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On the construction of extended problems and related functionals for general nonlinear equations

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Abstract

Starting from existing methods for the symmetrisation of general nonlinear, nonpotential operators (Tonti, Int. J. Engng. Sci. 22 (11–12) (1984) 1343–1371; Auchmuty, Nonlinear Anal. Theory Methods Appl. 12 (5) (1988) 531–564) this work discusses some alternative formulations and illustrates some significant implications of such methods, which should make them more suited to practical application. Further, a new class of the so-called "extended" functionals is proposed, much simpler to construct than the preceding ones. Even if the definition of the functionals requires the doubling of the unknown functions, the new unknowns have a precise physical meaning in the solution of the problem, which may help in the actual solution process. The application of the new method is illustrated by means of two examples in the field of continuum mechanics: the nonassociated elastic–plastic rate constitutive equations, and the nonlinear continuum dynamics equations with initial conditions. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The mathematical description of a physical phenomenon is usually given in terms of a so-called problem, i.e., a set of equations for instance written as follows:

$$\mathcal{N}(u) - P = 0 \tag{1.1}$$

in which u represents an unknown function, possibly vector or tensor valued, P represents a known function and $\mathcal{N}(\cdot)$ represents a generic nonlinear operator which transforms function u, defined in a suitable vector space U, into another function, defined

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in the vector space V which includes the known functions represented by P. Here, and in the following, all functions and operators may have the meaning of vectors of functions and matrices of operators.

A deeper understanding of the physical phenomenon might be obtained from a variational formulation, in the cases when such formulation can be written down. Variational formulations can be used as a starting point for theoretical analyses about existence and uniqueness of the solution, or as a starting point for the construction of numerical solution methods; with reference to the latter, the availability of a variational formulation allows the mathematician to compute error estimates. Starting from variational formulations, bounds on properties of the studied problem can often be constructed such as, for instance, when studying the linear and nonlinear behaviour of composite materials.

Nevertheless, it is not always simple to give a variational formulation to a mathematical problem. In particular, in the absence of symmetry of the operator governing the problem, with respect to a suitable bilinear form, it is impossible to construct the relevant variational formulation. This difficulty has prompted, already in the early 1950s, the study of symmetrisation methods, with the specific purpose of enabling to derive variational statements corresponding to general nonlinear problems. More specifically, the quest is for so-called "classical" variational statements, based on the concept of "classical" bilinear forms.

Here we refer to bilinear forms and to symmetry with respect to a bilinear form in the following sense (Vainberg, 1964; Yosida, 1965). Consider a function u belonging to a vector space U, and denote with the symbol $\mathscr{A}(\cdot)$ a generic operator acting on function u. Operator $\mathscr{A}(\cdot)$ transforms functions u defined in the domain $D(\mathscr{A}) \subset U$ into functions x defined in the range $R(\mathscr{A}) \subset V$, where V is another vector space. Note that, in what follows, we will always assume, without any loss of generality, that the domain of operator $\mathscr{A}(\cdot)$ is not restricted by any sort of initial/boundary conditions; we will always assume that these last are included into the definition of operator $\mathscr{A}(\cdot)$ itself.

We denote with $\mathscr{A}'_u(\cdot)$ the Gateaux derivative (if it exists) of operator $\mathscr{A}(\cdot)$ with respect to the function u, i.e.

$$\mathscr{A}'_u(u,z) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}[\mathscr{A}(u+\varepsilon z)]_{\varepsilon\to 0} = \mathscr{A}'_u z, \tag{1.2}$$

where z is another function defined in space U and ε is a scalar.

We denote with the symbol

$$\langle u,x\rangle \tag{1.3}$$

a nondegenerate bilinear form which puts into separating duality the linear spaces U and V. Note that, in the following, we will always make reference to symmetric bilinear forms, i.e., such that $\langle u,x\rangle = \langle x,u\rangle$.

In the case functions u and x depend both on position in a volume Ω and on time t, ranging from an initial time t_0 to a final time t_1, we label as "classical" a bilinear form of the following type:

$$\langle u,x\rangle = \int_{t_0}^{t_1}\int_{\Omega} ux \,\mathrm{d}\Omega\,\mathrm{d}t + \text{ possible boundary and initial terms.} \tag{1.4}$$

If operator $\mathscr{A}(\cdot)$ is linear, it is said to be symmetric with respect to the bilinear form (1.3) if the following holds:

$$\langle \mathscr{A}u, y\rangle = \langle u, \mathscr{A}y\rangle \quad \text{with } u, y \in D(\mathscr{A}). \tag{1.5}$$

If operator $\mathscr{A}(\cdot)$ is nonlinear, the concept of symmetry with respect to a bilinear form is referred to its Gateaux derivative, as follows:

$$\langle \mathscr{A}'_u z, y\rangle = \langle z, \mathscr{A}'_u y\rangle. \tag{1.6}$$

If operator $\mathscr{A}(\cdot)$ is symmetric with respect to a "classical" bilinear form of type (1.4), a problem governed by $\mathscr{A}(\cdot)$ admits a "classical" variational formulation.

An operator exhibiting symmetry with respect to a suitable bilinear form is also called "potential operator", in the sense that it admits a potential (i.e., an associated functional, whose gradient is the operator itself) defined by means of that particular bilinear form. In this paper we will use either of the terms "symmetric" and "potential" to indicate operators enjoying a potential associated with a "classical" bilinear form of type (1.4).

Morse and Feshbach (1953) proposed the so-called "adjoint operator method" as a tool to symmetrise a linear, nonsymmetric operator, in such a way as to obtain a "classical" variational formulation for the problem governed by that operator. This method, which requires the introduction of a new set of physically meaningless unknowns, was extended to the general, nonlinear case by Finlayson (1972). Applications of the adjoint operator method have been proposed, for instance, by Telega (1982) for nonassociated plasticity, and by Robinson and Yuen (1986) for integral equations; this method, by admission of its developers, has several drawbacks, and is just a last resort when everything else fails.

Gurtin (1964), Rafalski (1969a, b), Tonti (1973) and Reiss and Haug (1978) have all studied the case of linear, time-dependent problems, such as, for example, the case of continuum dynamics with initial conditions which destroy symmetry with respect to "classical" bilinear forms. Their variational formulations are therefore non-"classical" and, in general, difficult to apply for practical purposes.

