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An Explicit Formula for the Minimum Free Energy in Linear Viscoelasticity ⋆

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Abstract. A general explicit formula for the maximum recoverable work from a given state is derived in the frequency domain for full tensorial isothermal linear viscoelastic constitutive equations. A variational approach, developed for the scalar case, is here generalized by virtue of certain factorizability properties of positive-definite matrices. The resultant formula suggests how to characterize the state in the sense of Noll in the frequency domain. The property that the maximum recoverable work represents the minimum free energy according to both Graffi's and Coleman–Owen's definitions is used to obtain an explicit formula for the minimum free energy. Detailed expressions are presented for particular types of relaxation function.

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1. Introduction

The free energy is a key concept in thermodynamics. In particular, for materials with fading memory, extensive discussions on the definition of the free energy and investigations on some explicit formulas have been given in the literature, notably [2, 6, 7, 9, 10, 13, 14, 18, 19, 27, 33].

Important in the history of the development of the subject are the pioneering papers of Breuer and Onat [1], [2] on the determination of the maximum recoverable work from a given strain history for a linear viscoelastic material whose relaxation function is a finite linear combination of (scalar) decaying exponentials. They identify this quantity as a lower bound of the free energy of the material. Also, in [2], they clearly expose the non-uniqueness in the definition of free energy for such materials.

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However, for materials with fading memory, the first systematic work on the free energy is due to Coleman [3], who derived a list of its properties as consequences of the second law of thermodynamics. Much earlier, Volterra [33] already recognized the validity of those properties as reasonable for a functional to be a free energy in one-dimensional isothermal linear viscoelasticity. The properties derived by Coleman were taken as the defining characteristics of a free energy by Graffi [20, 21] and later by others [13, 14, 16, 27]. For this reason henceforth they will be referred as to *Graffi's definition of the free energy*. It is worth recalling that, in such a context, no specification of the concept of *state* was required.

A different definition of the free energy can be found in the general theory of thermodynamics developed by Coleman and Owen [6]. In fact, its application to linear viscoelasticity leads to a definition of a free energy as a function of the state of the material, that is, a lower potential of the work done during processes starting from the current state [10]. In the sequel, we will refer to this as the *Coleman–Owen's definition of the free energy*. A comparison between the two different definitions cited above can be found in [9, 10].

However, as shown in [6, 19], whichever definition is considered, the free energy turns out to be non-unique: in other words the properties that a functional must satisfy in order to be taken as a free energy do not lead to a unique expression. Indeed, many functionals satisfying Graffi's definition for the free energy have been given in the literature [13, 14, 19, 21, 33].

From the conceptual point of view, an important step has been the proof that there exist a maximum and a minimum free energy according to both the definitions. The characterization of these thermodynamic potentials for functions satisfying Graffi's definition has been given in [27] and for the ones satisfying Coleman–Owen's definition in [6, 10]. In particular, explicit forms of the maximum free energy obeying the first definition has been shown to hold under rather general assumptions [27], whereas the maximum Coleman–Owen's free energy in linear viscoelasticity has been identified with the minimum work done to approach any given state starting from the natural one [10]. On the other hand, for both definitions, the minimum free energy has been identified with the *maximum recoverable work*, so that the problem of finding an explicit formula of the minimum free energy has been identified in the literature with finding the expression of that quantity, although no explicit formula for it has been found in the linear viscoelastic tensorial case or for more general materials.

Actually, Day [7] found an expression for the maximum recoverable work for a rather general class of symmetric relaxation functions. However Day's expression does not completely solve the problem. In fact, his formula for the maximum recoverable work depends on the 'optimal continuation' (referred to as the reversible extension in [7]) of the strain history, which is the solution of a Wiener–Hopf equation (see e.g. [31] and references cited therein). However, the explicit form of this quantity was not derived. Therefore Day's formula represents an interesting characterization of the expression for the maximum recoverable work rather than an explicit formula in terms of the given strain.

A general expression for the isothermal minimum free energy of a linear viscoelastic material has been given recently [18] for the scalar case, in the frequency domain.

