

# Stability of strain gradient plastic materials

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## Abstract

A formulation of Fleck and Willis (2009a,b) for strain-gradient plasticity has been adapted to provide possible descriptions for materials that initially strain-harden but eventually soften. In the absence of gradient terms, such material is unstable for any wavelength and subject to localization in the softening regime. Gradient terms do not mitigate the basic (infinite-wavelength) material instability but they do inhibit the development of short-wavelength disturbances; they prevent localization but still may permit the development of narrow shear bands. In this work, the basic stability problem is studied via consideration of a small, generally time dependent, perturbation of an initially uniform state of deformation. The linearized problem for the perturbation is formulated for the general case of rate-dependent gradient plasticity but special attention is paid to the rate-independent limit. An interesting feature is that a qualitative difference is found between the effects of “energetic” and “dissipative” strain-gradient terms in this limit: energetic gradient terms permit the unbounded growth of any disturbance with wavelength larger than a critical value, whereas a disturbance of any finite wavelength in a medium with dissipative gradient terms can become unbounded only when the yield strength tends to zero.

*Keywords:* Strain-gradient plasticity, shear bands, non-locality, viscosity.

## 1 Introduction

It is well-known that a classic rate-independent associative plastic material becomes unstable within the softening regime (Rice, 1976): the incremental equations governing the quasi-static problem are no longer elliptic while in dynamic conditions at least one wave-speed becomes imaginary. Under this condition any boundary value problem becomes ill-posed

and the deformation localizes in a zero-thickness region (Bažant and Belytschko, 1985). For these reasons, any numerical analysis produces results that depend on the meshing procedure; in particular the width of the predicted localized zone is defined by the spacing of the discretization.

Needleman (1988) and Loret and Prévost (1991; 1993) showed that this problem can be resolved by admitting a viscosity parameter in the constitutive classic model (without the introduction of any internal characteristic length). The localization zone width then corresponds to the size of whatever imperfection was introduced in a static problem while it depends on the amount of viscosity in the dynamic problem (due to the consequent introduction of a non-microstructural internal length-scale).

Chambon et al. (1998) considered the localization within a one-dimensional problem for an elasto-plastic material characterized by non-local elasticity. They found that the introduction of a characteristic length does not lead automatically to the uniqueness of solution within the softening regime, but only a passage from infinite to a finite number of solutions.

Sluys et al. (1993) analyzed the in-plane wave propagation problem, considering a yield function dependent on the second gradient of the plastic strain. They showed that the introduction of the characteristic length leads to a localized region with finite width and the dependence on the wavenumber of the wave propagation velocity (and in particular of the existence of its real part). Qualitatively similar results have also been obtained for gradient damage models (Peerlings et al., 1996; see also the review paper by de Borst, 2001).

Our purpose is to study the instability and the localization phenomena for non-local plastic material whose response falls within the framework set out by Gudmundson (2004) and developed further by Fleck and Willis (2009a, 2009b), where the internal characteristic length can be described through either energetic and dissipative contributions. The novel feature that will emerge will be the qualitatively different influences of energetic and dissipative gradient terms. The usual bifurcation analysis that suffices in the case of classical plasticity (Anand et al., 1987; Bai, 1982) can be applied to the case of rate-independent gradient theory when the gradient terms are purely energetic but it becomes inapplicable in the presence of dissipative gradient terms. It becomes necessary to consider the development in time of a small perturbation. This can be pursued just as easily in the general case of rate-dependent material response. The resulting linear partial differential equations governing the perturbation have coefficients that depend on time, even in the rate-independent limit. The special case of the perturbation of a uniform monotonically-increasing simple shear deformation is considered in detail.

## 2 Strain gradient plasticity models

Consider a body occupying a domain  $V$  whose points are identified by the components of the position vector  $x_i$  ( $i = 1, 2, 3$ ). The total small strain tensor  $\varepsilon_{ij}$ , given as the symmetric

part of the gradient of displacement vector  $u_i$ ,

$$\varepsilon_{ij} = \frac{u_{i,j} + u_{j,i}}{2} \quad (2.1)$$

(where  $,j$  represents  $\partial/\partial x_j$ ), is additively decomposed into elastic and plastic parts,

$$\varepsilon_{ij} = \varepsilon_{ij}^{EL} + \varepsilon_{ij}^{PL}. \quad (2.2)$$

Considering as fundamental kinematic quantities the elastic strain  $\varepsilon_{ij}^{EL}$ , the plastic strain  $\varepsilon_{ij}^{PL}$ , and its gradient  $\varepsilon_{ij,k}^{PL}$  and defining as work-conjugate quantities, respectively, the symmetric Cauchy stress  $\sigma_{ij}$ , the generalized stress  $Q_{ij}$ , and the higher order stress  $\tau_{ijk}$ , the following principle of virtual work is postulated:

$$\int_V \{ \sigma_{ij} \delta \varepsilon_{ij}^{EL} + Q_{ij} \delta \varepsilon_{ij}^{PL} + \tau_{ijk} \delta \varepsilon_{ij,k}^{PL} \} dV = \int_V f_i \delta u_i dV + \int_S \{ T_i \delta u_i + t_{ij} \delta \varepsilon_{ij}^{PL} \} dS, \quad (2.3)$$

where repeated indices imply summation,  $f_i$  represents the body-force per unit volume and  $T_i$  the surface traction (both work-conjugate to  $u_i$ ) and  $t_{ij}$  represents the higher-order surface traction, work-conjugate to  $\varepsilon_{ij}^{PL}$ .

The principle of virtual work (2.3) implies the following equilibrium equations for points belonging to the volume  $V$

$$\begin{cases} \sigma_{ij,j} + f_i = 0, \\ Q_{ij} = (\text{dev} \boldsymbol{\sigma})_{ij} + \tau_{ijk,k} \end{cases} \quad \text{in } V \quad (2.4)$$

(where dev represents the deviatoric part), and for points on the boundary  $S$ ,

$$\begin{cases} T_i = \sigma_{ij} n_j, \\ t_{ij} = \tau_{ijk} n_k, \end{cases} \quad \text{on } S. \quad (2.5)$$

In the case of dynamics, the first of equations (2.4) is replaced by

$$\sigma_{ij,j} + f_i = \rho u_{i,tt}, \quad (2.6)$$

where  $\rho$  denotes the mass density; initial conditions have also to be specified. The main emphasis of this work is confined to the quasi-static case, in which inertia is disregarded; however some considerations about the dynamic case are made in Sect. 6.

Introducing the internal energy  $U(\boldsymbol{\varepsilon}^{EL}; \boldsymbol{\varepsilon}^{PL}; \nabla \boldsymbol{\varepsilon}^{PL})$  we define “energetic” stresses as

$$\sigma_{ij}^E = \frac{\partial U}{\partial \varepsilon_{ij}^{EL}}, \quad Q_{ij}^E = \frac{\partial U}{\partial \varepsilon_{ij}^{PL}}, \quad \tau_{ijk}^E = \frac{\partial U}{\partial \varepsilon_{ij,k}^{PL}}, \quad (2.7)$$

leaving the “dissipative” contributions to be defined as follows:

$$\sigma_{ij}^D = \sigma_{ij} - \sigma_{ij}^E, \quad Q_{ij}^D = Q_{ij} - Q_{ij}^E, \quad \tau_{ijk}^D = \tau_{ijk} - \tau_{ijk}^E. \quad (2.8)$$

The second law of thermodynamics (which reduces to positive plastic dissipation when, as now, thermal contributions are ignored) requires that

$$\sigma_{ij}^D \dot{\varepsilon}_{ij}^{EL} + Q_{ij}^D \dot{\varepsilon}_{ij}^{PL} + \tau_{ijk}^D \dot{\varepsilon}_{ij,k}^{PL} \geq 0 \quad \text{in } V \quad (2.9)$$

during any actual process. The stress terms  $\sigma_{ij}$ ,  $Q_{ij}$  and  $\tau_{ijk}$  are all assumed to be independent of elastic strain-rate  $\dot{\varepsilon}_{ij}^{EL}$ . Since this could assume any magnitude and direction, it follows that

$$\sigma_{ij}^D = 0, \quad \text{and} \quad \sigma_{ij} = \sigma_{ij}^E, \quad (2.10)$$

and the inequality (2.9) implies

$$Q_{ij}^D \dot{\varepsilon}_{ij}^{PL} + \tau_{ijk}^D \dot{\varepsilon}_{ij,k}^{PL} \geq 0 \quad \text{in } V. \quad (2.11)$$

A way to ensure satisfaction of inequality (2.11) is to introduce a dissipation potential  $\phi(\dot{\varepsilon}^{PL}; \nabla \dot{\varepsilon}^{PL})$  from which the dissipative stresses are derived as

$$Q_{ij}^D = \frac{\partial \phi}{\partial \dot{\varepsilon}_{ij}^{PL}}, \quad \tau_{ijk}^D = \frac{\partial \phi}{\partial \dot{\varepsilon}_{ij,k}^{PL}}. \quad (2.12)$$

The potential  $\phi$  can depend in addition on other arguments which define the history of the deformation, but the inequality (2.11) is satisfied so long as  $\phi$  is a convex function of the “rate” arguments given explicitly, and is zero when these arguments are zero.

### 3 Material stability

The influence of strain-gradient terms becomes significant only when plastic strain gradients become large. This happens, in particular, when classical plasticity predicts localization of deformation, which occurs when the equations governing the next increment of deformation cease to be elliptic (Rice, 1976). This is a property of the material as opposed to a property of the exact boundary-value problem under consideration and can be studied by reference to an infinite body subjected to uniform strain. It is appropriate, accordingly, to study the case of strain-gradient plastic material under the same conditions.

Although the main concern of this work is with rate-independent material response, it became apparent during the work that the approach usual for classical plasticity, of directly assessing the type of the equations of continuing equilibrium, would not suffice when “dissipative” gradient terms were present. It was found necessary, instead, to study a general time-dependent perturbation of a spatially-uniform state of deformation, itself evolving in time in such a way that all of the governing equations are satisfied. This can be done, just as easily, for the case of rate-dependent material response, in which the rate-independent limit is embedded as a special case.