Tonti (1984) has proposed a general technique to obtain variational formulations associated with any nonlinear problem; a similar method has been proposed also by Magri (1974) and Ortiz (1985), with reference to linear nonpotential operators only. This method is based on the choice of a suitable linear, invertible, symmetric kernel function $\mathscr{K}(\cdot)$, which allows the construction, in a systematic way, of "extended" functionals associated with any nonlinear problem. The practical use of this method rests on the choice of the kernel function, and little indication is given, by the general theory, on how to choose among the infinite number of possible kernels. In extreme summary, the extended functional proposed by Tonti, with reference to problem (1.1), is the following:

$$T[u] = \tfrac{1}{2}\langle \mathscr{N}(u), \mathscr{K}\mathscr{N}(u)\rangle - \langle P, \mathscr{K}\mathscr{N}(u)\rangle \tag{1.7}$$

whose minimum with respect to u gives the solution of the original problem, if it exists.

A few years later Auchmuty (1988) developed Tonti's ideas. In terms of the symbolism used in the present paper, and with a little less generality than that present in

Auchmuty (1988), Auchmuty's results can be seen as based on the splitting of the non-linear operator $\mathscr{N}(\cdot)$ into a linear, symmetric positive-definite part $\mathscr{S}(\cdot)$ and a residual part $\mathscr{R}(\cdot)$:

$$\mathscr{N}(\cdot) = \mathscr{S}(\cdot) + \mathscr{R}(\cdot). \tag{1.8}$$

Exploiting the consequences of this splitting, Auchmuty could somewhat extend Tonti's original theory, by considering, as the kernel function $\mathscr{K}(\cdot)$, the operator $\mathscr{S}^{-1}(\cdot)$; this lead to a reformulation of functional (1.7), in the single unknown function u.

More interestingly from the applicative viewpoint, Auchmuty could write, exploiting the theory of nonconvex duality, a dual extended functional, starting from a Lagrangian functional defined in terms of two unknown function, the original one u and an auxiliary one v. Again, in our notation, and with the same restricted generality as in Eq. (1.8), the Lagrangian functional introduced by Auchmuty is the following:

$$A[u, v] = \langle (v-u), [\mathscr{R}(u) - P] \rangle - \tfrac{1}{2}\langle u, \mathscr{S}u \rangle + \tfrac{1}{2}\langle v, \mathscr{S}v \rangle \tag{1.9}$$

and the solution of problem (1.1) is found by first computing the minimum with respect to u of the functional $A[u, v]$ (thus obtaining the above mentioned dual extended functional in the single unknown v), and then the maximum with respect to v.

Auchmuty gives the extended problem which corresponds to his version of Tonti's extended functional in the single unknown u; however, he writes the extended problem corresponding to functional (1.9) only for the linear case. He gives some examples of application of this theory, but he does not comment at all about the meaning of the auxiliary unknown function v, in relation to the meaning of the real one u.

Independent of the work by Auchmuty, Carini (1996, 1997), and later Carini and De Donato (1997) and Carini and Genna (1998) have developed this technique, with reference to several specific applications in the field of continuum mechanics. In the present paper we consider a generic nonlinear problem, governed by an operator which is nonsymmetric with respect to "classical" bilinear forms, and further develop all these ideas. First, we rewrite functional (1.9) in a somewhat simpler form, by starting from the identification of the extended problem corresponding to such a functional in the general, nonlinear case. From this we can also prove that the auxiliary unknown function v has a precise meaning in the solution of the extended problem.

Next, by means of a suitable, "degenerate" choice of the original operator splitting (1.8), we will show how to obtain a simple, new "classical" variational formulation for any nonlinear problem, which cannot be seen as a particular case of the general variational framework set up in Tonti (1984) and Auchmuty (1988), but which enjoys several properties of that framework. This new formulation is quite appealing from the applicative viewpoint, since it requires no manipulation of the equations defining the original problem, and requires no choice of linear, symmetric, positive-definite kernels and related bilinear forms. Two continuum mechanics examples illustrate the application of this new theory.

2. Auchmuty's Lagrangian functional rewritten, and the corresponding extended problem

Consider problem (1.1), and recall the operator splitting (1.8). Consider then the following definition of an extended operator $\mathscr{M}(\cdot)$:

$$\mathscr{M}(\cdot) = \begin{bmatrix} \mathscr{S}(\cdot) + \mathscr{R}(\cdot) + \mathscr{R}_u^{'*}(\cdot) & -\mathscr{R}_u^{'*}(\cdot) \\ -\mathscr{R}(\cdot) & -\mathscr{S}(\cdot) \end{bmatrix} \tag{2.1}$$

which allows the formulation of the following extended problem:

$$\mathscr{M}(w) = Q, \tag{2.2}$$

where the unknown function w and the known term Q are defined as follows:

$$w = \begin{bmatrix} u \\ v \end{bmatrix}, \quad Q = \begin{bmatrix} P \\ -P \end{bmatrix}, \tag{2.3}$$

where v is a new, auxiliary unknown function.

In Eq. (2.1) $\mathscr{R}_u^{'*}(\cdot)$ indicates the adjoint operator of the Gateaux derivative of operator $\mathscr{R}(\cdot)$. The adjoint operator of the Gateaux derivative of a generic nonlinear operator $\mathscr{N}(\cdot)$ is defined as the operator $\mathscr{N}_u^{'*}(\cdot)$ such that the following identity holds:

$$\langle y, \mathscr{N}_u' z\rangle = \langle \mathscr{N}_u^{'*} y, z\rangle. \tag{2.4}$$

In the case of a linear operator $\mathscr{L}$, its adjoint $\mathscr{L}^*$ is defined by the analogous identity

$$\langle y, \mathscr{L} u\rangle = \langle \mathscr{L}^* y, u\rangle. \tag{2.5}$$

Definition (2.1) implies the existence of the Gateaux derivative $\mathscr{R}_u'(\cdot)$ of the residual operator $\mathscr{R}(\cdot)$ of Eq. (1.8).