In the present paper, the same problem is solved for the full general tensorial case. The method used here is the same as that in [18], namely a variational technique. However, the choice of functional to maximize is motivated by showing the equivalence of some alternative definitions of the maximum recoverable work. Moreover such a maximization in the tensorial case relies crucially on certain results due to Gohberg and Kreĭn [17] concerning the factorizability of Hermitean matrices.

The resultant expression is shown to be a function of state in the sense of Noll [29], recently formulated in the context of linear viscoelasticity [9, 10]. Moreover, it turns out to satisfy both the above cited definitions of the free energy.

Detailed, explicit formulae are given for the material responses associated with particular classes of tensorial discrete spectrum models.

The layout of the paper is as follows. In Section 2, some notation is established, while in Section 3 the constitutive relationship of the material is discussed, together with the concept of state. In Section 4 the maximum recoverable work from a given state is considered in detail. Factorization of a quantity closely related to the tensorial loss modulus is considered in Section 5, which allows the determination of a general expression for the maximum recoverable work in terms of Fourier-transformed quantities in Section 6, from a variational argument. A result on the characterization of states in the sense of Noll [29] for viscoelastic materials in the frequency domain is proved in Section 7, with the aid of which the maximum recoverable work is shown to be a function of the state. Since the minimum free energy ψ_m is identified with the maximum recoverable work, the results of Sections 6 and 7 refer to ψ_m as well. In Section 8, the expression found in Section 6 is shown to have the properties of a free energy according to Graffi's definition [20, 21]. In Section 9, such an expression is shown to be a free energy in the sense of Coleman and Owen [6], by using a suitable norm on the space of the states. Various choices of norm, including the free energy itself, are compared. Explicit results for particular relaxation functions are presented in Section 10.

2. Notations

Let Sym be the space of symmetric second order tensors acting on $\mathcal{R}^3$ viz. Sym $:= \{\mathbf{M} \in \mathrm{Lin}(\mathcal{R}^3) : \mathbf{M} = \mathbf{M}^\top\}$, where the superscript '$^\top$' denotes the transpose. Operating on Sym is the space of the fourth order tensors Lin(Sym).

It is well known that Sym is isomorphic to $\mathcal{R}^6$. In particular, for every $\mathbf{L}, \mathbf{M} \in$ Sym, if $\mathbf{C}_i$, $i = 1, \ldots, 6$ is an orthonormal basis of Sym with respect to the usual inner product in Lin($\mathcal{R}^3$), namely tr($\mathbf{L}\mathbf{M}^\top$), it is clear that the representation

$$\mathbf{L} = \sum_{i=1}^{6} L_i \mathbf{C}_i, \qquad \mathbf{M} = \sum_{i=1}^{6} M_i \mathbf{C}_i \tag{2.1}$$

is such that $\mathrm{tr}(\mathbf{L}\mathbf{M}^\top) = \sum_{i=1}^{6} L_i M_i$. Therefore, henceforth we treat each tensor of Sym as a vector in $\mathcal{R}^6$ and denote by $\mathbf{L} \cdot \mathbf{M}$ the inner product between elements of Sym, viz.

$$\mathbf{L} \cdot \mathbf{M} = \mathrm{tr}(\mathbf{L}\mathbf{M}^\top) = \mathrm{tr}(\mathbf{L}\mathbf{M}) = \sum_{i=1}^{6} L_i M_i$$

and $|\mathbf{M}|^2 = \mathbf{M} \cdot \mathbf{M}$. As a consequence [24] any fourth order tensor $\mathbb{K} \in$ Lin(Sym) will be identified with an element of Lin($\mathcal{R}^6$) by the representation

$$\mathbb{K} = \sum_{i,j=1}^{6} K_{ij} \mathbf{C}_i \otimes \mathbf{C}_j \tag{2.2}$$

and $\mathbb{K}^\top$ means the transpose of $\mathbb{K}$ as an element of Lin($\mathcal{R}^6$). According to (2.2), the norm $|\mathbb{K}|$ of $\mathbb{K} \in$ Lin(Sym) may be given by

$$|\mathbb{K}|^2 = \mathrm{tr}\left(\mathbb{K}\mathbb{K}^\top\right) = \left(\sum_{i,j=1}^{6} K_{ij} K_{ij}\right).$$