Before proceeding, it is necessary to discuss briefly the influence of the history of the plastic deformation on the current response. This is accommodated by taking the dissipation potential  $\phi$  to depend on some history parameter  $\kappa$  which is expressible, at time  $t$ , as

$$\kappa(t) = \int_{t'=-\infty}^t K(\dot{\boldsymbol{\varepsilon}}^{PL}, \nabla \dot{\boldsymbol{\varepsilon}}^{PL}, \boldsymbol{\varepsilon}^{PL}, \nabla \boldsymbol{\varepsilon}^{PL}) dt'. \quad (3.1)$$

Thus,  $\phi = \phi(\dot{\boldsymbol{\varepsilon}}^{PL}, \nabla \dot{\boldsymbol{\varepsilon}}^{PL}; \kappa)$ , and its variation  $\delta\phi$  corresponding to a variation  $\delta\boldsymbol{\varepsilon}^{PL}$  of the plastic strain is

$$\delta\phi = \frac{\partial\phi}{\partial\dot{\varepsilon}_{ij}^{PL}} \delta\dot{\varepsilon}_{ij}^{PL} + \frac{\partial\phi}{\partial\varepsilon_{ij,k}^{PL}} \delta\varepsilon_{ij,k}^{PL} + \frac{\partial\phi}{\partial\kappa} \delta\kappa, \quad (3.2)$$

where

$$\delta\kappa = \int_{t'=-\infty}^t \left[ \frac{\partial K}{\partial\dot{\varepsilon}_{ij}^{PL}} \delta\dot{\varepsilon}_{ij}^{PL} + \frac{\partial K}{\partial\varepsilon_{ij,k}^{PL}} \delta\varepsilon_{ij,k}^{PL} + \frac{\partial K}{\partial\varepsilon_{ij}^{PL}} \delta\varepsilon_{ij}^{PL} + \frac{\partial K}{\partial\varepsilon_{ij,k}^{PL}} \delta\varepsilon_{ij,k}^{PL} \right] dt'. \quad (3.3)$$

The internal energy  $U$  will be taken to be independent of such history, and thus a function only of  $\boldsymbol{\varepsilon}^{EL}$ ,  $\boldsymbol{\varepsilon}^{PL}$  and  $\nabla \boldsymbol{\varepsilon}^{PL}$ .

Suppose, now, that fields of displacement  $\mathbf{u}(\mathbf{x}, t)$  and plastic strain  $\boldsymbol{\varepsilon}^{PL}(\mathbf{x}, t)$  that satisfy all of the governing equations for continuing equilibrium are subject to perturbations  $\delta\mathbf{u}(\mathbf{x}, t)$  and  $\delta\boldsymbol{\varepsilon}^{PL}(\mathbf{x}, t)$ . The perturbed fields satisfy the equations of continuing equilibrium if the perturbations satisfy the equations

$$\begin{cases} \delta\sigma_{ij,j} = 0, \\ \delta Q_{ij} = (\text{dev} \delta\boldsymbol{\sigma})_{ij} + \delta\tau_{ijk,k}, \end{cases} \quad (3.4)$$

corresponding to equations (2.4), where

$$\delta\mathbf{Q} = \delta\mathbf{Q}^E + \delta\mathbf{Q}^D, \quad \delta\boldsymbol{\tau} = \delta\boldsymbol{\tau}^E + \delta\boldsymbol{\tau}^D. \quad (3.5)$$

If the perturbations are small enough, the perturbations of the stress-like quantities satisfy the linearized relations

$$\begin{aligned} \delta\sigma_{ij} &= \frac{\partial^2 U}{\partial\varepsilon_{ij}^{EL} \partial\varepsilon_{lm}^{EL}} \delta\varepsilon_{lm}^{EL} + \frac{\partial^2 U}{\partial\varepsilon_{ij}^{EL} \partial\varepsilon_{lm}^{PL}} \delta\varepsilon_{lm}^{PL} + \frac{\partial^2 U}{\partial\varepsilon_{ij}^{EL} \partial\varepsilon_{lm,n}^{PL}} \delta\varepsilon_{lm,n}^{PL}, \\ \delta Q_{ij}^E &= \frac{\partial^2 U}{\partial\varepsilon_{ij}^{PL} \partial\varepsilon_{lm}^{EL}} \delta\varepsilon_{lm}^{EL} + \frac{\partial^2 U}{\partial\varepsilon_{ij}^{PL} \partial\varepsilon_{lm}^{PL}} \delta\varepsilon_{lm}^{PL} + \frac{\partial^2 U}{\partial\varepsilon_{ij}^{PL} \partial\varepsilon_{lm,n}^{PL}} \delta\varepsilon_{lm,n}^{PL}, \\ \delta\tau_{ijk}^E &= \frac{\partial^2 U}{\partial\varepsilon_{ij,k}^{PL} \partial\varepsilon_{lm}^{EL}} \delta\varepsilon_{lm}^{EL} + \frac{\partial^2 U}{\partial\varepsilon_{ij,k}^{PL} \partial\varepsilon_{lm}^{PL}} \delta\varepsilon_{lm}^{PL} + \frac{\partial^2 U}{\partial\varepsilon_{ij,k}^{PL} \partial\varepsilon_{lm,n}^{PL}} \delta\varepsilon_{lm,n}^{PL}, \end{aligned} \quad (3.6)$$

and

$$\begin{aligned}\delta Q_{ij}^D &= \frac{\partial^2 \phi}{\partial \dot{\varepsilon}_{ij}^{PL} \partial \dot{\varepsilon}_{lm}^{PL}} \delta \dot{\varepsilon}_{lm}^{PL} + \frac{\partial^2 \phi}{\partial \dot{\varepsilon}_{ij}^{PL} \partial \dot{\varepsilon}_{lm,n}^{PL}} \delta \dot{\varepsilon}_{lm,n}^{PL} + \frac{\partial^2 \phi}{\partial \dot{\varepsilon}_{ij}^{PL} \partial \kappa} \delta \kappa, \\ \delta \tau_{ijk}^D &= \frac{\partial^2 \phi}{\partial \dot{\varepsilon}_{ij,k}^{PL} \partial \dot{\varepsilon}_{lm}^{PL}} \delta \dot{\varepsilon}_{lm}^{PL} + \frac{\partial^2 \phi}{\partial \dot{\varepsilon}_{ij,k}^{PL} \partial \dot{\varepsilon}_{lm,n}^{PL}} \delta \dot{\varepsilon}_{lm,n}^{PL} + \frac{\partial^2 \phi}{\partial \dot{\varepsilon}_{ij,k}^{PL} \partial \kappa} \delta \kappa.\end{aligned}\quad (3.7)$$

In these relations, the partial derivatives are evaluated at the unperturbed arguments.

In the case of interest, that the unperturbed strain fields depend on time  $t$  but are independent of position  $\mathbf{x}$ , equations (3.4–3.7) comprise a system of linear partial differential equations whose coefficients depend only on  $t$ . They can be analyzed by Fourier transforming with respect to the spatial variables (or equivalently, seeking solutions whose spatial dependence is  $\exp(i\mathbf{k} \cdot \mathbf{x})$ ) to yield a system of ordinary differential equations with time-dependent coefficients. The next section pursues this further, in a particular case; more general implications will be discussed elsewhere.

## 4 Monotonic simple shear of an infinite medium

### 4.1 Constitutive model

For the examples to be considered, the constitutive model is chosen as follows. First, the internal energy function is specialized to the quadratic form

$$U(\boldsymbol{\varepsilon}^{EL}; \boldsymbol{\varepsilon}^{PL}; \nabla \boldsymbol{\varepsilon}^{PL}) = \frac{1}{2} \left( \varepsilon_{ij}^{EL} \mathbb{L}_{ijkl} \varepsilon_{kl}^{EL} + \widehat{\mu} \varepsilon_{ij}^{PL} \varepsilon_{ij}^{PL} + \bar{\mu} \ell_E^2 \varepsilon_{ij,k}^{PL} \varepsilon_{ij,k}^{PL} \right), \quad (4.1)$$

where  $\mathbb{L}_{ijkl}$  is the elastic constitutive fourth-order tensor,  $\widehat{\mu}$  and  $\bar{\mu}$  are stiffness moduli related to the plastic behaviour of the material (the former defines the kinematic hardening and the latter defines the non-local behaviour) and  $\ell_E$  is an “energetic” characteristic length-scale introduced for dimensional consistency.

From the potential (4.1) the energetic stresses (2.7) follow as

$$\sigma_{ij} = \mathbb{L}_{ijkl} \varepsilon_{kl}^{EL}, \quad Q_{ij}^E = \widehat{\mu} \varepsilon_{ij}^{PL}, \quad \tau_{ijk}^E = \bar{\mu} \ell_E^2 \varepsilon_{ij,k}^{PL}. \quad (4.2)$$

The tensor of elastic moduli will be required only to have orthotropic symmetry, to permit a state of simple shear; the actual values of all moduli will not be needed.

For the dissipation potential  $\phi$ , the particular form

$$\phi(\dot{\boldsymbol{\varepsilon}}^{PL}, \nabla \dot{\boldsymbol{\varepsilon}}^{PL}; \kappa) = \Sigma_0(\kappa) \phi_0(\dot{E}^P) \quad (4.3)$$

is chosen. The function  $\phi_0$  could be any convex function with  $\phi_0(0) = 0$  but here it is specialized to be

$$\phi_0(\dot{E}^P) = \frac{\dot{\varepsilon}_0}{N+1} \left( \frac{\dot{E}^P}{\dot{\varepsilon}_0} \right)^{N+1}, \quad (4.4)$$

where  $\dot{\varepsilon}_0$  and  $N$  are material constants describing the rate-dependence (the rate-independent limit is recovered when  $N$  approaches zero).

With a view towards incorporating rate-independent behaviour, the variable  $\dot{E}^P$  (which can be considered as an “effective plastic strain-rate”) is taken to be a convex positively-homogeneous function of degree 1 of the variables  $(\dot{\varepsilon}^{PL}, \ell_D \nabla \dot{\varepsilon}^{PL})$ , where  $\ell_D$  is a “dissipative” characteristic length-scale; here it is specialized to the form

$$\dot{E}^P = \sqrt{(\dot{\varepsilon}^P)^2 + (\ell_D \dot{\gamma}^P)^2}, \quad (4.5)$$

where

$$\dot{\varepsilon}^P = \sqrt{\frac{2}{3} \dot{\varepsilon}_{ij}^{PL} \dot{\varepsilon}_{ij}^{PL}}, \quad \dot{\gamma}^P = \sqrt{\dot{\varepsilon}_{ij,k}^{PL} \dot{\varepsilon}_{ij,k}^{PL}}. \quad (4.6)$$

Finally,  $\kappa$  is chosen to be either  $\varepsilon^P$  or  $E^P$ , given as

$$\varepsilon^P(t) = \int_{t'=-\infty}^t \dot{\varepsilon}^P dt', \quad E^P(t) = \int_{t'=-\infty}^t \dot{E}^P dt'. \quad (4.7)$$

It follows from the relations (2.12) and the choices of potential (4.3) and (4.4) that

$$\begin{aligned} Q_{ij}^D &= \frac{2}{3} \Sigma_0(\kappa) \left( \frac{\dot{E}^P}{\dot{\varepsilon}_0} \right)^N \frac{\dot{\varepsilon}_{ij}^{PL}}{\dot{E}^P}, \\ \tau_{ijk}^D &= \ell_D^2 \Sigma_0(\kappa) \left( \frac{\dot{E}^P}{\dot{\varepsilon}_0} \right)^N \frac{\dot{\varepsilon}_{ij,k}^{PL}}{\dot{E}^P}, \end{aligned} \quad (4.8)$$

which implies the scalar flow law<sup>1</sup>

$$\Sigma^D = \Sigma_0(\kappa) \left( \frac{\dot{E}^P}{\dot{\varepsilon}_0} \right)^N, \quad (4.9)$$

where

$$\Sigma^D \dot{E}^P = \sup \{ Q_{ij}^D \dot{\varepsilon}_{ij}^{PL} + \tau_{ijk}^D \dot{\varepsilon}_{ij,k}^{PL} \}, \quad (4.10)$$

the supremum being evaluated over all  $(\dot{\varepsilon}^{PL}, \nabla \dot{\varepsilon}^{PL})$  which deliver the specified value of  $\dot{E}^P$ .