Problem (2.1)–(2.2) is very similar to the extended problem recognised in Auchmuty (1988) as associated with the Lagrangian functional (1.9), in the case that the starting problem is linear. Here it is defined in a completely general, nonlinear context. Problem (2.1)–(2.2) admits a variational formulation associated to the following functional:

$$F[u,v] = \langle (u-v), [\mathscr{N}(u) - P]\rangle - \tfrac{1}{2}\langle (u-v), \mathscr{S}(u-v)\rangle. \tag{2.6}$$

The theoretical tools used to obtain result (2.6), starting from problem (2.1)–(2.2), are standard (see, for instance, Tonti, 1984; Volterra, 1913); functional $F[u,v]$ is actually a slightly modified version of Auchmuty's Lagrangian functional (1.9).

We can then prove the following statements.

Lemma 1. *Every solution u_0 of problem* (1.1) *is also part of the solution w_0 of problem* (2.2).

This is proved by writing problem (2.2) in expanded form

$$\mathscr{S}u + \mathscr{R}(u) + \mathscr{R}_u^{'*}u - \mathscr{R}_u^{'*}v = P, \tag{2.7}$$

$$-\mathscr{R}(u) - \mathscr{S}v = -P. \tag{2.8}$$

Recalling that the operator $\mathscr{S}(\cdot)$ is positive definite, one can express the unknown v as function of u from Eq. (2.8)

$$v = \mathscr{S}^{-1}P - \mathscr{S}^{-1}\mathscr{R}(u) \tag{2.9}$$

and, substituting this result into (2.7), one gets

$$\mathscr{S}u + \mathscr{R}(u) + \mathscr{R}_u^{'*}u - \mathscr{R}_u^{'*}\mathscr{S}^{-1}P + \mathscr{R}_u^{'*}\mathscr{S}^{-1}\mathscr{R}(u) = P \tag{2.10}$$

which is identically satisfied by any solution u_0 of problem (1.1), since the third, fourth and fifth terms on the left-hand side of Eq. (2.10) can be obtained from Eq. (1.1) by applying to it the operator $\mathscr{R}_u^{'*}\mathscr{S}^{-1}(\cdot)$.

Lemma 2. *If, and only if, u_0 is a solution of problem* (1.1) *then the following equality holds in the solution w_0 of problem* (2.2)

$$u_0 = v_0 \tag{2.11}$$

which gives a precise identity to the auxiliary unknown function v in the solution.

This is readily demonstrated: by writing Eq. (1.1) with $u = u_0$ and recalling the splitting (1.8), and considering Eq. (2.8) again written in the solution w_0 of problem (2.2), using Lemma 1, by adding the two equations, and recalling that operator $\mathscr{S}(\cdot)$ is definite positive, one arrives at conclusion (2.11).

Theorem 1. *Every solution u_0 of problem* (1.1) *corresponds to a stationarity point of functional* (2.6) *with respect to w.*

To prove this, consider the first variation of functional (2.6) with respect to its unknown function w:

$$\delta F = \langle(\delta u - \delta v), [\mathscr{N}(u) - P]\rangle + \langle(u - v), \delta\mathscr{N}(u)\rangle - \langle(\delta u - \delta v), \mathscr{S}(u - v)\rangle. \tag{2.12}$$

Owing to the symmetry of the operator $\mathscr{S}(\cdot)$ and to Lemma 2, this expression is zero in correspondence to the solution u_0 of problem (1.1), which proves the theorem.

Lemma 3. *Functional* (2.6) *assumes the zero value in each solution u_0 of problem* (1.1); *this is a straightforward consequence of result* (2.11).

Theorem 2. *Every function w_0 which makes stationary the functional $F[w]$ of Eq.* (2.6) *with $u_0 = v_0$ contains a solution u_0 of problem* (1.1), *and is such that the functional $F[w_0] = 0$.*

This can be proved by rewriting the first variation of the functional $F[w]$, Eq. (2.12), as follows:

$$\delta F = \langle\delta u, [\mathscr{N}(u) - P]\rangle + \langle\delta v, [-\mathscr{R}(u) - \mathscr{S}v + P]\rangle + \langle\delta\mathscr{R}(u), (u - v)\rangle. \tag{2.13}$$

If the functional $F[w]$ is stationary, then its first variation must vanish. Therefore, since the variation δv in Eq. (2.13) is arbitrary, Eq. (2.8) must hold, which gives result (2.9). Replacing it in Eq. (2.13) one obtains

$$\delta F = \langle \delta u, [\mathscr{N}(u) - P]\rangle + \langle \delta \mathscr{R}(u), [u - \mathscr{S}^{-1}P + \mathscr{S}^{-1}\mathscr{R}(u)]\rangle. \tag{2.14}$$

It is then possible to operate on the last term of Eq. (2.14), taking advantage of the symmetry and positive definiteness of operator $\mathscr{S}(\cdot)$, in the following way:

$$\begin{aligned} &\langle \delta \mathscr{R}(u), [u - \mathscr{S}^{-1}P + \mathscr{S}^{-1}\mathscr{R}(u)]\rangle \\ &\quad = \langle \delta \mathscr{R}(u), u\rangle - \langle \mathscr{S}^{-1}\delta \mathscr{R}(u), P\rangle + \langle \mathscr{S}^{-1}\delta \mathscr{R}(u), \mathscr{R}(u)\rangle \\ &\quad = \langle \mathscr{S}^{-1}\delta \mathscr{R}(u), [\mathscr{S}u + \mathscr{R}(u) - P]\rangle. \end{aligned} \tag{2.15}$$

Therefore, Eq. (2.14), written at a stationarity point of the functional $F[w]$, and recalling splitting (1.8), becomes

$$\delta F = \langle [\delta u + \mathscr{S}^{-1}\delta \mathscr{R}(u)],\ [\mathscr{N}(u) - P]\rangle = \langle \mathscr{S}^{-1}\delta \mathscr{N}(u),\ [\mathscr{N}(u) - P]\rangle = 0. \tag{2.16}$$

Then two possibilities exist:

1. $\mathscr{N}(u) - P = 0$. Then the function u solves problem (1.1), and it is easy to verify that the functional $F[w]$ vanishes in correspondence to such solution.
2. $\mathscr{N}(u) - P \neq 0$, $\delta \mathscr{N}(u) = 0$. Then the function u is not a solution of problem (1.1), which therefore, owing to Theorem 1, has no solution, even if problem (2.2) might have one, where equality (2.11) does not hold.