In the sequel we deal with complex valued tensors. We denote by Ω the complex plane and by Sym(Ω) and Lin(Sym(Ω)) the complexification respectively of Sym and Lin(Sym), namely the sets of the tensors represented respectively by the forms (2.1) and (2.2) with $L_i, M_i, K_{ij} \in \Omega$. The norms $|\mathbf{M}|$ and $|\mathbb{K}|$ of $\mathbf{M} \in$ Sym(Ω) and $\mathbb{K} \in$ Lin(Sym(Ω)) will be given respectively by

$$|\mathbf{M}|^2 = \left(\mathbf{M} \cdot \overline{\mathbf{M}}\right), \qquad |\mathbb{K}|^2 = \mathrm{tr}\left(\mathbb{K}\mathbb{K}^*\right) = \left(\sum_{i,j=1}^{6} K_{ij} \overline{K}_{ij}\right), \tag{2.3}$$

where the overhead bar indicates complex conjugate and $\mathbb{K}^* = \overline{\mathbb{K}}^\top$ is the Hermitian conjugate.

The above representations allows results of [17] to be easily extended to tensors belonging to Lin(Sym(Ω)).

The symbols $\mathcal{R}^+$ and $\mathcal{R}^{++}$ denote the nonnegative reals and the strictly positive reals, respectively, while $\mathcal{R}^-$ and $\mathcal{R}^{--}$ denote the nonpositive and strictly negative reals.

For any function $f\colon \mathcal{R} \to \mathcal{V}$, where $\mathcal{V}$ is a finite-dimensional vector space, in particular in the present context Sym or Lin(Sym), let f_F denote its *Fourier transform* viz. $f_F(\omega) := \int_{-\infty}^{\infty} f(s)\,\mathrm{e}^{-i\omega s}\,\mathrm{d}s$. Also, we define

$$f_+(\omega) := \int_0^\infty f(s)\,\mathrm{e}^{-i\omega s}\,\mathrm{d}s, \qquad f_-(\omega) := \int_{-\infty}^0 f(s)\,\mathrm{e}^{-i\omega s}\,\mathrm{d}s, \tag{2.4}$$

$$f_s(\omega) := \int_0^\infty f(s)\sin\omega s\,\mathrm{d}s, \qquad f_c(\omega) := \int_0^\infty f(s)\cos\omega s\,\mathrm{d}s. \tag{2.5}$$

The relation defining f_F, (2.4) and (2.5) are to be understood as applying to each component of the tensor quantities involved. The existence of the Fourier transforms is ensured if we assume that $f \in L^1(\mathcal{R},\mathcal{V}) \cup L^2(\mathcal{R},\mathcal{V})$ (or $f \in L^1(\mathcal{R}^\pm,\mathcal{V}) \cup L^2(\mathcal{R}^\pm,\mathcal{V})$ in the case of $f_\pm$). Also, recall that if $f \in L^2(\mathcal{R},\mathcal{V})$, then $f_F \in L^2(\mathcal{R},\mathcal{V})$ [30, 32]. Further assumptions will be imposed in the next section.

When $f\colon \mathcal{R}^+ \to \mathcal{V}$ we can always extend the domain of f to $\mathcal{R}$, by considering its *causal* extension *viz.*

$$f(s) = \begin{cases} f(s) & \text{for } s \geqslant 0, \\ 0 & \text{for } s < 0, \end{cases} \tag{2.6}$$

in which case

$$f_F(\omega) = f_+(\omega) = f_c(\omega) - i f_s(\omega).$$

We shall need to consider the Fourier transform of functions that do not go to zero at large times and thus do not belong to $L^1(\mathcal{R},\mathcal{V}) \cup L^2(\mathcal{R},\mathcal{V})$. In particular, let $f(s)$ in (2.6) be given by a constant $a \in \mathcal{V}$ for all s. The standard procedure is adopted of introducing an exponential decay factor, calculating the Fourier transform and then letting the time decay constant tend to infinity. Thus, we obtain

$$f_+(\omega) = \frac{a}{i\omega^-}, \qquad \omega^\pm = \lim_{\alpha\to 0^+}(\omega \pm i\alpha). \tag{2.7}$$

