State Observers and the Kalman filter

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Feedback System

State variable feedback system:
\[
\dot{x} = Ax + Bu
\]
\[
y = Cx
\]
Control feedback law: \( u = -Kx \)
\[
u = (r - Kx)
\]
The closed loop system is: \( \dot{x} = (A - BK)x + Br \)

The control law requires that all state variables are known
Feedback System

-Reference signal

**Feedback System Diagram:**

- **r**: Reference signal
- **u**: Input
- **B**: Input matrix
- **A**: State transition matrix
- **K**: State feedback gain matrix
- **x**: States
- **y**: Measurements
- **C**: Measurement matrix

Diagram components:

- **r** to **u** (input)
- **B** to **x** (input matrix to states)
- **x** (states)
- **A** (state transition matrix)
- **K** (state feedback gain matrix)
- **C** (measurement matrix)
- **y** (measurements)
Feedback System

Availability of State Variables

• Some of the state variables may not be measurable.

• The cost of many transducers may be too high.
State Observer

Therefore,
- we discard the assumption that all the state variables are measurable.

- we need something that gives an estimate of the state variables $x$ for feedback, from the measurement of the controlled output $y$ and from the input $u$.

This something is termed as the **State Observer**.
State Observer

• All observers use some form of mathematical model to produce an estimate of the actual state vector \( x \).

• \( \hat{x} \) is an estimate of \( x \); they are not equal.
• The observer dynamics will never be exactly equal to the system dynamics.

• It is assumed that the output measurements are noise free.
State Observer

The system must be *observable*: at any time $t_0$, the system state $x(t_0)$ can be exactly determined from observation of the output $y(t)$ over a finite time interval.

The system is completely observable if the $n \times n$ *observability matrix* $N$:

$$N = [C^T \ A^T C^T \ \ldots \ (A^T)^{n-1} C^T]$$

is *full rank*, i.e. is non-singular having a non-zero determinant.
Types:

- **Full-order** state observer
  - Estimates all of the state variables

- **Reduced order** state observer
  - Estimates some of the state variables
Full-order State Observer

\[ y = Cx \]

\( x \) - actual state variables

\( \hat{x} \) - estimated state

\( y \) - actual output
Luenberger Observer (1964)

The open-loop arrangement in the full-order state observer means that $x$ and $\hat{x}$ will gradually diverge.

If we estimate an output vector $\hat{y}$, we can use a closed-loo system to minimize the difference $y - \hat{y}$.
The Luenberger full order state observer

Full-order State Observer
Luenberger Observer (1964)

Feedback on the error between real measurement and estimated one
Full-order State Observer
Luenberger Observer (1964)

If the system of the Luenberger observer is defined by
\[ \dot{x} = Ax + Bu \]
\[ y = Cx \] (1)

Assume that the estimate \( \hat{x} \) of the state vector is:
\[ \dot{\hat{x}} = A\hat{x} + Bu + K_e(y - C\hat{x}) \]

Where \( K_e \) is the observer gain matrix.

Therefore, the equation of the full order state observer will be
\[ \dot{\hat{x}} = (A - K_eC)\hat{x} + Bu + K_e y \] (2)

Defining \( x - \hat{x} \) as the error vector \( e \), and subtracting (2) from the first equation in (1), we obtain:
\[ \dot{e} = (A - K_eC)e \]
Full-order State Observer
Luenberger Observer (1964)

Since \( \dot{e} = (A - K_e C)e \)

The *dynamic behaviour of the error vector depends upon the eigenvalues of* \( (A - K_e C) \).

As with any measurement system, these eigenvalues should allow the observer transient response to be more rapid than the system itself (typically a factor of 5), unless a filtering effect is required.
The problem of observer design is essentially the same as the regulator *pole placement problem*, and similar techniques may be used.

The following techniques may be used.

- *Direct comparison method*
- *Observable canonical form method*
- *Ackermann’s formula*
Full-order State Observer

Solving the eigenvalues of (A-KeC)

- Direct comparison method

If the *desired* locations of the closed-loop poles (eigenvalues) of the observer are:

\[ s = \mu_1, s = \mu_2, \ldots, s = \mu_n \]

Then the *characteristic polynomial* of (A-KeC) is:

\[
|sI - A + KeC| = (s - \mu_1)(s - \mu_2) \ldots (s - \mu_n) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1s + \alpha_0
\]
Observable canonical form method