Even if part of these propositions is implicitly contained in the works of Tonti (1984) and Auchmuty (1988), we feel it is worth proving them all in this context because the link of the extended problem (2.1)–(2.2) with the variational formulation (2.6) has never been proved in the general nonlinear case. In any event, some of the previous statements are new, and provide useful information about these techniques.

These results extend those of Auchmuty (1988), specially in that they give physical meaning to the auxiliary unknown function introduced to symmetrise the original problem. Moreover, the explicit construction of the extended problem of equations (2.1)–(2.2) allows to recognise the possibility of writing a new extended functional associated with problem (1.1), which cannot be seen as a special case of the existing theory. This functional will be discussed in the next section.

Here it may be worth pointing out some elementary special cases of the preceding theory. The first is the trivial case of a linear, positive-definite potential operator $\mathscr{N}(\cdot)$. Then the residual $\mathscr{R}(\cdot)$ vanishes, $\mathscr{S}(\cdot) = \mathscr{N}(\cdot)$ and problem (2.2) and its variational counterpart (2.6) reduce to two uncoupled formulations which include all "classical" extremum principles.

A second is that of a linear and positive definite, but nonpotential operator $\mathscr{N}(\cdot)$. Then it is possible to define the operator $\mathscr{S}(\cdot)$ as an equally weighted combination of the operator $\mathscr{N}(\cdot)$ and its adjoint $\mathscr{N}^*(\cdot)$, which gives rise to quite simple expressions of the associated functionals.

3. A new extended functional associated with problem (1.1)

Here we study a "special", degenerate case of the preceding theory, arising from the choice $\mathscr{S}(\cdot) = 0$, i.e., $\mathscr{R}(\cdot) = \mathscr{N}(\cdot)$. This case is named degenerate because it is not included in the range of validity of the theorems demonstrated in the preceding section, nor of those of Tonti (1984) and Auchmuty (1988). If one recalls that the validity of all the previous theory rests upon the positive definiteness of the operator $\mathscr{S}(\cdot)$ (see Eq. (2.9)), one recognises that the choice $\mathscr{S}(\cdot) = 0$ is outside the range of applicability of the theoretical framework illustrated before. Therefore, the validity of all the lemmas and theorems proved in Section 2 must be redemonstrated for this case.

The choice $\mathscr{S}(\cdot)=0$ leads to the writing of the extended operator $\mathscr{M}(\cdot)$ of Eq. (2.1) as

$$\mathscr{M}(\cdot) = \begin{bmatrix} \mathscr{N}(\cdot) + \mathscr{N}_u^{'*}(\cdot) & -\mathscr{N}_u^{'*}(\cdot) \\ -\mathscr{N}(\cdot) & 0 \end{bmatrix} \tag{3.1}$$

and to the writing of the associated functional, here named $G[w]$, as

$$G[u, v] = \langle (u - v), [\mathscr{N}(u) - P] \rangle. \tag{3.2}$$

To prove that functional (3.2) does furnish a variational formulation of problem (1.1), we must prove the same five statements enunciated in the preceding section.

Lemmas 1–3 can be proved simultaneously by writing the expanded form of Eq. (2.2) with operator $\mathscr{M}(\cdot)$ of Eq. (3.1):

$$\mathscr{N}(u) + \mathscr{N}_u^{'*}u - \mathscr{N}_u^{'*}v = P, \tag{3.3}$$

$$-\mathscr{N}(u) = -P. \tag{3.4}$$

Note that Eq. (3.4) is now uncoupled from (3.3), unlike the case with the corresponding more general equations (2.7) and (2.8).

Eq. (3.4) immediately proves Lemma 1; by substituting it into Eq. (3.3) (which implies that problem (1.1) has a solution) one sees that if the kernel of operator $\mathscr{N}_u^{'*}(\cdot)$ is equal to zero (such as, for instance, when $\mathscr{N}_u^{'*}(\cdot)$ is invertible), then Eq. (2.11) holds, which proves Lemma 2. Lemma 3 then follows immediately.

Theorem 1 is proved by taking the first variation of functional (3.2) with respect to the function w:

$$\delta G = \langle (\delta u - \delta v), [\mathscr{N}(u) - P] \rangle + \langle (u - v), \delta\mathscr{N}(u) \rangle \tag{3.5}$$

which vanishes in the solution of problem (1.1), where Eq. (2.11) also holds.

Theorem 2 can here be stated in a much stronger way: Eq. (3.5), the variation δv being arbitrary, shows that functional (3.2) is stationary if and only if Eq. (1.1) is satisfied. Moreover, if the kernel of $\mathscr{N}_u^{'*}(\cdot)$ is equal to zero, at each stationarity point of functional (3.2) Eq. (2.11) holds.

Functional (3.2), nonlinear in the unknown function u, is linear in the auxiliary one v for any fixed function u; its properties follow from those of operator $\mathscr{N}(\cdot)$. Its stationary point corresponds therefore, in general, to a nonoriented saddle point; this also implies that its stationarity is associated with two uncoupled problems in u and v.

Its practical application is much easier than that of the more general expression (2.6), since it does not require any a priori choice of kernel operator, and associated bilinear form, thus leaving total freedom of action. Of course, in the definition of the bilinear form to be used in constructing functional (3.2) one has to guarantee the absence of degeneracy in the sense illustrated, for instance, in Tonti (1984).

From the applicative viewpoint, therefore, we feel that it should be convenient to start by exploring functional (3.2), and to turn to the more sophisticated (2.6) only in order to use the symmetric linear kernel $\mathscr{S}(\cdot)$ as a preconditioning operator of the extended problem. This issue has already been considered, for specific examples, for instance in Carini et al. (1995) and in Carini and Genna (2001a).

4. Illustrative examples

We illustrate the application of the preceding theory by means of two example problems within the context of solid continuum mechanics. Both problems are governed by operators which are nonsymmetric with respect to "classical" bilinear forms and, for this reason, they have not been given, so far, a "classical", nonextended variational formulation.