The corresponding result for a constant function defined on $\mathcal{R}^-$ is obtained by taking the complex conjugates of this relationship. Also, if f is a function defined on $\mathcal{R}^-$ and if $\lim_{s\to-\infty} f(s) = b \in \mathcal{V}$ where the components of the function $g\colon \mathcal{R}^- \to \mathcal{V}$ defined by $g(s) = f(s) - b$ belong to $L^2(\mathcal{R}^-,\mathcal{V})$, then

$$f_F(\omega) = g_F(\omega) - \frac{b}{i\omega^+}. \tag{2.8}$$

Again, taking complex conjugates gives the result for functions defined on $\mathcal{R}^+$. This procedure amounts to defining the Fourier transform of such functions as the limit of the transforms of a sequence of functions in $L^2(\mathcal{R}^-, \mathcal{V})$. The limit is to be taken after integrations over ω are carried out if the ω^{-1} results in a singularity in the integrand. Generally, in the present application, the ω^{-1} produces no such singularity and the limiting process is redundant.

The complex plane, denoted by Ω, will play an important role in our discussions. We define the following sets:

$$\Omega^+ = \{\zeta \in \Omega : \Im_m \zeta \geqslant 0\}, \qquad \Omega^{(+)} = \{\zeta \in \Omega : \Im_m \zeta > 0\}.$$

Analogous meanings are assigned to Ω^- and $\Omega^{(-)}$.

The quantities $f_\pm$ defined by (2.4) are analytic in $\Omega^{(\mp)}$ respectively. This analyticity is extended by assumption to $\Omega^\mp$. The function f_+ may be defined by (2.4) and analytic on a portion of Ω^+ if for example f decays exponentially at large times. Over the remaining portion of Ω^+, on which the integral definition is meaningless, f_+ is defined by analytic continuation.

3. Relaxation Functions, Histories, Processes and States

A linear viscoelastic material is described by the classical Boltzmann–Volterra constitutive equation between the second order symmetric stress tensor $\mathbf{T}(t) \in \mathrm{Sym}$ and the second order symmetric strain tensor $\mathbf{E}$: $(-\infty, t] \to \mathrm{Sym}$, of the form

$$\mathbf{T}(t) = \mathbb{G}_0\mathbf{E}(t) + \int_0^\infty \dot{\mathbb{G}}(s)\mathbf{E}(t-s)\,\mathrm{d}s. \tag{3.1}$$

The fourth order tensor valued function $\dot{\mathbb{G}}$: $\mathcal{R}^+ \to \mathrm{Lin(Sym)}$ is assumed to belong to $L^1(\mathcal{R}^+, \mathrm{Lin(Sym)}) \cap L^2(\mathcal{R}^+, \mathrm{Lin(Sym)})$ and so is integrable. Its primitive, the *relaxation function* $\mathbb{G}$: $\mathcal{R}^+ \to \mathrm{Lin(Sym)}$ is absolutely continuous and can be defined as

$$\mathbb{G}(t) := \mathbb{G}_0 + \int_0^t \dot{\mathbb{G}}(s)\,\mathrm{d}s,$$

where $\mathbb{G}_0 = \mathbb{G}(0)$ is the *instantaneous elastic modulus*. Moreover there exists the limit

$$\mathbb{G}_\infty := \lim_{t\to\infty} \mathbb{G}(t) \in \mathrm{Lin(Sym)},$$

where $\mathbb{G}_\infty$ is the *equilibrium elastic modulus*. The following property is assumed

$$0 < \left|\int_0^\infty s\dot{\mathbb{G}}(s)\,\mathrm{d}s\right| < \infty. \tag{3.2}$$

