Boundary elements for non-linear elasticity

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Abstract. A boundary element technique is employed to analyze perturbations in terms of small elastic deformations superimposed upon a given homogeneous strain. Plane strain deformations are considered of an incompressible hyperelastic solid within the elliptic range. The numerical method is based on a recently developed Green's function and boundary integral equations for non-linear incremental elastic deformations. The proposed approach is shown to yield bifurcation loads. In particular strain localizations are analyzed as a special case of instability, and they are found to occur in the elliptic range as induced by perturbations.

Introduction

Large strain effects are of great importance in a number of engineering problems. In particular, pre-stress influences the behaviour of microeletromechanical system, geological formations, biological tissues, and structural elements, such as seismic insulators and rubber bearings.

Referring to incompressible materials, Biot [1] has shown that elastic first-order incremental plane strain deformations superimposed upon a given homogeneous strain are governed by two incremental shear moduli, functions of the current stretch. A Green's function and a boundary integral formulation have been obtained in [2], assuming the Biot's constitutive framework.

Both Green's function and integral formulation are adopted for constructing a boundary element technique (BEM) suitable to analyze incremental problems of non-linear elasticity. In the present article a general numerical scheme is formulated to handle generic boundary value problems with prescribed nominal tractions or displacement boundary conditions.

Constitutive Assumptions

We refer to the constitutive framework introduced by [1], which embraces a broad class of material behaviours including hyper and hypo elasticity, and the loading branch of elastoplasticity. In a Lagrangean formulation of field equations, with the current state taken as reference, the constitutive equations for the nominal stress tensor rate $\dot{t}_{ij}$ can be written in the form

$$\dot{t}_{11} = (2\mu_* - \frac{\sigma}{2} - p)\, v_{1,1} + \dot{p}\,, \qquad \dot{t}_{22} = (2\mu_* - \frac{\sigma}{2} - p)\, v_{2,2} + \dot{p}\,,$$

$$\dot{t}_{12} = (\mu + \frac{\sigma}{2})\, v_{2,1} + (\mu - p)\, v_{1,2}\,, \qquad \dot{t}_{21} = (\mu - \frac{\sigma}{2})\, v_{1,2} + (\mu - p)\, v_{2,1}, \qquad (1)$$

where μ and μ_* are the incremental shear moduli corresponding to, and at 45° to, the Eulerian principal axes, p is the in-plane hydrostatic stress, $\sigma = \sigma_1 - \sigma_2$ the deviatoric stress, and σ_1 and σ_2 the principal components of the Cauchy stress.

The Green’s Function and Boundary Integral Equations

With reference to an infinite medium characterized by constitutive eq (1), the

Green’s function set $\{ v_i^g, \dot{\pi}^g \}$ is [2]

$$v(r,\theta) = \log \hat{r} \int_0^{\pi} \frac{\sin[\alpha + (1-m)\pi/2] \ \cos[\alpha + (2-g)\pi/2]}{2\pi^2 \mu(1+k)\,\Lambda(\alpha)} d\alpha$$

$$+ \int_0^{\pi} \frac{\sin[\alpha + \vartheta + (1-m)\pi/2]\cos[\alpha + \vartheta + (2-g)\pi/2]\log|\cos\alpha|}{2\pi^2 \mu(1+k)\,\Lambda(\alpha+\vartheta)} d\alpha \qquad (2)$$

and

$$\dot{\pi}^g(r,\theta) = \frac{\cos[\theta - (g-1)]}{2\pi r}$$

$$+ \ P.V. \int_0^{\pi} \frac{\sin^2[\alpha + \vartheta + (g-1)\pi/2]\cos[\alpha + \vartheta + (g-1)\pi/2]}{2\pi^2 r(1+k)\Lambda(\alpha)\cos\alpha} \Gamma(\alpha) d\alpha \qquad (3)$$

where P.V. stands for Cauchy principal value, r and θ are the generic polar coordinates singling out the generic point with respect to the position $\mathbf{y}$ of the concentrated force, indices i and g range between 1 and 2, and

$$\Lambda(\alpha) = (\cos^2\alpha - \gamma_1 \sin^2\alpha)\,(\cos^2\alpha - \gamma_2 \sin^2\alpha)$$

$$\Gamma(\alpha) = 2(\mu_* / \mu - 1)(2\cos 2\alpha - 1) - k\,,$$

where $\gamma_{1,2}$ define the regime of the problem (elliptic, parabolic and hyperbolic), and the non dimensional parameter $k = \sigma/2\mu$ characterises the pre-stress.

Boundary Element Formulation

We make reference to mixed boundary value problems in which velocities v_i and incremental nominal tractions $\dot{\tau}$ are prescribed functions defined on separate portions ∂B_v and ∂B_τ, respectively, of the boundary ∂B of a solid B, currently in a state of homogeneous, finite deformation.

In this context, an integral representation exist relating the velocity (at points interior, exterior or at the boundary of the body) to the boundary values of nominal traction rates and velocities

$$C_i^g v_i(\mathbf{y}) = \int_{\partial B} v_j^g(\mathbf{x},\mathbf{y})\,\dot{t}_{ij} n_i \, dl_x - \int_{\partial B} v_j \, \dot{t}_{ij}^g(\mathbf{x},\mathbf{y}) n_i \, dl_x \qquad (4)$$

where

$$C_i^g = \lim_{\varepsilon\to 0} \int_{\partial C\varepsilon} \dot{\tau}_i^g(\mathbf{x},\mathbf{y}) n_i \, dl_x$$

is the so-called **C**–matrix, depending on the material parameters, state of pre-stress and the geometry in the neighborhood of the point **y**, and K_{ijkl} is the fourth–order incremental constitutive tensor relating velocity gradient $v_{i,j}$ to incremental nominal stress $\dot{t}_{ij}$.