The first problem is defined by the rate elastic–plastic constitutive equations in the nonassociated case, i.e., when the flow rule which governs the rate plastic strains depends on a potential function different from the yield function. In this case the lack of symmetry is due only to the difference between the gradients of the yield function and the plastic potential. We will make reference to generic, vector valued yield functions and plastic potentials.

The basic equations can be written as follows (see, for instance, Maier, 1969):

$$\dot{\varphi}_\alpha = \dot{b}_\alpha - A_{\alpha\beta}\dot{\lambda}_\beta, \tag{4.1}$$

$$\dot{\lambda}_\alpha \geq 0, \tag{4.2}$$

$$\dot{\varphi}_\alpha \leq 0, \tag{4.3}$$

$$[\dot{\varphi}_\alpha \dot{\lambda}_\alpha] = 0 \quad \text{(no sum on } \alpha), \tag{4.4}$$

where $\varphi_\alpha(\sigma_{ij})$ indicates the components of a generic, vector valued yield function (a superimposed dot here indicates rate quantities; $\alpha, \beta = 1, \ldots, Y$, where Y is the number of the equations defining the yield function and σ_{ij} is the stress tensor); $\dot{\lambda}_\alpha$ indicates the components of the plastic multiplier vector, defined by the following equation:

$$\dot{\varepsilon}^{\mathrm{p}}_{ij} = \frac{\partial \psi_\alpha}{\partial \sigma_{ij}} \dot{\lambda}_\alpha = M_{\alpha ij}\dot{\lambda}_\alpha; \quad \dot{\lambda}_\alpha \geq 0 \quad \alpha = 1, \ldots, Y, \tag{4.5}$$

where $\dot{\varepsilon}^{\mathrm{p}}_{ij}$ is the rate plastic strain and $\psi_\alpha(\sigma_{ij})$ indicates a plastic potential, which governs the flow rule, in general not coinciding with the yield function φ_α but having the same number of components (see Maier, 1969). The vector $\dot{b}_\alpha$ in Eq. (4.1) is defined as

$$\dot{b}_\alpha = \frac{\partial \varphi_\alpha}{\partial \sigma_{ij}} D_{ijhk}\dot{\varepsilon}_{hk} = N_{\alpha ij} D_{ijhk}\dot{\varepsilon}_{hk}, \tag{4.6}$$

where $\dot{\varepsilon}_{hk}$ is the prescribed strain increment and D_{ijhk} is the tensor of the elastic moduli. The tensor $A_{\alpha\beta}$ in Eq. (4.1) is defined as

$$A_{\alpha\beta} = N_{\alpha ij} D_{ijhk} M_{\beta hk} \tag{4.7}$$

and is, in general, nonsymmetric. It becomes symmetric in the case $\varphi_\alpha = \psi_\alpha$, i.e., when $N_{\alpha ij} = M_{\alpha ij}$.

Among several possibilities of expressing problem (4.1)–(4.4) in an operatorial form, we use here, for the sake of clarity, the simplest possible, even if the results which can be obtained are not the most interesting. Several other statements can be given, both to the rate elastic–plastic constitutive equations and to the finite elastic–plastic problem, as well as to the structural elastic–plastic case, using different approaches; to this specific field of application is devoted a forthcoming paper (Carini and Genna, 2001b).

Here we re-cast problem (4.1)–(4.4) in the following way:

$$A_{\alpha\beta} l_\beta^2 - \dot{b}_\alpha - f_\alpha^2 = 0, \tag{4.8}$$

$$[f_\alpha^2 l_\alpha^2] = 0 \quad \text{(no sum on } \alpha), \tag{4.9}$$

where the two unknowns $\dot{\lambda}_\alpha$ and $-\dot{\varphi}_\alpha$ have been replaced by the nonnegative unknowns l_α^2 and f_α^2, respectively. In the following, we will implicitly assume that no sum will be intended for repeated indices within the square brackets.

According to the theory illustrated in Section 3, a variational statement corresponding to problem (4.8)–(4.9) can be obtained by writing explicitly Eq. (3.2), introducing the auxiliary unknowns $\tilde{l}_\alpha$ and $\tilde{f}_\alpha$:

$$G_1(l_\alpha, f_\alpha, \tilde{l}_\alpha, \tilde{f}_\alpha) = (l_\alpha - \tilde{l}_\alpha)(A_{\alpha\beta} l_\beta^2 - \dot{b}_\alpha - f_\alpha^2) + (f_\alpha - \tilde{f}_\alpha)[f_\alpha^2 l_\alpha^2]. \tag{4.10}$$

Before proving that the stationarity conditions of function (4.10) do indeed correspond to the required equations, as given by the general theory, it is interesting to observe that the same theory indicates the possibility of writing a fully equivalent, but different function, associated with problem (4.8)–(4.9), in the following form:

$$G_2(l_\alpha, f_\alpha, \tilde{l}_\alpha, \tilde{f}_\alpha) = (f_\alpha - \tilde{f}_\alpha)(A_{\alpha\beta} l_\beta^2 - \dot{b}_\alpha - f_\alpha^2) + (l_\alpha - \tilde{l}_\alpha)[f_\alpha^2 l_\alpha^2], \tag{4.11}$$

i.e., by inverting the position of the two unknowns in vectors u and v of the general theory relative to that done in writing Eq. (4.10).

We show that the stationarity conditions of function (4.10) do correspond to problem (4.8)–(4.9) by writing the relevant derivatives as follows:

$$\frac{\partial G_1}{\partial l_\alpha} = (A_{\alpha\beta} l_\beta^2 - \dot{b}_\alpha - f_\alpha^2) + (l_\beta - \tilde{l}_\beta)[2A_{\beta\alpha} l_\alpha] + 2[l_\alpha f_\alpha^2 (f_\alpha - \tilde{f}_\alpha)] = 0, \tag{4.12}$$

$$\frac{\partial G_1}{\partial \tilde{l}_\alpha} = -(A_{\alpha\beta} l_\beta^2 - \dot{b}_\alpha - f_\alpha^2) = 0, \tag{4.13}$$

$$\frac{\partial G_1}{\partial f_\alpha} = -2[f_\alpha(l_\alpha - \tilde{l}_\alpha)] + 2[f_\alpha l_\alpha^2(f_\alpha - \tilde{f}_\alpha)] + [f_\alpha^2 l_\alpha^2] = 0, \tag{4.14}$$

$$\frac{\partial G_1}{\partial \tilde{f}_\alpha} = -[f_\alpha^2 l_\alpha^2] = 0. \tag{4.15}$$

Eqs. (4.13) and (4.15) coincide with Eqs. (4.8) and (4.9), respectively, and therefore furnish by themselves the same solution (unique or nonunique, if a solution exists), as that of the original problem.