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<sup>1</sup>This can be seen by considering the dual potential (Legendre transform)

$$\phi^*(\mathbf{Q}^D, \boldsymbol{\tau}^D; \kappa) = \sup_{(\dot{\varepsilon}^{PL}, \nabla \dot{\varepsilon}^{PL})} \{ Q_{ij}^D \varepsilon_{ij}^{PL} + \tau_{ijk}^D \varepsilon_{ij,k}^{PL} - \phi(\dot{\varepsilon}^{PL}, \nabla \dot{\varepsilon}^{PL}; \kappa) \},$$

evaluating the supremum sequentially. Equation (4.10) in fact defines  $\Sigma^D$  for any chosen function  $\dot{E}^P$ , homogeneous of degree 1.

Introducing the effective generalized and higher-order dissipative stresses  $Q_e^D$  and  $\tau_e^D$  defined as

$$Q_e^D = \sqrt{\frac{3}{2}Q_{ij}^D Q_{ij}^D}, \quad \tau_e^D = \sqrt{\tau_{ijk}^D \tau_{ijk}^D}, \quad (4.11)$$

the definition (4.10) gives

$$\Sigma^D = \sqrt{(Q_e^D)^2 + \left(\frac{\tau_e^D}{\ell_D}\right)^2}. \quad (4.12)$$

In the special case of rate-independent behaviour ( $N \rightarrow 0$ ), the flow law gives

$$\begin{aligned} \dot{E}^P &\geq 0 \text{ if } \Sigma^D = \Sigma_0(\kappa); \\ \dot{E}^P &= 0 \text{ if } \Sigma^D < \Sigma_0(\kappa); \end{aligned} \quad (4.13)$$

implying the hardening law

$$\dot{\Sigma}^D = \frac{Q_e^D \dot{Q}_e^D + \tau_e^D \dot{\tau}_e^D / \ell_D^2}{\Sigma_0} = h(\kappa) \dot{\kappa} \quad (4.14)$$

for continued plastic flow, where

$$h(\kappa) = \Sigma'_0(\kappa). \quad (4.15)$$

It should perhaps be emphasized that the general framework could accommodate features in addition to those adopted in this section. In particular, an internal energy function  $U$  could be defined which does not just consist of three independent functions of elastic strain, plastic strain and plastic strain-gradient, and the dependence on the plastic strain and its gradient need not be quadratic, or even homogeneous. Likewise, the dissipation potential  $\phi$  could be more general, even while retaining the form (4.4). Rate-independence in the limit  $N \rightarrow 0$  would be retained by the selection of any function  $\dot{E}^P$  that is homogeneous of degree 1 in the variables  $(\dot{\epsilon}^{PL}, \ell_D \nabla \dot{\epsilon}^{PL})$ ; the corresponding function  $\Sigma^D$  would still be given by (4.10). In particular, Evans and Hutchinson (2009) have discussed the virtues of a family of homogeneous functions for defining  $\dot{E}^P$ , though still not allowing for coupling between plastic strain-rate and its gradient. The main point of the present work, however, is to obtain some preliminary indication of possible differences between energetic and dissipative gradient terms.

## 4.2 Equations governing the perturbation

Take the unperturbed deformation to be simple shear so that

$$\begin{aligned} u_1 &= \Gamma(t) x_2, & u_2 &= u_3 = 0, \\ \varepsilon_{12}^{PL} &= \varepsilon_{12}^{PL}(t), & \varepsilon_{11}^{PL} &= \varepsilon_{22}^{PL} = 0, & \varepsilon_{i3}^{PL} &= 0, \end{aligned} \quad (4.16)$$



where  $\Gamma(t)$  represents the imposed shear distortion; it is assumed that the deformation is monotonic,  $\dot{\Gamma}(t) > 0$  and correspondingly  $\dot{\varepsilon}_{12}^{PL} \geq 0$ . The equilibrium equation (2.4)<sub>2</sub> in the case of homogeneous deformation/stress state imposes

$$\sigma_{12}(t) = Q_{12}(t). \quad (4.17)$$

In what follows,  $\kappa$  (chosen to be either  $\varepsilon^P$  or  $E^P$ , with  $\dot{E}^P$  having the special form (4.5)) takes the value  $2\varepsilon_{12}^{PL}/\sqrt{3}$  for the simple shear deformation, since  $\dot{\varepsilon}_{12}^{PL} \geq 0$ . With the constitutive equations (4.2) and (4.8), equation (4.17) leads to a relation between  $\Gamma(t)$  and  $\varepsilon_{12}^{PL}(t)$ ,

$$\sigma_{12}(t) \equiv 2\mu[(\Gamma/2) - \varepsilon_{12}^{PL}] = \hat{\mu}\varepsilon_{12}^{PL} + \frac{1}{\sqrt{3}} \left( \frac{2}{\sqrt{3}} \frac{\dot{\varepsilon}_{12}^{PL}}{\dot{\varepsilon}_0} \right)^N \Sigma_0(\kappa), \quad (4.18)$$

where  $\mu$  is the relevant elastic shear modulus. Then, all of the governing equations are satisfied.

Both for simplicity and for its likely relevance, the perturbation is assumed to involve only perturbations  $\delta u_1$  of the non-zero displacement  $u_1$  and  $\delta\varepsilon_{12}^{PL}$  of the non-zero plastic strain. Furthermore, the perturbation in the plastic strain is taken to have the form

$$\delta\varepsilon_{12}^{PL} = F(t)e^{ikx_2}, \quad (4.19)$$

so that the perturbation of the total strain  $\delta\varepsilon_{12}$  has the corresponding form

$$\delta\varepsilon_{12} = G(t)e^{ikx_2}. \quad (4.20)$$

Using the constitutive models (4.1) and (4.3) in the linearized constitutive relations (3.6) and (3.7), together with (3.4)<sub>2</sub>, give

$$\delta\sigma_{12} = \left( \frac{2\Sigma_0\phi_0''}{3} + \ell_D^2 k^2 \frac{\Sigma_0\phi_0'}{E^P} \right) \delta\varepsilon_{12}^{PL} + \left( \hat{\mu} + \bar{\mu}\ell_E^2 k^2 + \frac{2\Sigma_0'\phi_0'}{3} \right) \delta\varepsilon_{12}^{PL}, \quad (4.21)$$

while the equilibrium equation (3.4)<sub>1</sub> reduces to

$$k\delta\sigma_{12} = 0. \quad (4.22)$$

Thus, so long as  $k \neq 0$ , equilibrium requires that  $\delta\sigma_{12} = 0$ ; explicitly, with  $\phi_0$  given by (4.4),

$$\frac{\Sigma_0}{\dot{\varepsilon}_0} \left( \frac{2N}{3} + \ell_D^2 k^2 \right) \left( \frac{2\dot{\varepsilon}_{12}^{PL}}{\sqrt{3}\dot{\varepsilon}_0} \right)^{N-1} \dot{F} + \left( \hat{\mu} + \frac{2h}{3} \left( \frac{2\dot{\varepsilon}_{12}^{PL}}{\sqrt{3}\dot{\varepsilon}_0} \right)^N + \bar{\mu}\ell_E^2 k^2 \right) F = 0. \quad (4.23)$$

The additional relation

$$\delta\sigma_{12} = 2\mu\delta\varepsilon_{12}^{EL} = 2\mu(\delta\varepsilon_{12} - \delta\varepsilon_{12}^{PL}) \equiv 2\mu(G - F)e^{ikx_2} \quad (4.24)$$

fixes

$$G(t) = F(t). \quad (4.25)$$

In the case  $k = 0$  (so that no gradients are involved), the equilibrium condition (4.22) provides no restriction but it is reasonable to require that  $G(t) = 0$ , so that the specified mean strain is maintained. Equation (4.23) with the zero on its right side replaced by  $-2\mu F \equiv \delta\sigma_{12}$  provides the required differential equation for  $F$ . It is of little interest because in practice it is always going to imply stability against perturbations with  $k = 0$ . This case will not be considered further.

### 4.3 Implications for stability

Equation (4.23) is a first-order differential equation whose general solution is

$$F(t) = A \exp \left( - \int_{t'=0}^t \frac{\hat{\mu} + \frac{2h}{3} \left( \frac{2\dot{\varepsilon}_{12}^{PL}}{\sqrt{3}\dot{\varepsilon}_0} \right)^N + \bar{\mu}\ell_E^2 k^2}{\frac{\Sigma_0}{\dot{\varepsilon}_0} \left( \frac{2N}{3} + \ell_D^2 k^2 \right) \left( \frac{2\dot{\varepsilon}_{12}^{PL}}{\sqrt{3}\dot{\varepsilon}_0} \right)^{N-1}} dt' \right), \quad (4.26)$$

where  $A = F(0)$ . The convention of linearized stability theory is that the unperturbed deformation defined by  $\varepsilon_{12}^{PL}(t)$  is stable if  $F(t)$  remains finite for all  $t$ : any small perturbation is predicted by the linearized governing equation to remain small and the linearized formulation at least is consistent. If, conversely,  $F(t)$  is predicted to become unbounded, the unperturbed deformation is considered to be unstable – though in this case the linearized formulation becomes inconsistent and merely demonstrates that some analysis of the nonlinear exact equation governing the perturbation is required.

It should perhaps be emphasized that equation (4.23) was derived under the assumption that  $\dot{\varepsilon}_{12}^{PL}(t) \geq 0$  for all  $t$ , and all that follows is subject to this assumption. If  $\Sigma_0(\kappa)$  is an increasing function of  $\kappa$ , it follows that the unperturbed deformation is always stable. If  $h(\kappa)$  can become negative as  $\kappa$  increases, then instability at least is possible, depending on the unperturbed plastic strain-rate  $\dot{\varepsilon}_{12}^{PL}$  and on the wavenumber of the perturbation,  $k$ . Some special cases are now considered.