The Green's function (2)-(3) and the integral equation (4) are the starting point to derive the *collocation boundary element method*. To this purpose, the boundary ∂B is divided into linear elements and velocities and nominal tractions are discretized adopting linear shape functions.

Collocating the discretize form of eq (4) at each node along the two directions x_1 and x_2 yields an algebraic system that can be written in a form

$$\mathbf{H}\ \hat{\mathbf{v}} = \mathbf{G}\ \hat{\boldsymbol{\tau}} \qquad (5)$$

where vectors $\hat{\mathbf{v}}$ and $\hat{\boldsymbol{\tau}}$ collect the nodal boundary value of velocity and incremental nominal traction. Solution of system (5), after re-arrangement of data and unknowns, gives the nodal velocities on ∂B$_\tau$, and the nominal traction rates on ∂B$_v$.

In a second step, after system (5) has been solved, the internal fields of velocity, pressure rate and nominal stress rate can be computed.

Numerical examples

The first example concerns the so-called `van Hove condition', where the solid is subjected to prescribed displacements over the entire boundary and the current deformation (and stress) is homogeneous. In these conditions, starting from an unloaded configuration, shear bands occur as the first possible bifurcation.

We analyze this situation for the square elastic block shown in Figure 1, characterised by the ratio $\mu_*/\mu = 0.25$ (corresponding to the elliptic complex regime) and homogeneously deformed in a state of uniaxial tension and compression. All displacements are prescribed on the boundary, so that the solution is known unless an arbitrary value of homogeneous pressure.

A perturbation is given by prescribing the triangular distribution of velocity sketched in Figure 1.

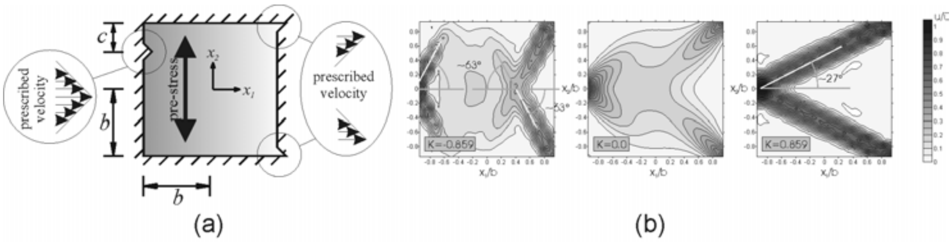


Fig. 1. Van Hove condition: loading geometry (a) and level sets of modulus of velocity (b) at different values of pre-stress *k*.

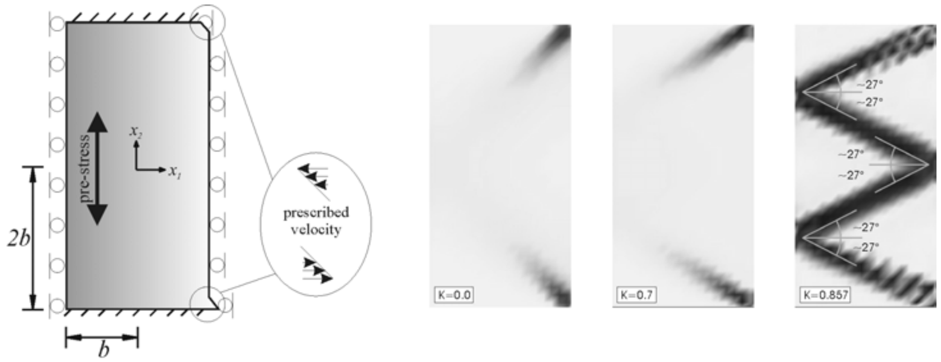
The level sets of the modulus of the velocity for three different values of pre-stress $k = \{-0.859, 0, 0.859\}$, is reported in Fig. 1b relative to c/b=1/2 in Fig. 1a. The values ±0.859 are close to the boundary of loss of ellipticity (±0.866), where shear bands become possible, inclined at an angle $\eta = 27.367°$, with respect to the direction of tensile stress.

This aproach may provide an explanation of the fact that shear banding is a preferred instability when compared to other diffuse bifurcations, possible at loss of ellipticity under van Hove conditions.

The van Hove setting is very peculiar and provides the maximum possible `confinement' to a material sample. A relaxation of this severe configuration was proposed by [4] and will be called `weak van Hove' conditions in the following.

It can be seen that shear banding is not evident until $k = 0.7$ but it appears clearly for $k = 0.857$, which is close to the boundary of ellipticity. ‘Reflection’ of shear bands at the boundary emerges as a peculiar deformation pattern.

Similar deformation mechanisms have been also observed in different contexts (porous plastic materials; dynamics of visco-plastic solids), and may explain pattern formation in biological system or geological structures. The localization of deformation may also suggest possible technological applications.



(a) (b)

Fig. 2. Weak van Hove condition: loading geometry (a) and level sets of velocity modulus (b) at different values of pre-stress k

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