If Eqs. (4.13) and (4.15) hold, the remaining two equations (4.12) and (4.14) reduce to

$$\frac{\partial G_1}{\partial l_\alpha} = (l_\beta - \tilde{l}_\beta)[2A_{\beta\alpha} l_\alpha] = 0, \tag{4.16}$$

$$\frac{\partial G_1}{\partial f_\alpha} = [2 f_\alpha \tilde{l}_\alpha] = 0. \tag{4.17}$$

Here one immediately observes that the vector $\tilde{f}_\alpha$ is undetermined; also the vector $\tilde{l}_\alpha$ might have undetermined components, depending on the solution of the starting problem (4.8)–(4.9) and on the aspect of the governing matrix $A_{\alpha\beta}$. However, if, for instance, in the solution of problem (4.8)–(4.9) the component $f_\alpha \neq 0$, then Eq. (4.17) implies $\tilde{l}_\alpha = 0$ (and therefore $\tilde{l}_\alpha = l_\alpha$), which also satisfies Eq. (4.16); in any case it is apparent that a full solution with l_α and f_α satisfying Eqs. (4.8)–(4.9) (or (4.13) and (4.15)), and with $\tilde{l}_\alpha = l_\alpha$ and $\tilde{f}_\alpha = f_\alpha$, satisfies all the equations of the extended problem (4.12)–(4.15).

It is important to note that, as always happens within this theory for the case of functional (3.2), the actual equations, defining the original problem, are obtained by making stationary the functional $G[w]$ with respect to the auxiliary unknowns only, and that they are uncoupled from the remaining ones. Therefore, from a practical viewpoint, the doubling of unknowns plays no role, since the computational effort required to compute the unknown functions involves only a number of equations equal to that of the original unknowns.

The second example makes reference to the nonlinear continuum dynamic problem under initial conditions, within the simplifying context of small strains and displacements. This problem has been addressed in a previous paper (Carini and Genna, 1998), by starting from functional (2.6). Here we wish to illustrate the use of the new functional (3.2), to obtain a variational formulation of the Total Potential Energy type, i.e., one in which the only unknown function is the displacement field $u_j(t)$, and all the relevant geometric boundary and initial conditions are satisfied a priori.

The field equations describing this problem, framed in this particular way, can be written, with reference to an orthogonal cartesian reference frame x_i, $i=1,2,3$, in terms of the following operators:

$$\text{Equilibrium operator: } \mathscr{E}\sigma_{ij} = \frac{\partial \sigma_{ij}}{\partial x_i}, \tag{4.18}$$

$$\text{Compatibility operator: } \mathscr{C}u_j = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{4.19}$$

$$\text{Nonlinear constitutive law: } \sigma_{ij} = \Psi_{ij}(\varepsilon_{hk}). \tag{4.20}$$

The field equations describing the continuum nonlinear dynamic problem in a Navier-type formulation are then

$$-\mathscr{E}\Psi_{ij}(\mathscr{C}u_j(t)) + \rho\ddot{u}_j(t) - b_j(t) = 0 \quad \text{in } \Omega \times \mathscr{T}, \tag{4.21}$$

$$n_i\Psi_{ij}(\mathscr{C}u_j(t)) - p_j(t) = 0 \quad \text{in } \Gamma_p \times \mathscr{T}, \tag{4.22}$$

$$u_j(t) = \bar{u}_j(t) \quad \text{in } \Gamma_u \times \mathscr{T}, \tag{4.23}$$

$$u_j(t=0) = u_j^0 \quad \text{in } \Omega, \tag{4.24}$$

$$\dot{u}_j(t=0) = \dot{u}_j^0 \quad \text{in } \Omega. \tag{4.25}$$

Here t indicates time, a superimposed dot means derivative with respect to time, ρ is the mass density, $b_j(t)$ are the body forces, $p_j(t)$ the surface tractions, n_i the outward normals to the exterior surface of the body, whose volume is indicated by Ω and whose external surface Γ is subdivided into a loaded part Γ_p and a constrained part Γ_u. $\mathscr{T}$ indicates the time interval of interest, for simplicity assumed to be defined as $0 \le t \le T$; $\bar{u}_j(t)$ are the prescribed displacements in Γ_u and $u_j^0, \dot{u}_j^0$ are the prescribed initial displacements and velocities.

Eqs. (4.21)–(4.25) can be put into a form of type (1.1) by means of the following definitions:

$$u = \begin{bmatrix} u_j(t) \\ z_j(t) \end{bmatrix} \tag{4.26}$$

(please note that the symbol u with no indices represents only the vector of the unknown functions as in Eq. (1.1)) where

$$z_j(t) = \rho\dot{u}_j(t), \tag{4.27}$$

$$\mathscr{N}(\cdot) = \begin{bmatrix} \mathscr{A}(\cdot) \\ \mathscr{B}(\cdot) \\ \mathscr{D}^0(\cdot) \end{bmatrix}, \qquad P = \begin{bmatrix} f \\ g \\ d^0 \end{bmatrix} \tag{4.28}$$

with

$$\mathscr{A}(\cdot) = \begin{bmatrix} -\mathscr{E}\Psi_{ij}(\mathscr{C}(\cdot)) & \dfrac{\mathrm{d}(\cdot)}{\mathrm{d}t} \\ -\dfrac{\mathrm{d}(\cdot)}{\mathrm{d}t} & \dfrac{1}{\rho} \end{bmatrix} \quad \text{in } \Omega \times \mathscr{T}, \tag{4.29}$$