For a generalized transfer function shown above, the observable canonical form of the state equation may be written as

\[
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{x}_2 \\
    \vdots \\
    \dot{x}_n
\end{bmatrix} =
\begin{bmatrix}
    0 & 0 & \ldots & 0 & -a_0 \\
    1 & 0 & \ldots & 0 & -a_1 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \ldots & 1 & -a_{n-1}
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_n
\end{bmatrix} +
\begin{bmatrix}
    b_0 \\
    b_1 \\
    \vdots \\
    b_{n-1}
\end{bmatrix} u
\]

\[
\begin{bmatrix}
    \tilde{A} \\
    \tilde{B}
\end{bmatrix} =
\begin{bmatrix}
    Q^{-1} A Q \\
    Q^{-1} B
\end{bmatrix}
\]

\[
\tilde{C} = C Q
\]
**Full-order State Observer**

Solving the eigenvalues of \((A-K_eC)\)

\(Q\) is the *transformation matrix* that transforms the system state equation into the observable canonical form:

\[ Q = (WN^T)^{-1} \]

\(N\) is the *observability* matrix and \(W\) is defined as:

\[
W = \begin{bmatrix}
a_1 & a_2 & \ldots & a_{n-1} & 1 \\
a_2 & a_3 & \ldots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n-1} & 1 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0
\end{bmatrix}
\]

and contains coefficients of the generalized transfer function \(G(s)=Y(s)/U(s)\). If the equation is in the observable canonical form then \(Q = I\).

The value of the *observer gain* matrix \(K_e\) may be written as

\[
K_e = Q \begin{bmatrix} \alpha_0 - a_0 \\ \alpha_1 - a_1 \\ \vdots \\ \alpha_{n-1} - a_{n-1} \end{bmatrix}
\]
Full-order State Observer
Solving the eigenvalues of \((A-K_eC)\)

• Ackermann’s Formula

- only applicable to systems where \(u(t)\) and \(y(t)\) are scalar quantities.

The observer gain matrix may be calculated as follows

\[
K_e = \phi(A)N^{-1}[\begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \end{bmatrix}]^T
\]

or alternatively

\[
K_e = \phi(A) \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix}^{-1} \begin{bmatrix} 0 \\
0 \\
\vdots \\
1
\end{bmatrix}
\]

where \(\phi(A)\) is defined as

\[
\phi(A) = A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_1A + \alpha_0I
\]
Full-order State Observer

Effect on a closed-loop system

Closed-loop control system with full order state feedback

The equations of this closed-loop system are

\[ \dot{x} = Ax + Bu \]
\[ y = Cx \]
Full-order State Observer
Effect on a closed-loop system

The equations of this closed-loop system are
\[
\dot{x} = Ax + Bu \\
y = Cx
\]
Control is implemented using observed state variables. \( u = -K\dot{x} \)

Error equation, \( e(t) = x(t) - \dot{x}(t) \), then \( \dot{x}(t) = x(t) - e(t) \)

Combining above equations: \( \dot{x} = Ax - BK(x - e) \)
\[
= (A - BK)x + BKe
\]
and remembering that \( \dot{e} = (A - K_eC)e \)
the closed loop dynamic of the observer is:
\[
\begin{bmatrix}
\dot{x} \\
\dot{e}
\end{bmatrix} =
\begin{bmatrix}
A - BK & BK \\
0 & A - K_eC
\end{bmatrix}
\begin{bmatrix}
x \\
e
\end{bmatrix}
\]

The system characteristic equation is \( |s|A + BK||s|I - A + K_eC| = 0 \)

*The system characteristic equation shows that the desired closed-loop poles for the control system are not changed by the introduction of the state observer.*
Reduced-order State Observer

A reduced-order state observer does not measure all the state variables.

If the state vector is of nth order and the measured output vector is of mth order, then it is only necessary to design an (n - m)th order state observer.

Consider the case of the measurement of a single state variable $x_1(t)$. The output equation is therefore

$$y = x_1 = Cx = [1 \ 0 \ldots \ 0]x$$

Partition the state vector

$$x = \begin{bmatrix} x_1 \\ x_e \end{bmatrix}$$

where $x_e$ are the state variables to be observed.
Reduced-order State Observer

Partition the state equations

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_c
\end{bmatrix}
= \begin{bmatrix}
a_{11} & A_{1e} \\
A_{c1} & A_{cc}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
+ \begin{bmatrix}
b_1 \\
B_c
\end{bmatrix} u
\]

If the desired eigenvalues for the reduced-order observer are

\[s = \mu_1e, s = \mu_2e, \ldots, s = \mu_{(n-1)e}\]

Then it can be shown that the characteristic equation for the reduced-order observer is

\[|sI - A_{ee} + K_e A_{1e}| = (s - \mu_1e)(s - \mu_2e)\ldots(s - \mu_{(n-1)e})\]

\[= s^{n-1} + \alpha_{(n-2)e}s^{n-2} + \cdots + \alpha_{1e}s + \alpha_{0e}\]

In the above equation $A_{ee}$ replaces $A$ and $A_{1e}$ replaces $C$ in the full-order observer.