#### 4.3.1 Classical plasticity (no gradient dependence)

Suppose first that  $\ell_E = \ell_D = 0$ . In this case any dependence on the wavenumber  $k$  disappears in the differential equation (4.23). The ode (4.23) retains its form except in the rate-independent limit ( $N = 0$ ), when it degenerates to an algebraic equation. In this case, no perturbation with any wavenumber  $k \neq 0$  is possible, except at a plastic strain at which

$$\hat{\mu} + \frac{2h}{3} = 0, \quad (4.27)$$

which is the classical condition for localization. The original problem is well-posed only up to the plastic strain at which (4.27) is satisfied. As observed by Needleman (1988), admission

of any amount of rate-dependence ( $N > 0$ ) regularizes the perturbation: assuming that  $h$  decreases monotonically as  $\kappa$  increases,  $F(t)$  reduces with  $t$  until the condition (4.27) is reached, and grows thereafter. Perturbations with all wavenumbers  $k > 0$  behave in the same way.

### 4.3.2 Rate-independent limit of gradient theory

In the rate-independent limit ( $N \rightarrow 0$ ) the governing equation (4.23) becomes

$$\frac{\sqrt{3}}{2} \ell_D^2 k^2 \Sigma_0 \frac{dF}{d\varepsilon_{12}^{PL}} + \left( \hat{\mu} + \frac{2h}{3} + \bar{\mu} \ell_E^2 k^2 \right) F = 0, \quad (4.28)$$

having written  $\dot{F}$  as  $\dot{\varepsilon}_{12}^{PL} dF/d\varepsilon_{12}^{PL}$ . Its solution (so long as  $\ell_D k \neq 0$ ) is

$$F(\varepsilon_{12}^{PL}) = A \exp \left( -\frac{2}{\sqrt{3} \ell_D^2 k^2} \int_0^{\varepsilon_{12}^{PL}} \frac{\hat{\mu} + \frac{2}{3} h \left( \frac{2\varepsilon_{12}^{PL}}{\sqrt{3}} \right) + \ell_E^2 k^2}{\Sigma_0 \left( \frac{2\varepsilon_{12}^{PL}}{\sqrt{3}} \right)} d\varepsilon_{12}^{PL} \right). \quad (4.29)$$

If  $\ell_D = 0$ , equation (4.28) degenerates to an algebraic equation that admits a non-zero solution only when

$$\hat{\mu} + \frac{2h}{3} + \bar{\mu} \ell_E^2 k^2 = 0. \quad (4.30)$$

This is analogous to the localization equation (4.27) of classical plasticity, except for the dependence on wavenumber  $k$ . The presence of an energetic characteristic length  $\ell_E$  delays the appearance of perturbations with high wavenumbers (Fig. 1). The perturbation can be regularized by admitting some rate-dependence ( $N > 0$ ). If this is done, a perturbation with wavenumber  $k$  decays until the condition (4.30) is reached, and grows thereafter. The rate of decay (or growth) depends on the rate of the original plastic strain.

Alternatively, however, it is possible to retain rate-independence ( $N = 0$ ) but to allow a dissipative gradient term ( $\ell_D > 0$ ), yielding the solution (4.29). This solution reduces as  $\varepsilon_{12}^{PL}$  increases (implying stability) for wavenumbers  $k$  for which

$$\hat{\mu} + \frac{2h}{3} + \bar{\mu} \ell_E^2 k^2 \geq 0 \quad (4.31)$$

for all  $\varepsilon_{12}^{PL}$ . It remains bounded even if this restriction is not met, if the integral in (4.29) converges. Formally, in this situation, the unperturbed solution is stable but the practical issue is whether or not the perturbation becomes unacceptably large as  $\varepsilon_{12}^{PL}$  increases.

For the purpose of illustration, some calculations have been performed for material whose characterizing stress  $\Sigma_0$  is chosen to be a quadratic function of the plastic deformation:

$$\Sigma_0(\kappa) \equiv \Sigma_0(2\varepsilon_{12}^{PL}/\sqrt{3}) = \frac{\sigma_0}{3} \left( 4 - \frac{(\varepsilon_{12}^{PL} - \varepsilon_c)^2}{\varepsilon_c^2} \right). \quad (4.32)$$

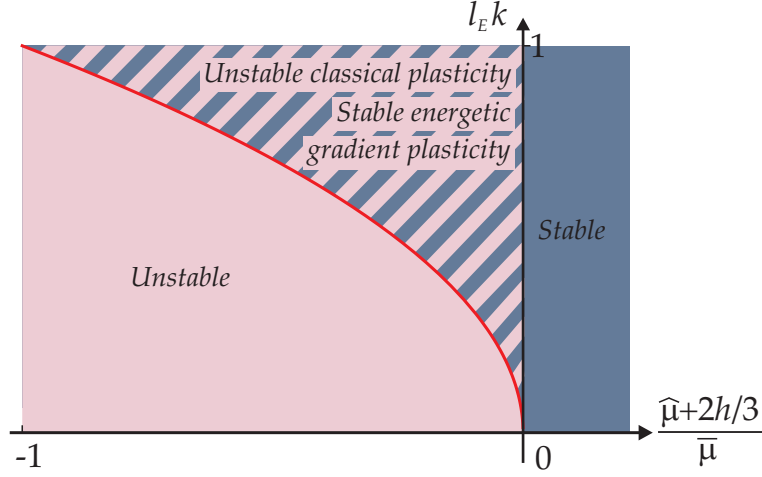


Figure 1: Stable and unstable regions for a rate-independent material in presence of energetic characteristic length  $\ell_E$ , eqn. (4.30). The presence of  $\ell_E$  increases the region where the perturbation is stable: the higher the wavenumber (smaller the wavelength), the higher the negative hardening for which it becomes unstable.

Thus, the material hardens up to a plastic shear strain  $\varepsilon_{12}^{PL} = \varepsilon_c$  and softens thereafter, losing all strength at  $\varepsilon_{12}^{PL} = 3\varepsilon_c$ , as depicted in Fig. 2. The “kinematic hardening” modulus  $\hat{\mu}$  is taken to be zero, and  $\bar{\mu} = \sigma_0/\varepsilon_c$  (this, in effect, fixes the scale for  $\ell_E$ ).

In a formal sense, as  $\varepsilon_{12}^{PL} \rightarrow 3\varepsilon_c$ , the unperturbed deformation is unstable (even with rate-dependence) against any perturbation whose wavenumber  $k$  fails to satisfy the restriction (4.31) as  $\varepsilon_{12}^{PL} \rightarrow 3\varepsilon_c$ : the integral in (4.26) tends to  $-\infty$  and  $F(\varepsilon_{12}^{PL})$  correspondingly blows up. When  $\ell_E \neq 0$ , the restriction (4.31) remains satisfied for sufficiently large wavenumbers  $k$ ; then, the integral in (4.26) tends to  $+\infty$  and  $F(\varepsilon_{12}^{PL}) \rightarrow 0$ . Both of these behaviours are exhibited by the solution (4.34) and are illustrated in Fig. 3(b). This is an artificial feature, however, associated with the simple choice (4.32) of  $\Sigma_0$ . The question of practical interest is whether or not the perturbation grows significantly within some given range of  $\varepsilon_{12}^{PL}$  – for instance up to  $2\varepsilon_c$ .

When  $\Sigma_0$  is given by (4.32), the hardening function  $h$  is

$$h(2\varepsilon_{12}^{PL}/\sqrt{3}) = \frac{\sqrt{3}}{2} \frac{d\Sigma_0(2\varepsilon_{12}^{PL}/\sqrt{3})}{d\varepsilon_{12}^{PL}} = \frac{\sigma_0}{\sqrt{3}} \left( \frac{\varepsilon_c - \varepsilon_{12}^{PL}}{\varepsilon_c^2} \right) \quad (4.33)$$

and the solution (4.29) yields

$$\frac{F(\varepsilon_{12}^{PL})}{F(0)} = \left[ \frac{3}{\left(3 - \frac{\varepsilon_{12}^{PL}}{\varepsilon_c}\right) \left(1 + \frac{\varepsilon_{12}^{PL}}{\varepsilon_c}\right)} \right] \frac{2}{3\ell_D^2 k^2} \left[ \frac{3 - \frac{\varepsilon_{12}^{PL}}{\varepsilon_c}}{3 \left(1 + \frac{\varepsilon_{12}^{PL}}{\varepsilon_c}\right)} \right] \frac{\sqrt{3}(\hat{\mu} + \bar{\mu}\ell_E^2 k^2)\varepsilon_c}{2\sigma_0\ell_D^2 k^2}. \quad (4.34)$$

Figure 3 gives plots of this solution, for a range of values of  $\ell_D k$ , in two cases: (a)  $\ell_E = 0$  and (b)  $\ell_E = \ell_D$ . All of the features identified qualitatively are realised. In particular, when  $\ell_D k$  is small ( $\ell_D k = 0.1$  is illustrated), the curves in both cases (a) and (b) display a very sharp minimum, indicating that the unperturbed field is very stable against such a perturbation when  $\varepsilon_{12}^{PL}$  is smaller than the value at which the minimum is attained, and very unstable when  $\varepsilon_{12}^{PL}$  is larger than that value. Moreover, the presence of a characteristic length  $\ell_E$  shifts to higher values the plastic deformation  $\varepsilon_{12}^{PL}$  at which the minimum amplitudes of perturbations are attained.

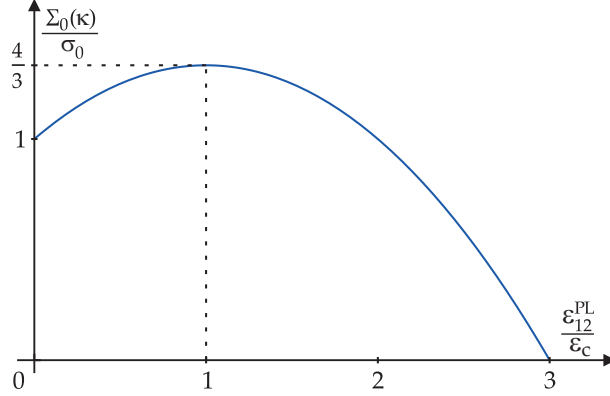


Figure 2: Normalized plot of yield stress  $\Sigma_0(\kappa)$ , eqn. (4.32), versus plastic strain  $\varepsilon_{12}^{PL}$ , in the case of simple shear ( $\kappa = 2\varepsilon_{12}^{PL}/\sqrt{3}$ ).

It should be noted that the perturbation theory has been developed under the assumption that  $|\delta\dot{\varepsilon}_{12}^{PL}| \ll \dot{\varepsilon}_{12}^{PL}$ , and hence can be valid only if  $\dot{\varepsilon}_{12}^{PL}$  is bounded below by some positive number. In fact, the term multiplying  $\dot{F}$  in equation (4.23) blows up as  $\dot{\varepsilon}_{12}^{PL} \rightarrow 0$  for all  $N < 1$ . In the case of rate-independent behaviour ( $N = 0$ ), it is possible to contemplate the time-dependent perturbation of a stationary state, for which  $\dot{\varepsilon}_{12}^{PL} = 0$  after some time  $t_0$  say. The assumed linearization of the perturbation of  $\dot{E}^P$ , but not that of  $E^P$ , fails for  $t > t_0$ . Except when  $\ell_D = 0$ , the equation governing the perturbation is intrinsically nonlinear. Its study is beyond the scope of the present work.