$$\mathscr{B}(\cdot) = \begin{bmatrix} n_i\Psi_{ij}(\mathscr{C}(\cdot)) & 0 \\ -n_i & 0 \end{bmatrix} \quad \begin{matrix} \text{in } \Gamma_p \times \mathscr{T}, \\ \text{in } \Gamma_u \times \mathscr{T}, \end{matrix} \tag{4.30}$$

$$\mathscr{D}^0(\cdot) = \begin{bmatrix} 0 & \mathscr{I}(\cdot) \\ -\mathscr{I}(\cdot) & 0 \end{bmatrix} \quad \text{in } \Omega,\ t = 0, \tag{4.31}$$

where $\mathscr{I}(\cdot)$ indicates the identity operator $\mathscr{I}(\cdot) = (\cdot)$;

$$f = \begin{bmatrix} b_j(t) \\ 0 \end{bmatrix} \quad \text{in } \Omega \times \mathscr{T}, \tag{4.32}$$

$$g = \begin{bmatrix} p_j(t) \\ -n_i \bar{u}_j \end{bmatrix} \quad \begin{matrix} \text{in } \Gamma_p \times \mathscr{T}, \\ \text{in } \Gamma_u \times \mathscr{T}, \end{matrix} \tag{4.33}$$

$$d^0 = \begin{bmatrix} -\rho \dot{u}_j^0 \\ u_j^0 \end{bmatrix} \quad \text{in } \Omega. \tag{4.34}$$

The construction of functional (3.2) within a Total Potential Energy framework requires the definition of a kinematically admissible auxiliary function v, defined as

$$v = \begin{bmatrix} \tilde{u}_j(t) \\ \tilde{z}_j(t) \end{bmatrix} \tag{4.35}$$

with

$$\tilde{z}_j(t) = \rho \dot{\tilde{u}}_j(t) \tag{4.36}$$

which, in the solution of problem (4.21)–(4.25), equals the function u of Eq. (4.26). The kinematic admissibility here means that the following conditions must be prescribed a priori on function v:

$$\tilde{u}_j(t) = \bar{u}_j(t) \quad \text{in } \Gamma_u \times \mathscr{T}, \tag{4.37}$$

$$\tilde{u}_j(t=0) = u_j^0 \quad \text{in } \Omega, \tag{4.38}$$

$$\tilde{z}_j(t=0) = \dot{u}_j^0 \quad \text{in } \Omega. \tag{4.39}$$

At this point it is possible to apply definition (3.2) to obtain a "classical" variational formulation of problem (4.21)–(4.25). Here we want to use a "classical" bilinear form, of type (1.4); definitions (4.26)–(4.34) suggest the following choice:

$$\langle v, \mathscr{N}(u) \rangle = \int_0^T \int_\Omega v \mathscr{A}(u)\, \mathrm{d}\Omega\, \mathrm{d}t + \int_0^T \int_\Gamma v \mathscr{B}(u)\, \mathrm{d}\Gamma\, \mathrm{d}t + \int_\Omega v \mathscr{D}^0 u\, \mathrm{d}\Omega \tag{4.40}$$

which in this case turns out to be nondegenerate.

In order to write its final form one needs to perform integrations by parts on terms of the type $u_j \rho \ddot{u}_j$, and to use the divergence theorem in a very usual way, such as described, for instance, in Carini and Genna (1998) when dealing with the same problem using a different functional.

The final result is the following:

$$E[u_j,\tilde u_j] = -\int_0^T\int_\Omega [\dot u_j(t)-\dot{\tilde u}_j(t)]\rho\dot u_j(t)\,\mathrm{d}\Omega\,\mathrm{d}t + \int_0^T\int_\Omega [\varepsilon_{ij}(u_l)-\tilde\varepsilon_{ij}(\tilde u_l)]\Psi_{ij}(\varepsilon_{hk})\,\mathrm{d}\Omega\,\mathrm{d}t$$

$$-\int_0^T\int_\Omega [u_j(t)-\tilde u_j(t)]b_j(t)\,\mathrm{d}\Omega\,\mathrm{d}t - \int_0^T\int_{\Gamma_p} [u_j(t)-\tilde u_j(t)]p_j(t)\,\mathrm{d}\Gamma\,\mathrm{d}t$$

$$+\int_\Omega [u_j(T)-\tilde u_j(T)]\rho\dot u_j(T)\,\mathrm{d}\Omega$$

under the constraints

$$u_j(t) = \bar u_j(t) \quad \text{in } \Gamma_u\times\mathscr{T},$$

$$\tilde u_j(t) = \bar u_j(t) \quad \text{in } \Gamma_u\times\mathscr{T},$$

$$u_j(0) = u_j^0 \quad \text{in } \Omega,$$

$$\dot u_j(0) = \dot u_j^0 \quad \text{in } \Omega,$$

$$\tilde u_j(0) = u_j^0 \quad \text{in } \Omega,$$

$$\dot{\tilde u}_j(0) = \dot u_j^0 \quad \text{in } \Omega. \tag{4.41}$$

This result, somehow similar to the corresponding one obtained in Carini and Genna (1998) (their Eq. (6.4)), derives from a functional different from that used in Carini and Genna (1998): here we have used the simple form (3.2), whereas in Carini and Genna (1998) use has been made of the general form (2.6), together with a very specific choice of the kernel function $\mathscr{S}(\cdot)$; as a consequence, beside several differences in the exterior aspect of the two results, here functional (4.41) is linear in $\tilde u_j$ (the other was quadratic also in the auxiliary unknown variable), and here no a priori constraints must be prescribed on unknown functions at time T, as was required by the formulation presented in Carini and Genna (1998).

If one computes the stationarity conditions of functional (4.41), one recovers Eqs. (4.21) and (4.22), written for both unknown functions $u_j(t)$ and $\tilde u_j(t)$.

5. Extension of the results of Section 3 and link with the adjoint operator method

The new symmetrisation technique described in Section 3 is amenable to a generalisation which, even if not as strictly linked with the starting problem (1.1) as problem (2.1)–(2.2) and functional (2.6), may prove interesting in practical applications, and which allows to establish a link with the original adjoint operator method of Morse and Feshbach (1953). Such generalisation is fully outside the theoretical framework of Tonti (1984) and Auchmuty (1988).