The reduced-order observer gain matrix $K_e$ can be obtained using methods discussed earlier in case of full-order observer.
State estimation is the process of extracting a best estimate of a variable from a number of measurements that contain noise.

The classical problem of obtaining a best estimate of a signal by combining two noisy continuous measurements of the same signal was first solved by Weiner (1949).

His solution required that both the signal and noise be modelled as random process with known statistical properties.

Kalman and Bucy (1961) extended this work by designing a state estimation process based upon an optimal minimum variance filter, generally referred to as a Kalman filter.
In the design of state observers, it was assumed that the measurements \( y = Cx \) were noise free; however, this is not usually the case: also measurements are corrupted by noise. The observed state vector \( \hat{x} \) may also be contaminated with noise. Hence, a filter is required to remove the effect of noise. The Kalman filter is used for this purpose: noisy data in - less noisy data out; but delay is the price for filtering; need of the process model for describing how states change over time.
The Kalman Filter
Introduction

KF operates by:
• Predicting the new state and its uncertainty;
• Correcting with the new measurements.
The Kalman Filter
Single Variable Estimation Problem

Let $A_x$ be a measurement of a parameter $x$ and let its variance $P_a$ be given by

$$P_a = E\left\{ (A_x - \bar{A}_x)^2 \right\}$$

where $\bar{A}_x$ is the mean and $E\{ \}$ is the expected value.

Let $B_x$ be a measurement from another system of the same parameter and the variance $P_b$ is

$$P_b = E\left\{ (B_x - \bar{B}_x)^2 \right\}$$

Assume that $x$ can be expressed by the parametric relationship

$$x = A_xK + B_x(1 - K) \tag{1}$$

where $K$ is any weighting factor between 0 and 1.

The problem is to derive a value of $K$ which gives an optimal combination of $A_x$ and $B_x$ and hence the best estimate of measured variable $x$, which is given the symbol $\hat{x}$. 

The Kalman Filter
Single Variable Estimation Problem

If \( P \) is the variance of the weighted mean,
\[
P = E\{(x - \bar{x})^2\}
\]  

(2)

The optimal value of \( K \) is the one that yields the minimum variance, i.e.
\[
\frac{dP}{dK} = 0
\]

Substitution of equation (1) into (2) gives
\[
P = K^2 P_A + (1 - K)^2 P_B
\]

Hence \( K \) is given by
\[
\frac{d}{dK} \left\{ K^2 P_A + (1 - K)^2 P_B \right\} = 0
\]

From which
\[
K = \frac{P_B}{P_A + P_B}
\]  

(3)
Substitution of equation (3) into (1) provides

\[ \hat{x} = A_x - \left( \frac{P_A}{P_A + P_B} \right)(A_x - B_x) \]

or

\[ \hat{x} = A_x - K(A_x - B_x) \]

\( K \) is the Kalman gain and the total error variance expected is

\[ P = P_A - K(P_A - P_B) \]

so that

\[ \hat{x} = x + P_A - K(P_A - P_B) \quad (4) \]

Equation (4) is illustrated in the next Figure.
The Kalman Filter
Single Variable Estimation Problem

\[ \hat{x} = x + P_A - K(P_A - P_B) \]

Fig. Integration of two measurement systems to obtain optimal estimate.
The Kalman Filter
Model and initialization

Process and measurement model:
\[
\begin{align*}
\{ x_{k+1} &= Ax_k + Bu_k + w_k \\
 z_k &= Cx_k + Du_k + v_k \\
\end{align*}
\]

- \( w_k \in \mathbb{R}^n \) is the process white noise
  \( w_k \sim N(0, R) \) with:
  \[
  \begin{align*}
  E[w_k] &= 0 \\
  E[w_k w_j^T] &= \begin{cases} 
  0, & k \neq j \\
  R, & k = j 
  \end{cases} 
  \end{align*}
  \]

- \( v_k \in \mathbb{R}^p \) is the measurement white noise
  \( v_k \sim N(0, Q) \) with:
  \[
  \begin{align*}
  E[v_k] &= 0 \\
  E[v_k v_j^T] &= \begin{cases} 
  0, & k \neq j \\
  Q, & k = j 
  \end{cases} 
  \end{align*}
  \]