We will later make use of $\check{\mathbb{G}}$: $\mathcal{R}^+ \to \mathrm{Lin(Sym)}$, defined as

$$\check{\mathbb{G}}(t) := \mathbb{G}(t) - \mathbb{G}_\infty\,. \tag{3.3}$$

It is easy to check that (3.2) is equivalent to

$$0 < \left| \int_0^\infty \check{\mathbb{G}}(s)\, \mathrm{d}s \right| < \infty. \tag{3.4}$$

The Fourier transform of $\dot{\mathbb{G}}(t)$, namely $\dot{\mathbb{G}}_F(\omega) = \dot{\mathbb{G}}_c(\omega) - i\dot{\mathbb{G}}_s(\omega)$, for real ω, belongs to $L^2(\mathcal{R}, \mathrm{Lin}(\mathrm{Sym}(\Omega)))$, according to our earlier assumptions. It is clear that $\dot{\mathbb{G}}_c(\omega)$ is even as a function of ω and $\dot{\mathbb{G}}_s(\omega)$ is odd. The quantity $\dot{\mathbb{G}}_s(\omega)$ therefore vanishes at the origin. In fact, as a consequence of our assumption of analyticity of Fourier transformed quantities on the real axis of Ω, it vanishes at least linearly at the origin. The leftmost inequality in (3.2) implies that it vanishes no more strongly than linearly. It is worth noting that the rightmost inequality in (3.2) follows from the assumption that $\dot{\mathbb{G}}_F$ is analytic (and therefore differentiable) at the origin.

Thermodynamic properties of the linear viscoelastic materials imply (see [15, 16] and references therein)

$$\mathbb{G}_0 = \mathbb{G}_0^\top, \qquad \mathbb{G}_\infty = \mathbb{G}_\infty^\top$$

and

$$\dot{\mathbb{G}}_s(\omega)\mathbf{E} \cdot \mathbf{E} < 0 \quad \forall \mathbf{E} \in \mathrm{Sym} \setminus \{\mathbf{0}\} \quad \forall \omega \in \mathcal{R}^{++}. \tag{3.5}$$

An important consequence of (3.5) is [16]

$$\dot{\mathbb{G}}(0)\mathbf{E} \cdot \mathbf{E} \leqslant 0 \quad \forall \mathbf{E} \in \mathrm{Sym} \setminus \{\mathbf{0}\} \tag{3.6}$$

and also the following formula [16]

$$\mathbb{G}_\infty - \mathbb{G}_0 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\dot{\mathbb{G}}_s(\omega)}{\omega}\, \mathrm{d}\omega\ , \tag{3.7}$$

which leads to the inequality

$$\mathbb{G}_0\mathbf{E} \cdot \mathbf{E} > \mathbb{G}_\infty\mathbf{E} \cdot \mathbf{E} \ \ \forall \mathbf{E} \in \mathrm{Sym} \setminus \{\mathbf{0}\}. \tag{3.8}$$

Here we assume a stronger relation than (3.6), namely

$$\dot{\mathbb{G}}(0)\mathbf{E} \cdot \mathbf{E} < 0, \quad \forall \mathbf{E} \in \mathrm{Sym} \setminus \{\mathbf{0}\} \tag{3.9}$$

and, on the basis of arguments presented in [23, 8], the equilibrium elastic modulus of a viscoelastic solid is assumed to be positive definite, i.e.

$$\mathbb{G}_\infty\mathbf{E} \cdot \mathbf{E} > 0, \quad \forall \mathbf{E} \in \mathrm{Sym} \setminus \{\mathbf{0}\}.$$

For simplicity, in this paper we let $\mathbb{G}(t)$ be symmetric for all values of t, even though this assumption will be explicitly invoked only from Section 6 onwards. Moreover, for the result of Section 5, we need $\ddot{\mathbb{G}}$ to exist and to be integrable (so that it belongs to $L^1(\mathcal{R}^+, \mathrm{Lin(Sym)})$ but not necessarily $L^2(\mathcal{R}^+, \mathrm{Lin(Sym)})$).