## 5 Monotonic simple shear of a homogeneous strip

Consider now a homogeneous strip, occupying the domain  $\{\mathbf{x} : -H < x_2 < H, -\infty < x_1, x_3 < \infty\}$ , subject to the boundary conditions

$$u_1(x_2 = \pm H) = \pm\Gamma(t)H, \quad \tau_{122}(x_2 = \pm H) = 0, \quad (5.1)$$

other components of  $\mathbf{u}$  and  $\boldsymbol{\tau}$  being identically zero.



there can be sharp elastic-plastic boundaries. Across any such boundary with unit normal  $n_i$  say, the quantities  $u_i$  and  $\sigma_{ij}n_j$  must be continuous. In addition, for strain-gradient material,  $\varepsilon_{ij}^{PL}$  and  $\tau_{ijk}n_k$  must be continuous.<sup>3</sup> For the solutions considered here, this means that  $u_1$ ,  $\varepsilon_{12}$ ,  $\varepsilon_{12}^{PL}$  and  $\varepsilon_{12,2}^{PL}$  must be continuous. Equilibrium of the perturbed field requires that

$$\frac{d}{dx_2} \left\{ \bar{\mu} \ell_E^2 \frac{d^2(\delta\varepsilon_{12}^{PL})}{dx_2^2} - (\hat{\mu} + 2h/3) \delta\varepsilon_{12}^{PL} \right\} = 0. \quad (5.4)$$

This is obtained from taking  $k$  times equation (4.28) with  $\ell_D = 0$  and the original  $d/dx_2$  replacing  $ik$ , in conformity with the full equilibrium condition (4.22). Considering constant material parameters ( $\ell_E, \bar{\mu}, \hat{\mu}, h$ ) along the variable  $x_2$  during the unperturbed simple shear deformation process, the ode (5.4) has general solution<sup>4</sup>

$$\delta\varepsilon_{12}^{PL}(x_2) = \begin{cases} C_1 + C_2 \cosh(\alpha x_2) + C_3 \sinh(\alpha x_2), & \alpha = \left( \frac{\hat{\mu} + 2h/3}{\bar{\mu} \ell_E^2} \right)^{1/2} & \text{if } \hat{\mu} + 2h/3 > 0, \\ D_1 + D_2 \cos(kx_2) + D_3 \sin(kx_2), & k = \left( \frac{-(\hat{\mu} + 2h/3)}{\bar{\mu} \ell_E^2} \right)^{1/2} & \text{if } \hat{\mu} + 2h/3 < 0, \end{cases} \quad (5.5)$$

Suppose, now, that a uniform state of deformation has been maintained by some means, up to a strain for which  $\hat{\mu} + 2h/3 < 0$ , i.e. into the softening range of the material's response, and consider the next small increment of the boundary displacement, defined by  $\delta\Gamma > 0$  (the exact time-dependence is immaterial due to rate-independence). In this regime, the distribution of incremental plastic deformation can be non-unique. Therefore we distinguish two main cases, illustrated in Fig. 4.

Case  $\mathcal{A}$ . The simplest possible increment of deformation is a continuation of the uniform solution<sup>5</sup> (i.e.  $D_2 = D_3 = 0$ ):

$$\delta\varepsilon_{12} = \frac{\delta\Gamma}{2}, \quad \delta\varepsilon_{12}^{PL} = \frac{2\mu}{2\mu + \hat{\mu} + 2h/3} \frac{\delta\Gamma}{2}. \quad (5.6)$$

---

<sup>3</sup>There is in general the complication that  $\tau_{ijk}^D$  is not defined constitutively in an elastic domain but can be any field that satisfies (2.4) and the continuity condition for higher-order traction (Fleck and Willis, 2009a,b). This is not an issue for the solutions presented here.

<sup>4</sup>The general solution (5.5) can be used to analyze a perturbation from a state in which the hardening  $h$  is piecewise constant, so that  $\hat{\mu} + 2h/3$  is positive except in some interval (or intervals) in which it is negative, to provide some insight into the effect of a small imperfection (or imperfections), as considered by Loret and Prévost (1991) for classical plasticity. In this case, within the zone where  $\hat{\mu} + 2h/3$  is positive, the perturbation state becomes non-uniform since, to match the interface condition between the two zones, the constants  $D_2$  and  $D_3$  must be different from zero.

<sup>5</sup>Considering a perturbation applied to a state for which  $\hat{\mu} + 2h/3 > 0$ , the constants  $C_2$  and  $C_3$  of solution (5.5)<sub>1</sub> are required to be null in order to satisfy the boundary conditions and therefore in this case the solution is unique and corresponds to the uniform distribution. Therefore, similarly to case  $\mathcal{A}$ , this case also leads to solution (5.6).

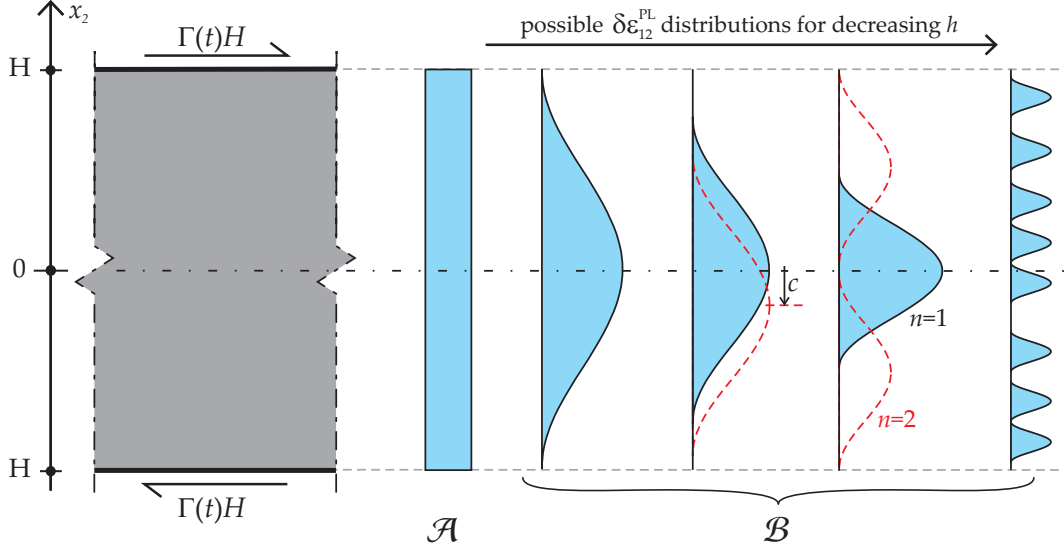


Figure 4: Strip of width  $2H$  subject to a monotonically-increasing simple shear deformation through the displacement  $\pm\Gamma(t)H$  on its boundary. Possible uniform (case  $\mathcal{A}$ ) and non-uniform (case  $\mathcal{B}$ ) distributions for the increment in the plastic deformation  $\delta\varepsilon_{12}^{PL}$  at decreasing value of hardening  $h$ .

Since it is required that  $\delta\varepsilon_{12}^{PL} \geq 0$  for plastic loading, no such solution exists unless

$$-2\mu < \hat{\mu} + 2h/3 < 0. \quad (5.7)$$

The perturbation in the shear stress can be obtained through the elastic constitutive relation:

$$\delta\sigma_{12} = 2\mu (\delta\varepsilon_{12} - \delta\varepsilon_{12}^{PL}) = \frac{2\mu(\hat{\mu} + 2h/3)}{2\mu + \hat{\mu} + 2h/3} \frac{\delta\Gamma}{2} < 0, \quad (5.8)$$

showing the unloading character of stress within the softening regime.

The equivalent incremental shear stiffness  $\mu^*$ , given as the ratio of the increment in shear stress  $\delta\sigma_{12}$  to the mean value  $\delta\bar{\varepsilon}_{12}$  of the increment in shear strain (equal to  $\delta\varepsilon_{12}$  due to its uniformity), is

$$\mu^* = \frac{\delta\sigma_{12}}{2\delta\bar{\varepsilon}_{12}} = \frac{\hat{\mu} + 2h/3}{2\mu + \hat{\mu} + 2h/3} \mu < 0, \quad (5.9)$$

showing that

$$\lim_{\hat{\mu} + 2h/3 \rightarrow -2\mu} \mu^* \rightarrow -\infty. \quad (5.10)$$

Case  $\mathcal{B}$ . Restricting for now attention to symmetric solutions with respect to  $x_2 = 0$  (i.e.  $D_3 = 0$ ), a non-uniform perturbation can exist in a subset  $(-b, b)$  of the width when



$D_1 = D_2 = A$ , so that

$$\delta\varepsilon_{12}^{PL} = \begin{cases} A[1 + \cos(\pi x_2/b)] & \text{for } |x_2| < b, \\ 0 & \text{for } b < |x_2| < H, \end{cases} \quad (5.11)$$

where

$$b = \frac{\pi}{k} = \pi\ell_E \left( \frac{\bar{\mu}}{-(\hat{\mu} + 2h/3)} \right)^{1/2} \quad (5.12)$$

and  $A > 0$  (to ensure plastic loading).

This perturbation and its derivative are continuous and so constitute a weak solution of (5.4). The perturbation of the Cauchy stress in the interval  $(-b, b)$  is

$$\delta\sigma_{12} = (\hat{\mu} + 2h/3)A \quad (5.13)$$

and hence, in  $(-b, b)$ ,

$$\delta\varepsilon_{12} = \left( \frac{\hat{\mu} + 2h/3}{2\mu} \right) A + \delta\varepsilon_{12}^{PL}. \quad (5.14)$$

The value (5.13) of  $\delta\sigma_{12}$  can be maintained outside the interval  $(-b, b)$  by taking

$$\delta\varepsilon_{12} = \left( \frac{\hat{\mu} + 2h/3}{2\mu} \right) A. \quad (5.15)$$

Thus,  $\delta\varepsilon_{12}$  is given by (5.14) for all  $x_2 \in (-H, H)$  and so is continuous. The mean value of  $\delta\varepsilon_{12}$  across the whole strip  $-H < x_2 < H$  is now

$$\delta\bar{\varepsilon}_{12} \equiv \frac{\delta\Gamma}{2} = \left( \frac{\hat{\mu} + 2h/3}{2\mu} + \frac{b}{H} \right) A, \quad (5.16)$$

thus providing an expression for the ‘‘amplitude’’  $A$  given in terms of  $\delta\Gamma$ .

It is necessary that  $b \leq H$ , and also, for the consistency of the assumed regions of plastic loading and elastic unloading, that  $A > 0$ . Thus, taking account of the definition (5.12) of  $b$ , the perturbation described by eqn. (5.11) can exist only when

$$\frac{\bar{\mu}}{2\mu} \left( \frac{-(\hat{\mu} + 2h/3)}{\bar{\mu}} \right)^{3/2} < \frac{\pi\ell_E}{H} \leq \left( \frac{-(\hat{\mu} + 2h/3)}{\bar{\mu}} \right)^{1/2}. \quad (5.17)$$

There can be no such solution (regardless of the value of  $\ell_E/H$ ) unless condition (5.7) is satisfied.