- \( x_0 \in \mathbb{R}^n \) is the initial state white noise
  \( x_0 \sim N(\bar{x}_0, P_0) \) with:
  \[
  \begin{align*}
  E[x_0] &= \bar{x}_0 \\
  E[(x - \bar{x}_0)(x - \bar{x}_0)^T] &= P_0 \geq 0 \\
  E[w_k x_0^T] &= E[v_k x_0^T] = 0, \quad \forall \ k \in \mathbb{Z}
  \end{align*}
  \]
The Kalman Filter
Prediction-Correction

Time Update (prediction)
Future state
\[ \hat{x}_k^- = A\hat{x}_{k-1}^+ \]
Error covariance
\[ P_k^- = A P_{k-1}^+ A^T + Q \]

Measurement Update (correction)
Gain
\[ \bar{K}_k = P_k^- C^T [CP_k^- C^T + R]^{-1} \]
Update state estimate
\[ \hat{x}_k^+ = \hat{x}_k^- + \bar{K}_k [z_k - C \hat{x}_k^-] \]
Update covariance
\[ P_k^+ = [I - \bar{K}_k C] P_k^- \]

Gain in the measurement space
transition
uncertainty
Actual measurement
Predicted measurement
Consider a plant that is subject to a Gaussian sequence of disturbances \( w(kT) \) with disturbance transition matrix \( C_u(T) \). Measurements \( z(k+1)T \) contain a Gaussian noise sequence \( v(k+1)T \) as shown in the following Figure.

State time evolution
\[
\begin{align*}
    x_{k+1} &= Ax_k + Bu_k + w_k \\
    z_k &= Cx_k + Du_k + v_k
\end{align*}
\]

Measurements vector
The Kalman Filter
Multivariable Estimation Problem

The general form of the Kalman filter usually contains a discrete model of the system together with a set of *recursive equations that continuously update the Kalman gain matrix K and the system covariance matrix P*. The state estimate:

\[ \hat{x}(k+1|k+1) \]

Is obtained by calculating the predicted state

\[ \hat{x}(k+1|k) \]

from:

\[ \hat{x}(k+1|k)T = A(T)\hat{x}(k/k)T + B(T)u(kT) \]  \hspace{1cm} (1)

And the determining the estimated state at time \((k+1)T\) using:

\[ \hat{x}(k+1/k+1)T = \hat{x}(k+1|k)T + K(k+1)(z(k+1)T - C(T)\hat{x}(k+1/k)T) \]  \hspace{1cm} (2)

*The term \((k+1/k)\) means data at time \(k+1\) based on information available at time \(k\).*
The Kalman Filter
Multivariable Estimation Problem

The vector measurement is given by:

\[ z(k + 1)^T = C(T)x(k + 1)^T + v(k + 1)^T \]  \hfill (3)

The \textit{Kalman gain} matrix \( K \) is obtained from a set of recursive equations that commence from some initial covariance matrix \( P(k|k) \):

\[ P(k + 1/k) = A(T)P(k/k)A^T(T) + C_d(T)QC_d^T(T) \]  \hfill (4)

\[ K(k + 1) = P(k + 1/k)C^T(T)\{C(T)P(k + 1/k)C^T(T) + R\}^{-1} \]  \hfill (5)

\[ P(k + 1/k + 1) = \{I - K(k + 1)C(T)\}P(k + 1/k) \]  \hfill (6)

where:

- \( z(k+1)^T \) is the measurement vector;
- \( C(T) \) is the measurement matrix;
- \( v(k+1)^T \) is a Gaussian noise sequence;
- \( C_d(T) \) is the disturbance transition matrix;
- \( Q \) is the disturbance noise covariance matrix;
- \( R \) is the measurement noise covariance matrix.
The recursive process continues by substituting the covariance matrix \( P(k + 1/k + 1) \) computed in equation (6) back into (4) as \( P(k/k) \) until \( K(k+1) \) settles to a steady value.

\[
\dot{x}(k+1/k)T = A(T)\dot{x}(k/k)T + B(T)u(kT)
\]
\[
P(k + 1/k) = A(T)P(k/k)A^T(T) + C_d(T)QC_d^T(T)
\]
\[
K(k + 1) = P(k + 1/k)C^T(T)\{C(T)P(k + 1/k)C^T(T) + R\}^{-1}
\]
\[
\dot{x}(k + 1/k + 1)T = \dot{x}(k + 1/k)T + K(k + 1)\{z(k + 1)T - C(T)\dot{x}(k + 1/k)T\}
\]
\[
P(k + 1/k + 1) = \{I - K(k+1)C(T)\}P(k + 1/k)
\]
Reference

• Burns R. S., Advanced Control Engineering, 2001, Butterworth-Heinemann, Jordan Hill, Oxford OX2 8DP.