Any strain tensor $\mathbf{E}$: $(-\infty, t] \to \mathrm{Sym}$ can be associated with the couple $(\mathbf{E}(t), \mathbf{E}^t)$ where $\mathbf{E}(t) \in \mathrm{Sym}$ denotes the *instantaneous* or *current value* of the strain and $\mathbf{E}^t$: $\mathcal{R}^{++} \to \mathrm{Sym}$ denotes the *past history* defined as

$$\mathbf{E}^t(s) := \mathbf{E}(t-s), \quad s > 0.$$

In order that the stress $\mathbf{T}$ be well defined, and to allow a definition of equivalent states, all the admissible past histories must belong to the following vector space (see [10])

$$\Gamma := \left\{ \mathbf{E}^t\colon \mathcal{R}^{++} \to \mathrm{Sym};\ \left| \int_0^\infty \dot{\mathbb{G}}(s+\tau)\mathbf{E}^t(s)\,\mathrm{d}s \right| < \infty \quad \forall \tau \geqslant 0 \right\}. \quad (3.10)$$

Further, to ensure the existence of Fourier transforms and their time derivatives, and to allow the application of Plancherel's theorem, we assume

$$\mathbf{E}^t \in L^1(\mathcal{R}^{++}, \mathrm{Sym}) \cap L^2(\mathcal{R}^{++}, \mathrm{Sym}) \cap SBV(\mathcal{R}^{++}, \mathrm{Sym}),$$

and continuous from the left. The meaning of the space SBV is recalled in the appendix at the end of the paper.

The Boltzmann–Volterra constitutive equation (3.1) defines the linear functional $\tilde{\mathbf{T}}$: $\mathrm{Sym} \times \Gamma \to \mathrm{Sym}$ such that

$$\tilde{\mathbf{T}}(\mathbf{E}(t), \mathbf{E}^t) := \mathbb{G}_0\mathbf{E}(t) + \int_0^\infty \dot{\mathbb{G}}(s)\mathbf{E}^t(s)\,\mathrm{d}s. \quad (3.11)$$

The integral in (3.11) can be extended to $\mathcal{R}$ by identifying $\dot{\mathbb{G}}$ with its odd extension and taking $\mathbf{E}^t$ to be zero on $\mathcal{R}^{--}$. Then, noting that $\dot{\mathbb{G}}_F(\omega) = -2i\dot{\mathbb{G}}_s(\omega)$, Plancherel's theorem yields

$$\tilde{\mathbf{T}}(\mathbf{E}(t), \mathbf{E}^t) = \mathbb{G}_0\mathbf{E}(t) + \frac{i}{\pi}\int_{-\infty}^\infty \dot{\mathbb{G}}_s(\omega)\mathbf{E}^t_+(\omega)\,\mathrm{d}s, \quad (3.12)$$

which is equivalent to a relation given in [13].

REMARK 3.1. Given the couple $(\mathbf{E}(t), \mathbf{E}^t)$ and the strain continuation defined by $\mathbf{E}(t+a) = \mathbf{E}(t),\ \forall a \in \mathcal{R}^+$, it is easy to check that the related stress is given by

$$\mathbf{T}(t+a) = \mathbb{G}(a)\mathbf{E}(t) + \int_0^\infty \dot{\mathbb{G}}(s+a)\mathbf{E}^t(s)\,\mathrm{d}s.$$

It has been shown ([10], Proposition 2.2(ii)) that the integrability property $\dot{\mathbb{G}} \in L^1(\mathcal{R}^+, \mathrm{Lin(Sym)})$ ensures that, for every $\varepsilon > 0$, there exists $a(\varepsilon, \mathbf{E}^t)$ sufficiently large such that

$$\left| \int_0^\infty \dot{\mathbb{G}}(s+a)\mathbf{E}^t(s)\,\mathrm{d}s \right| < \varepsilon, \quad \forall a > a(\varepsilon, \mathbf{E}^t). \tag{3.13}$$

Therefore, (3.13) can be thought of as an equivalent form of the fading memory property. It follows that $\lim_{a\to\infty} \mathbf{T}(t+a) = \mathbb{G}_\infty \mathbf{E}(t)$.