It is remarked now that exactly the same profile of plastic strain perturbation can be supported on any interval  $(c-b, c+b)$  for any  $c \in (b-H, H-b)$ , Fig. 4, and therefore losing any symmetry. The perturbation of total strain still conforms to (5.14) and the restrictions (5.17) continue to apply. Now, however, suppose that  $b < H/n$  for some integer  $n \geq 2$ . It

becomes possible to fit up to  $n$  intervals (i.e. number of waves), each of length  $2b$ , into the strip, with plastic strain perturbation supported on these intervals. The only difference is that the “amplitude”  $A$  has to satisfy (5.16) with  $b$  replaced by  $nb$ , and analogously with eqn. (5.17) in order to have that a perturbation comprising  $n$  distinct incipient shear bands is possible,<sup>6</sup> namely

$$\frac{\bar{\mu}}{2\mu} \left( \frac{-(\hat{\mu} + 2h/3)}{\bar{\mu}} \right)^{3/2} < \frac{n\pi\ell_E}{H} \leq \left( \frac{-(\hat{\mu} + 2h/3)}{\bar{\mu}} \right)^{1/2}. \quad (5.18)$$

In this case, the equivalent incremental shear stiffness  $\mu^*$  becomes

$$\mu^* = \frac{1}{1 - \frac{n\pi\ell_E}{H} \frac{2\mu}{\bar{\mu}} \left( \frac{\bar{\mu}}{-(\hat{\mu} + 2h/3)} \right)^{3/2}} \mu < 0, \quad (5.19)$$

showing that  $\mu^* \rightarrow -\infty$  in the limit when the left inequality of eqn. (5.18) becomes an equality, similarly to the homogeneous case. Moreover, when the right inequality of (5.18) becomes an equality, the equivalent incremental shear stiffness  $\mu^*$  obtained in case  $\mathcal{A}$ , eqn. (5.9), is the same as that obtained in case  $\mathcal{B}$ , eqn. (5.19).

Fig. 5 shows domains in an  $X$ - $Y$  plane, with  $X = (\hat{\mu} + 2h/3)/\bar{\mu}$  and  $Y = \pi\ell_E/H$ , in which the various solutions exist or do not exist, with  $\bar{\mu}$  set equal to  $2\mu$ . We can note that when the inequality (5.7) is violated, so that the uniform incremental solution cannot exist, (5.18) cannot be satisfied and therefore no incremental solution of the type (5.11) exists either. The obtained equations and figure show how, in the limit case of classical plasticity ( $\ell_E = 0$ ), it becomes possible to develop any number  $n$  of waves characterized by any width  $2b$  in materials with any negative value of  $\hat{\mu} + 2h/3$ . A similar result was obtained by Loret and Prévost (1991) in the case of classical plasticity considering a piecewise constant (positive and negative) distribution of hardening.

## 5.2 Some computations for the case $\ell_E = 0$

As noted immediately following equation (5.3), in the case that  $\ell_E = 0$ , perturbations of all allowed wavelengths (i.e. with any value of  $n$ ) grow as soon as  $\hat{\mu} + 2h/3$  becomes negative. It is nevertheless true that the rate of growth is more rapid for small  $n$  than for large  $n$  (see, for example, equation (4.29)) so it is to be expected that a perturbation triggered by a small defect will have a tendency to spread out as it develops. A competing effect arises from the fact that the yield stress  $\Sigma_0$  reduces as the plastic strain increases. While some “weakly-nonlinear” analysis based on the perturbation theory might be possible, it is more efficient (and less restrictive) simply to perform finite-element computations. This subsection presents some of the results that have been obtained.

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<sup>6</sup>All  $A$ 's have to be the same, because  $\delta\sigma_{12}$  has to be constant and equation (5.13) has to be satisfied in each interval.

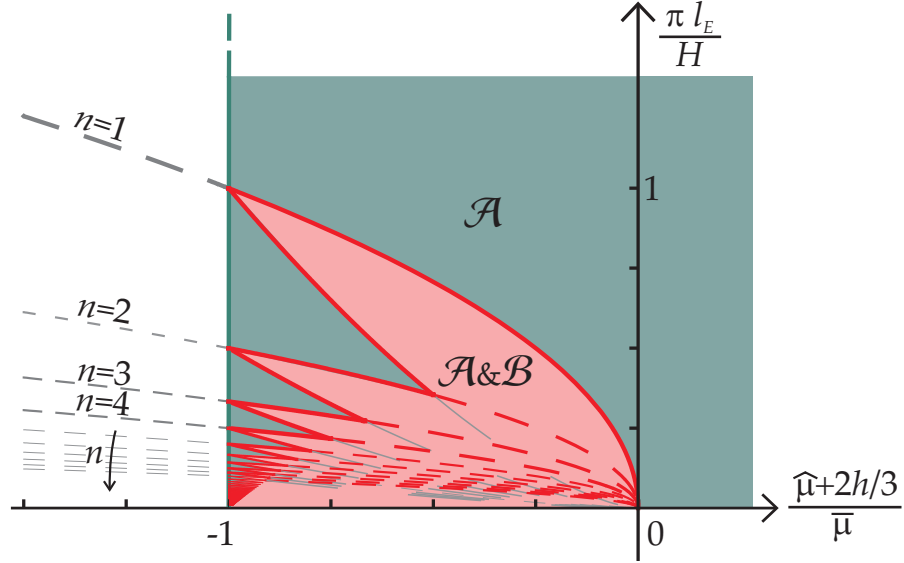


Figure 5: Domains in which the solution of the incremental problem exists and is unique (case  $\mathcal{A}$  alone), is non unique (cases  $\mathcal{A}$  and  $\mathcal{B}$ ) and does not exist (white) in the case of  $\bar{\mu} = 2\mu$ . The zone below the curves identified with  $n$  represents the domain where  $nb < H$ .

Similarly to eqn. (4.32), the characterizing stress  $\Sigma_0$  is chosen to have the form

$$\Sigma_0(x_2) = \sigma_0 + h_0(x_2)\varepsilon^P(x_2) + h'_0 \frac{(\varepsilon^P(x_2))^2}{2}, \quad (5.20)$$

(for large  $\ell_D$  slight differences in the results can be obtained replacing  $\varepsilon^P$  with  $E^P$ ) so that the hardening is given by

$$h(x_2) = \frac{\partial \Sigma_0}{\partial \varepsilon^P} = h_0(x_2) + h'_0 \varepsilon^P(x_2). \quad (5.21)$$

To trigger the occurrence of instability, as the only parameter varying across the width, we consider a piecewise-uniform distribution of the initial hardening value  $h_0(x_2)$ , requiring it to take the value  $h_0(0)$  everywhere except in regions of imperfection (either 2 or 32, each of length  $2a$ ). In such regions, it is reduced to  $d_1 h_0(0)$  except in  $(2a, 4a)$ , where  $h_0$  is reduced to  $d_2 h_0(0)$ , as shown in Fig. 6.  $d_2$  is chosen smaller than  $d_1$  so that the region  $(2a, 4a)$  is the most defective, thereby breaking any symmetry of the problem.

The material model has been implemented in a plane strain mixed finite element program employing bilinear quadrilateral elements.<sup>7</sup> The mixed finite element formulation consists in writing the two field equations (2.4) in weak form and then making use of the material constitutive laws (2.7), (2.8), (2.10) and (2.12) to obtain a system of equations for the displacement

<sup>7</sup>In the present application, the elements become one-dimensional, linear in the variable  $x_2$ .

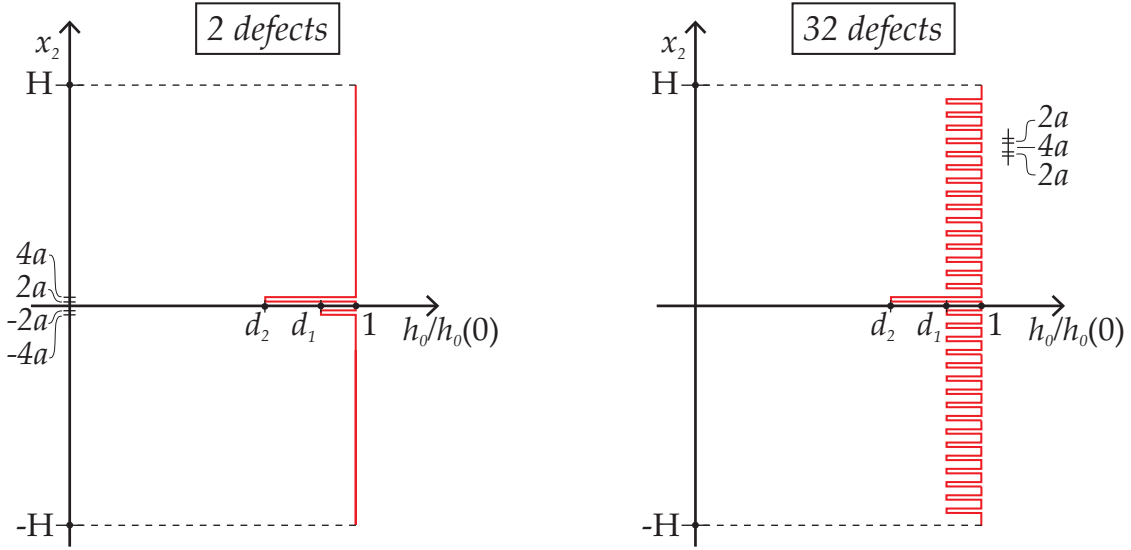


Figure 6: Initial piecewise constant distributions of positive hardening  $h_0(x_2)$  across the width of the sample in presence of 2 and 32 defects of width  $2a$ . The presence of a defect with greater imperfection ( $d_2 < d_1$ ) is introduced in order to avoid any symmetry in the solution.

increments and the increments of the components of the plastic strain rate (see Danas et al., Niordson and Legarth (2010) for more details). A small amount of rate-dependence was admitted by accepting a small positive value of  $N$ , in order to avoid having to ensure that the state of stress remains exactly on the yield surface during plastic loading. This has some effect on the rate of development of long-wavelength components in the perturbation but less effect on the short-wavelength components that are of most interest. The two field equations (2.4) in weak form are solved successively using a forward Euler integration scheme, such that equation (2.4)<sub>1</sub> gives the solution for the displacement increments which are then used in relation (2.4)<sub>2</sub> to solve for the increments of the components of the plastic strain rate. Global convergence of the field equations (2.4) is reached when the relative difference of two successive increments of the effective plastic strain rate  $\dot{E}^P$ , eqn. (4.5), is less than a tolerance set here equal to  $10^{-2}$ . Several simulations for different meshes and time-steps have been performed checking the convergence of the results. Here we report just the results relative to the more accurate one corresponding to a mesh with  $n_{el} = 20000$  (uniform in the width, therefore 200 elements within each defected zone) and a time-step of  $\Delta t = 10^{-3}\text{sec}$ .<sup>8</sup>

