closed interval defined by the snap-back modulus and the critical plastic modulus for strain localization in the Hill comparison solid. However, the possibility of strain localization with elastic unloading on both sides of the band cannot be excluded, in principle, to occur prior to strain localization in the Hill comparison solid. This circumstance, which is excluded in the infinitesimal theory (for positive definite incremental elastic tensor), is a consequence of the "spurious stiffness" effect caused by the plastic nonassociativity (MRÓZ [1963,1966]).

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APPENDIX: INVERSE OF ACOUSTIC TENSOR $\bar{\mathbf{A}}_h(\mathbf{n})$

In the following, it will be verified that eqn (47) gives the inverse of the elastoplastic acoustic tensor. It suffices to show that

$$[\bar{\mathbf{A}}_h^{-1}(\mathbf{n})\bar{\mathbf{A}}_h(\mathbf{n})]\mathbf{v} = \mathbf{v}, \quad \forall \mathbf{v} \in \mathcal{V}. \tag{A.1}$$

By using the definitions (18), (5) and (24), one obtains:

$$\begin{aligned}
[\bar{\mathbf{A}}_h^{-1}(\mathbf{n})\bar{\mathbf{A}}_h(\mathbf{n})]\mathbf{v} &= \left[\mathbf{I} + \frac{\bar{\mathbf{A}}_e^{-1}(\mathbf{n})\mathbb{E}[\mathbf{P}]\mathbf{n} \otimes \mathbb{E}[\mathbf{Q}]\mathbf{n}}{g - g^h(\mathbf{n})}\right]\bar{\mathbf{A}}_e^{-1}(\mathbf{n})\bar{\mathbf{A}}_h(\mathbf{n})\mathbf{v} \\
&= \left[\mathbf{I} + \frac{\bar{\mathbf{A}}_e^{-1}(\mathbf{n})\mathbb{E}[\mathbf{P}]\mathbf{n} \otimes \mathbb{E}[\mathbf{Q}]\mathbf{n}}{g - g^h(\mathbf{n})}\right]\bar{\mathbf{A}}_e^{-1}(\mathbf{n}) \\
&\quad \times \left[\bar{\mathbf{A}}_e(\mathbf{n})\mathbf{v} - \frac{1}{g}(\mathbb{E}[\mathbf{Q}]\mathbf{n}\cdot\mathbf{v})\mathbb{E}[\mathbf{P}]\mathbf{n}\right] \\
&= \mathbf{v} + \frac{(\mathbb{E}[\mathbf{Q}]\mathbf{n}\cdot\mathbf{v})}{g(g - g^h(\mathbf{n}))}\bar{\mathbf{A}}_e^{-1}(\mathbf{n})\mathbb{E}[\mathbf{P}]\mathbf{n} \\
&\quad \times \left[g^h(\mathbf{n}) - \mathbb{E}[\mathbf{Q}]\mathbf{n}\cdot\bar{\mathbf{A}}_e^{-1}(\mathbf{n})\mathbb{E}[\mathbf{P}]\mathbf{n}\right] = \mathbf{v},
\end{aligned}$$

for every $\mathbf{v} \in \mathcal{V}$.