An extended problem connected with problem (1.1), and different from both (2.1)–(2.2) and (3.1), can be constructed by considering a nonlinear but potential operator $\mathscr{P}(\cdot)$. By definition, if an operator $\mathscr{P}(\cdot)$ is potential, it can be seen as the gradient of a functional $Z[\,\cdot\,]$. A possible way to construct such an operator, starting from a given nonlinear, nonpotential operator $\mathscr{A}(\cdot)$, is the following:

$$\mathscr{P}(\cdot) = \alpha(\mathscr{A}(\cdot) + \mathscr{A}_u^{'*}(\cdot)), \tag{5.1}$$

α being an arbitrary constant. By construction, such an operator $\mathscr{P}(\cdot)$ is the gradient of the following functional $Z[\,\cdot\,]$:

$$Z[\,\cdot\,] = \alpha\langle \mathscr{A}(\cdot), \cdot\rangle. \tag{5.2}$$

Of course, operator $\mathscr{P}(\cdot)$ can be chosen in any other way which guarantees symmetry with respect to a bilinear form.

A new extended problem, associated with problem (1.1), of the form (2.2) can be constructed by means of the following definitions:

$$\mathscr{M}(\cdot) = \begin{bmatrix} \mathscr{P}(\cdot) & -\mathscr{N}_u^{'*}(\cdot) \\ -\mathscr{N}(\cdot) & 0 \end{bmatrix}, \qquad w = \begin{bmatrix} u \\ v \end{bmatrix}, \qquad Q = \begin{bmatrix} \tilde{P} \\ -P \end{bmatrix}, \tag{5.3}$$

where v is the customary auxiliary unknown function and $\tilde{P}$ is an arbitrary known function.

The extended problem defined by Eqs. (2.2) and (5.3) explicitly contains problem (1.1) (its second equation), and is symmetric by construction. Therefore it is possible to write its corresponding functional as follows:

$$H[w] = Z[u] - \langle u, \tilde{P}\rangle - \langle v, [\mathscr{N}(u) - P]\rangle. \tag{5.4}$$

It is immediate to verify that the stationarity conditions of functional $H[w]$ of Eq. (5.4) correspond to the solution of problem (1.1). In this case, however, the auxiliary unknown function v has no physical meaning, in general. Also, the stationarity conditions of functional (5.4) become a rigorous $\min_u$, $\max_v$ problem when the functional $Z[u]$ is convex.

Functional (5.4) is the general expression of a class of functionals not included into Tonti's and Auchmuty's framework. Although the introduction of a physically meaningless new unknown function and known term makes this development more a mathematical device for symmetrisation, than a tool for improving the understanding of the original problem (1.1), its derivation is useful to discover the links between the existing theories and the new one proposed in Section 3. In fact, functional (3.2) is recovered by starting from functional (5.4) and setting

$$Z[u] = \langle \mathscr{N}(u), u\rangle; \;\; \tilde{P} = P \tag{5.5}$$

which corresponds to making precisely choice (5.1), with $\mathscr{A}(\cdot) = \mathscr{N}(\cdot)$ and $\alpha = 1$, for the nonlinear potential operator $\mathscr{P}(\cdot)$. Instead, by setting $Z[u] = 0$ in Eq. (5.4) it is easy to recover the adjoint operator method of Morse and Feshbach (1953).

In summary, the general expression (5.4) defines a class of extended functionals associated with problem (1.1) which includes interesting special cases, and which is not included in the parallel general framework of Tonti and Auchmuty, whose implications have been discussed in Section 2 of the present paper. Our feeling is that all the special cases in which it is impossible to understand the physical meaning of the auxiliary unknown function should lead to formulations with unneeded extra burden, and should not have, therefore, useful practical applications. On the other side, a careful choice of both the potential operator $\mathscr{P}(\cdot)$ and the arbitrary known term $\tilde{P}$ of Eq. (5.3) might lead to extended formulations in which the set of auxiliary unknowns acquires interesting meaning, thus allowing to gain further insight into the original problem features.

6. Conclusions

This work describes developments of methods for the symmetrisation of nonpotential operators which allow (i) better understanding of the features of established extended variational formulations, and (ii) the development of new, very simple extended functionals associated with any nonlinear problem, governed by a generic, nonpotential operator. The theory allows to write explicitly the extended problem associated with all functionals, which may be of help in the formal operations involved in the symmetrisation of the operator.

The main disadvantage of these techniques is the doubling of the unknown variables; however, owing to the well-defined meaning of the auxiliary set of unknowns in the solution, given by Eq. (2.11), it is often possible to deal, during the actual solution process, with the real unknown function only, thus reducing the computational burden to the order of magnitude of that required by conventional strategies, not based on variational formulations.

This theory should have several possible applications. Our interest is in continuum mechanics problems; work under publication (Carini and Genna, 2001a,b) concerns the use of functionals of the form (2.6) or (3.2) both in the case of nonassociated plasticity and for the case of structural dynamics, with the aim of obtaining a new time-integration method; some very preliminary results in this last field have already been published (Carini et al., 1996). Another field of research based on this theory might be that of bounding the dynamic linear/nonlinear behaviour of composite materials.

Other useful bounds, based on the variational formulations discussed in this paper, might be obtained for the viscoelastic behaviour of both homogeneous and composite materials. Results of this type, based on the usual adjoint operator method, have been obtained by Robinson and Yuen (1986).

Even if the theories of Tonti and Auchmuty have been known for some time, to the authors' knowledge their practical application, at least in the engineering field, is rather unexplored. Beside the quoted work by Ortiz and Carini and coworkers, and some further contributions by Alliney and Tralli (1986) and Amadio and Rajgelj (1992), we know of no further applications of this specific theory. For this reason we feel that any work devoted to the clarification of the basic theory, and to the exemplification of its practical application — often rather cumbersome, specially if starting from the initial, very mathematical form of the theory itself — should lead to an improvement of the understanding of the physics of the studied problems.

Finally, we wish to recall the possibility of further refining all these variational formulations by transforming saddle point functionals into convex functionals by means of the preconditioning method proposed by Bramble and Pasciak (1988).

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