The following parameters were employed in the computations:

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<sup>8</sup>Movies about simulations are available at the following web-site address: <http://www.ing.unitn.it/dims/ssmg>

*Material properties*

|                       |             |                   |                  |                      |              |         |   |
|-----------------------|-------------|-------------------|------------------|----------------------|--------------|---------|---|
| $H$ [ $\mu\text{m}$ ] | $\mu$ [GPa] | $\hat{\mu}$ [GPa] | $\sigma_0$ [MPa] | $h_0(x_2 = 0)$ [GPa] | $h'_0$ [GPa] | $N$ [-] | $\dot{\varepsilon}_0$ [ $\text{s}^{-1}$ ] |
| 1.5                   | 34          | 0                 | 100              | 5                    | -250         | 0.02    | $10^{-3}$                                 |

*Defect properties*

|                              |                          |                            |
|------------------------------|--------------------------|----------------------------|
| $d_1$<br>("ordinary" defect) | $d_2$<br>(larger defect) | $a$<br>(defect semi-width) |
| 98%                          | 97.5%                    | 0.01 $H$                   |

Thus, in the "non-defective" material, the value of  $\varepsilon_{12}^{PL}$  corresponding to zero hardening is  $\varepsilon_c = \sqrt{3}/100$ , and  $\Sigma_0 = 0$  when  $\varepsilon_{12}^{PL} = (1 + \sqrt{3})\varepsilon_c$ .

*Boundary conditions:*

$\Gamma(t) = 10^{-4} t$  (referring to displacement field (5.1)). Thus,  $\dot{\Gamma} = \dot{\varepsilon}_0/10$  (constant in time).

Also, the higher-order tractions are taken to be zero:  $\tau_{122}(\pm H) = 0$ .

## Results

The development of the plastic strain  $\varepsilon_{12}^{PL}(x_2, t)$  across the width of the strip, for values of the parameter  $\ell_D = \{0.01; 0.1; 1; 10; 100\} a$ , is shown in Fig. 7 in the case of two defects (as shown on the left of Fig. 6); the corresponding plots of the applied stress  $\sigma_{12}$  against the mean value of strain  $\bar{\varepsilon}_{12} = \Gamma(t)/2$  are shown in the top left figure, up to the points at which the curves dropped too steeply to allow the computation to proceed. The labels  $A, B, C$ , etc. identify points on the  $\sigma_{12}-\bar{\varepsilon}_{12}$  curves for which corresponding plastic strain profiles are reported; the labels  $Z_i$  ( $i = 1, \dots, 4$ ) identify the points where the simulation stops. The length scale  $\ell_D$  has no appreciable effect on the "macroscopic" stress-strain curve, except for delaying the final gross instability<sup>9</sup> where no further positive increment in boundary displacement  $\delta\Gamma$  could be imposed. The two individual defects are resolved when  $\ell_D \ll a$  but the smoothing effect of larger  $\ell_D$  is already evident when  $\ell_D = a$ . The very large value  $\ell_D = 100a$  ( $= H$ ) permits no perceptible non-uniformity of deformation across the strip so that, for any practical purpose, the state of uniform deformation is stable.

Similar trends are observed (Fig. 8) in the case of 32 defects, distributed evenly across the strip (as shown on the right in Fig. 6). Resolution of individual defects is (almost) lost for  $\ell_D = a$  but the uniform deformation is clearly unstable (also for  $\ell_D = 10a$ ), the perturbation tending to concentrate around the site of the worst defect.

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<sup>9</sup>more strictly, the failure of the computation.

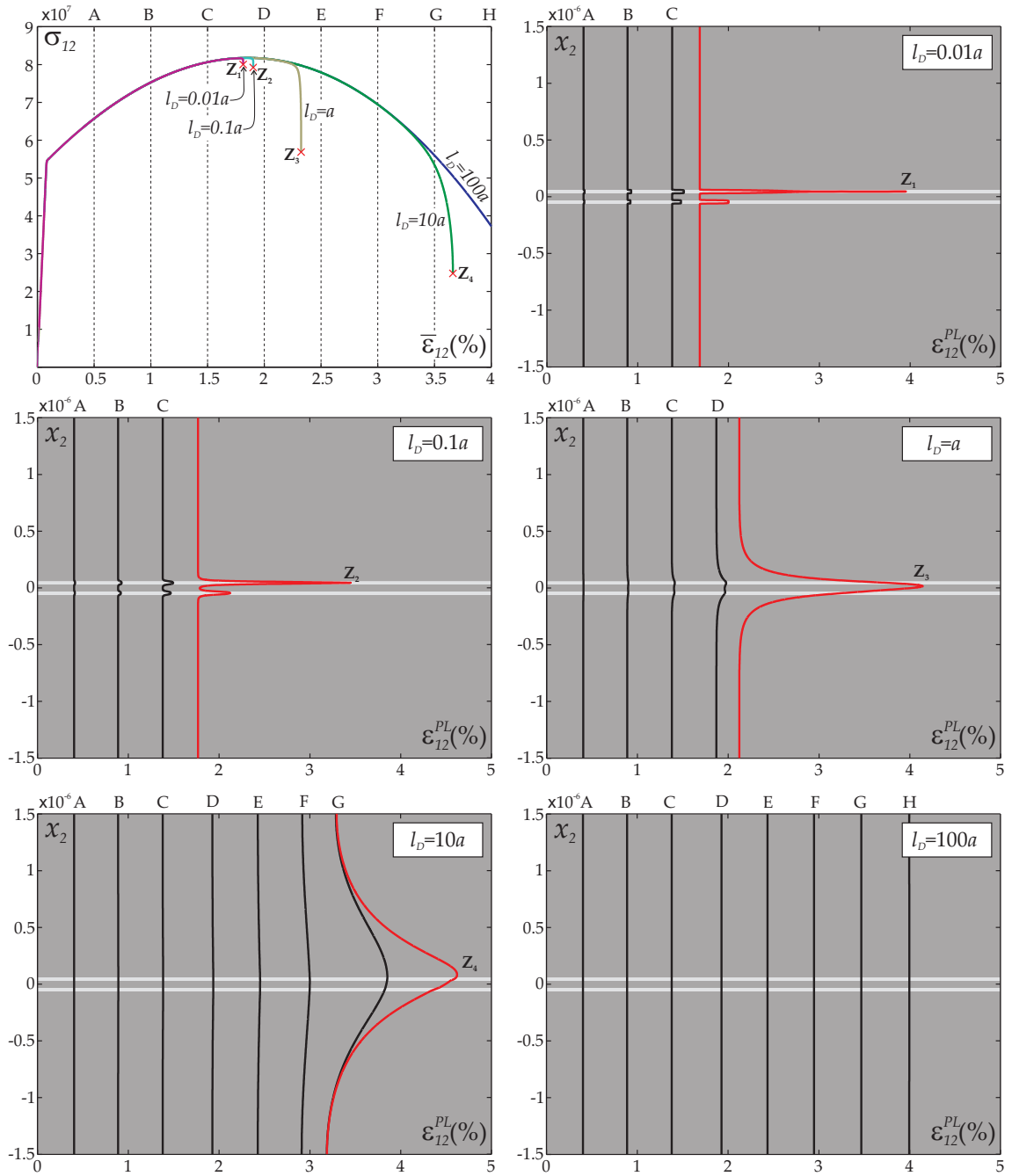


Figure 7: Development of the plastic strain  $\varepsilon_{12}^{PL}(x_2, t)$  across the width of the strip with increasing  $\Gamma(t)$ , in the presence of two defects, for various values of  $\ell_D$ . The first graph shows the corresponding applied stress  $\sigma_{12}$  versus mean strain  $\bar{\varepsilon}_{12}$  curves, up to the points that they become so unstable that computation cannot be continued.

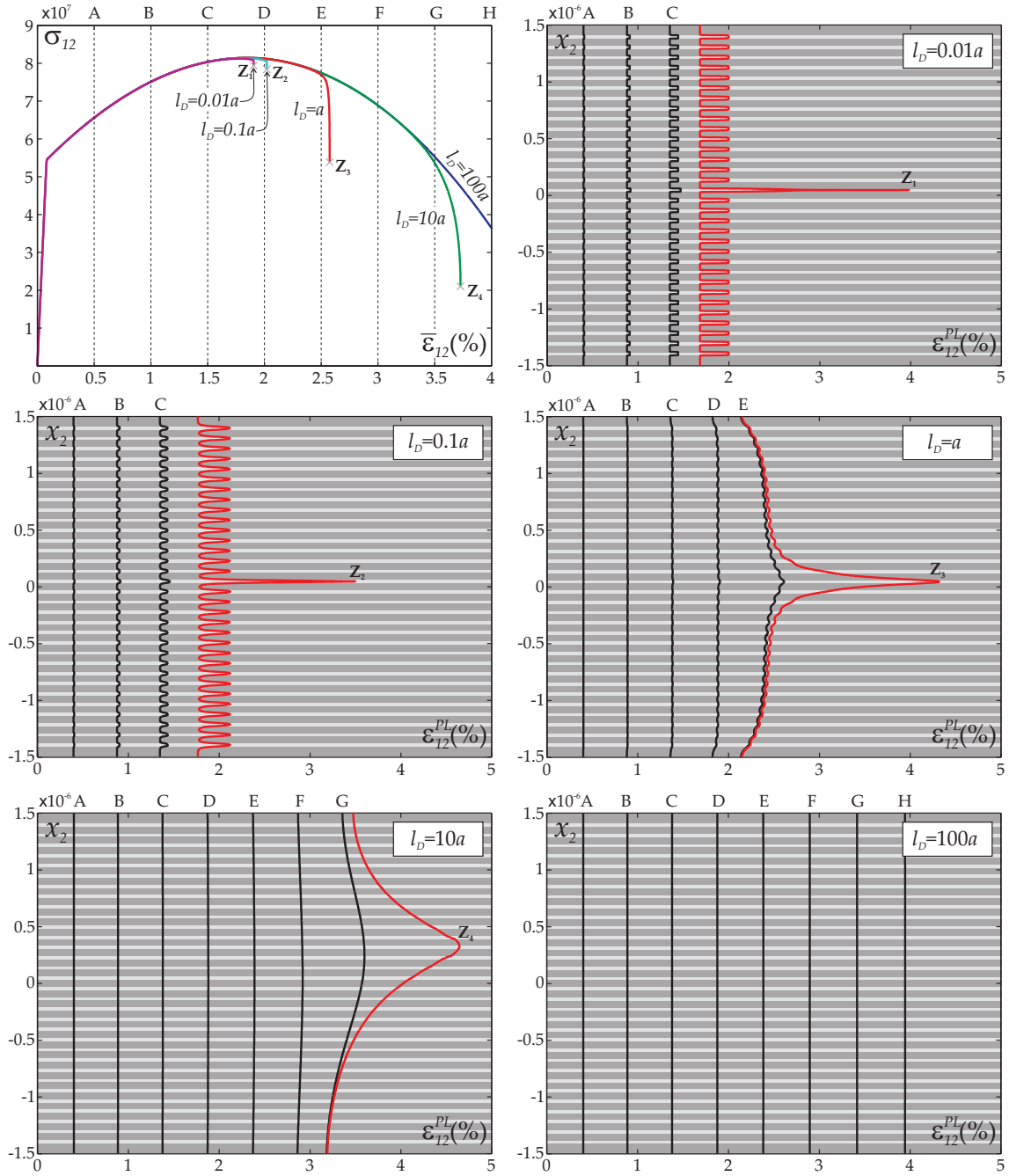


Figure 8: As for Fig. 7, except that there are 32 defects.

## 6 Discussion

This work has considered the stability of deformation of a medium that conforms to the Gudmundson/Fleck–Willis formulation of gradient plasticity and displays strain-softening behaviour. Rate-dependence was allowed for but the main emphasis was on rate-independent response. The approach was to consider a small perturbation of a time-dependent but spatially uniform state of deformation. The equations governing the perturbation were taken in linearized form. They are linear partial differential equations with coefficients that depend on time but are spatially uniform; hence they can be reduced, via Fourier analysis, to ordinary differential equations in time. They were studied explicitly in the case of simple shear deformation. When there are no gradient terms ( $\ell_E = \ell_D = 0$ ), the uniform deformation is unstable in the sense that any small perturbation tends to grow once the strain-softening regime is reached, so long as there is some rate-dependence. This is illustrated in Fig. 1. In the absence of rate-dependence, no perturbation can exist except at the exact instant that the medium is about to soften (the localization condition (4.27) is satisfied). In fact, beyond this point the problem becomes dynamically ill-posed; this is discussed further below. When just energetic gradient terms are admitted ( $\ell_E \neq 0$ ), a perturbation of infinite wavelength (zero wavenumber  $k$ ) grows as soon as condition (4.27) is met but the growth of a perturbation with wavenumber  $k > 0$  is delayed until condition (4.30) is met. The rate-independent limit again is singular, in the sense that a perturbation of wavenumber  $k$  can *only* exist at the instant that (4.30) is satisfied. In contrast, when dissipative gradient terms are present, the rate-independent limit is no longer singular (except at zero wavenumber), Fig. 3.

These phenomena have influence on the uniqueness and stability of deformation of finite bodies. This was illustrated in Section 5, which considered simple shear of a strip of finite width. In the case of purely energetic gradient terms, perturbations allowed by the boundary conditions have finite wavelength, and these cannot grow until condition (4.30) is met; thus, the onset of instability (more strictly, loss of uniqueness) is delayed. It is also possible, sufficiently after the point at which (4.30) is satisfied, to find perturbations for which the continued plastic deformation is localized within some interval (or intervals), with unloading occurring elsewhere (Fig. 4). Existence of all (quasi-static) solutions is lost when the material softens sufficiently steeply, consistent with Fig. 1. The analysis was restricted to considering a small increment of deformation; evidently, however, once the process of non-uniform deformation has started, regions in which the plastic deformation is greatest soften more and the tendency towards localization of the deformation is increasingly enhanced.

The results of computations were reported for monotonically-increasing simple shear of a finite strip composed of material with dissipative gradient terms and containing imperfections. A small amount of rate-dependence was admitted, for computational convenience, but the results that were obtained are considered to be essentially the same as rate-independent. The influence of the dissipative length scale  $\ell_D$  is less easy to assess in the context of perturbation theory but the computations show that in practice its effect is very similar to that of  $\ell_E$ . Figures 7 and 8 show how deformation increasingly tends to localize near the site of the



greatest imperfection, until final instability (i.e. loss of existence of the quasi-static solution) occurs, when it becomes impossible to impose any further boundary displacement.

Finally, brief mention will be made of the influence of dynamics on the development of a perturbation. In the case of simple shear, a perturbation is governed by the equation of motion,

$$\frac{\partial(\delta\sigma_{12})}{\partial x_2} = \rho \frac{\partial^2(\delta u_1)}{\partial t^2}, \quad (6.22)$$

and the constitutive relation

$$\begin{aligned} \delta\sigma_{12} &= \mu \left( \frac{\partial(\delta u_1)}{\partial x_2} - 2\delta\varepsilon_{12}^{PL} \right) \\ &= \frac{\Sigma_0}{\dot{\varepsilon}_0} \left( \frac{2\dot{\varepsilon}_{12}^{PL}}{\sqrt{3}\dot{\varepsilon}_0} \right)^{N-1} \left( \frac{2N}{3} - \ell_D^2 \frac{\partial^2}{\partial x_2^2} \right) \frac{\partial(\delta\varepsilon_{12}^{PL})}{\partial t} + \left[ \hat{\mu} + \frac{2h}{3} \left( \frac{2\dot{\varepsilon}_{12}^{PL}}{\sqrt{3}\dot{\varepsilon}_0} \right)^N - \bar{\mu}\ell_E^2 \frac{\partial^2}{\partial x_2^2} \right] \delta\varepsilon_{12}^{PL}, \end{aligned} \quad (6.23)$$

where the last equality comes from (4.23) with  $ik$  replaced by  $\partial/\partial x_2$ . Equations (6.22)-(6.23) lead to the following quasi-linear partial differential equation for  $\delta u_1(x_2, t)$

$$\begin{aligned} &\left\{ \frac{\Sigma_0}{\dot{\varepsilon}_0} \left( \frac{2\dot{\varepsilon}_{12}^{PL}}{\sqrt{3}\dot{\varepsilon}_0} \right)^{N-1} \left( \frac{2N}{3} - \ell_D^2 \frac{\partial^2}{\partial x_2^2} \right) \frac{\partial}{\partial t} + \left[ \hat{\mu} + \frac{2h}{3} \left( \frac{2\dot{\varepsilon}_{12}^{PL}}{\sqrt{3}\dot{\varepsilon}_0} \right)^N - \bar{\mu}\ell_E^2 \frac{\partial^2}{\partial x_2^2} \right] \right\} \\ &\quad \times \left( \frac{\partial^2(\delta u_1)}{\partial x_2^2} - \frac{\rho}{\mu} \frac{\partial^2(\delta u_1)}{\partial t^2} \right) = 2\rho \frac{\partial^2(\delta u_1)}{\partial t^2}, \end{aligned} \quad (6.24)$$

whose type is determined by the highest derivatives present. The perturbation in plastic strain follows from

$$\frac{\partial(\delta\varepsilon_{12}^{PL})}{\partial x_2} = \frac{1}{2} \left( \frac{\partial^2(\delta u_1)}{\partial x_2^2} - \frac{\rho}{\mu} \frac{\partial^2(\delta u_1)}{\partial t^2} \right). \quad (6.25)$$

Referring to the following families of curves in the  $x_2$ - $t$  plane:

$$\Upsilon_{1,2}(x_2, t) : \frac{dx_2}{dt} = \pm c_e, \quad \Upsilon_3(x_2, t) : dt = 0, \quad \Upsilon_4(x_2, t) : dx_2 = 0,$$

where  $c_e = \sqrt{\mu/\rho}$  is the velocity of the elastic shear waves, from considering the pde (6.24) we remark that the problem is well-posed in the following cases:

- If  $\ell_D \neq 0$ , the pde is of fifth order. It is totally hyperbolic, with characteristics  $\Upsilon_1$ ,  $\Upsilon_2$ ,  $\Upsilon_3$  (twice), and  $\Upsilon_4$ .
- If  $\ell_D = 0$  but  $\ell_E \neq 0$ , independently of  $N$ , the pde is of fourth order and its characteristics are  $\Upsilon_1$ ,  $\Upsilon_2$ , and  $\Upsilon_3$  (twice).
- If  $\ell_D = \ell_E = 0$  but  $N \neq 0$ , the pde is of third order and its characteristics are  $\Upsilon_1$ ,  $\Upsilon_2$ , and  $\Upsilon_4$ .

If all of  $\ell_D$ ,  $\ell_E$  and  $N$  are zero, the pde (6.24) is of second order, with characteristics

$$\frac{dx_2}{dt} = \pm c_e \sqrt{\frac{\hat{\mu} + 2h/3}{2\mu + \hat{\mu} + 2h/3}}, \quad (6.26)$$

showing that it becomes elliptic (so the problem is ill-posed) when inequality (5.7) holds.

Similarly to the quasi-static case treated in Sect. 4.2, looking for a perturbation in the displacement field in the form

$$\delta u_1(x_2, t) = U(t)e^{ikx_2}, \quad (6.27)$$

the pde (6.24) simplifies to the following third order ode for the function  $U(t)$ ,

$$\alpha(t)\ddot{U}(t) + [\beta(t) + 2\mu]\dot{U}(t) + c_e^2 k^2 \alpha(t)U(t) + c_e^2 k^2 \beta(t)U(t) = 0, \quad (6.28)$$

where

$$\alpha(t) = \frac{\Sigma_0}{\dot{\epsilon}_0} \left( \frac{2\dot{\epsilon}_{12}^{PL}}{\sqrt{3}\dot{\epsilon}_0} \right)^{N-1} \left( \frac{2N}{3} + \ell_D^2 k^2 \right), \quad \beta(t) = \hat{\mu} + \frac{2h}{3} \left( \frac{2\dot{\epsilon}_{12}^{PL}}{\sqrt{3}\dot{\epsilon}_0} \right)^N + \bar{\mu} \ell_E^2 k^2, \quad (6.29)$$

while equation (6.25) becomes

$$\frac{\partial (\delta \varepsilon_{12}^{PL})}{\partial x_2} = -\frac{1}{2} \left( \frac{1}{c_e^2} \ddot{U}(t) + k^2 U(t) \right) e^{ikx_2}. \quad (6.30)$$

Considering the case  $\ell_D = N = 0$ , since  $\alpha(t) = 0$ , the third order ode (6.28) degenerates to the second order equation

$$[\beta(t) + 2\mu]\ddot{U}(t) + c_e^2 k^2 \beta(t)U(t) = 0. \quad (6.31)$$

In the special case of a constant unperturbed state (i.e. no variation in time of the hardening  $h$  and therefore constant  $\beta$ ) and zero kinematic hardening,  $\hat{\mu} = 0$ , this ode gives the same (exponential) solution as was obtained by Sluys et al. (1993, their eqns. (28)–(32)). Indeed, their model (without introducing higher order stresses but considering only a yield function depending on the plastic strain and its second gradient) is equivalent to our case  $\ell_E \neq 0$  and null values for  $\ell_D$  and  $N$ .

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