The concepts of process and state for a linear viscoelastic solid have been discussed by various authors [10, 22, 29]. We briefly recall some basic points.

REMARK 3.2. According to the definition given in [16], for each fixed couple $(\mathbf{E}(t), \mathbf{E}^t) \in \mathrm{Sym} \times \Gamma$, related to the strain $\mathbf{E}\colon (-\infty, t] \to \mathrm{Sym}$, a process P of finite duration d, acting on $\mathbf{E}$, is defined as the function $\dot{\mathbf{E}}_P\colon (0,d) \to \mathrm{Sym}$ associated with any $\mathbf{E}_P \in \mathrm{SBV}([0,d], \mathrm{Sym})$ (see the Appendix at the end of the paper). In particular we remember that, by representation (A.4), (A.5), we have

$$\mathbf{E}_P(\tau) = \mathbf{E}(t) + \int_0^\tau \dot{\mathbf{E}}_P(s)\,\mathrm{d}s \quad \tau \in [0,d], \tag{3.14}$$

with the properties

$$\lim_{\tau\to 0^+} \mathbf{E}_P(\tau) = \mathbf{E}(t), \qquad \lim_{s\to\tau^+} \mathbf{E}_P(s) = \mathbf{E}_P(\tau) \quad \forall \tau \in (0,d).$$

The strain $\mathbf{E}_f(\tau') = (\mathbf{E}_P * \mathbf{E})(\tau')$, $\tau' \leqslant t+d$, yielded by $\mathbf{E}^t$ and $\dot{\mathbf{E}}_P$, is given by

$$\begin{aligned} \mathbf{E}_f(t+d-s) &= (\mathbf{E}_P * \mathbf{E})(t+d-s) \\ &:= \begin{cases} \mathbf{E}_P(d-s) & 0 \leqslant s < d, \\ \mathbf{E}(t+d-s) & s \geqslant d. \end{cases} \end{aligned} \tag{3.15}$$

Thus, $\mathbf{E}_f$ is related to the couple $(\mathbf{E}_P(d), (\mathbf{E}_P * \mathbf{E})^{t+d})$.

We denote by Π the set of all processes of finite duration.

REMARK 3.3. Observe that, according to [10], a process P of duration d can also be defined by a couple $P := (\mathbf{B}, \mathbf{K})$ with

$$\mathbf{B} = \mathbf{E}_P(d) - \mathbf{E}(t), \qquad \mathbf{K}(d-\tau) = \mathbf{E}_P(\tau) - \mathbf{E}(t) \quad \tau \in (0,d), \tag{3.16}$$

with the properties

$$\lim_{\tau\to d^-} \mathbf{K}(\tau) = 0, \qquad \lim_{s\to\tau^-} \mathbf{K}(s) = \mathbf{K}(\tau) \quad \forall \tau \in (0,d).$$

For sake of precision we must say that in [10] the first term of the couple is **K** acting on the past history of strain, and the second one is **B** acting on the current value of the strain; according to our choice, in this paper the first term is the one relative to the current value.

It is easy to check that such a definition is consistent with the previous one, in fact (3.15) is equivalent to (3.7) and (3.8) of [10] in view of (3.16).

DEFINITION 3.1. Two pairs $(\mathbf{E}_1(t), \mathbf{E}_1^t)$ and $(\mathbf{E}_2(t), \mathbf{E}_2^t)$ are said to be equivalent if for every $\dot{\mathbf{E}}_P\colon (0,\tau] \to \mathrm{Sym}$ and for every $\tau \geqslant 0$, they satisfy [11]

$$\tilde{\mathbf{T}}(\mathbf{E}_{P1}(\tau), (\mathbf{E}_{P1} * \mathbf{E}_1)^{t+\tau}) = \tilde{\mathbf{T}}(\mathbf{E}_{P2}(\tau), (\mathbf{E}_{P2} * \mathbf{E}_2)^{t+\tau}), \tag{3.17}$$

where $\mathbf{E}_{P1}, \mathbf{E}_{P2}$ are given by (3.14) for $\mathbf{E}(t)$ replaced by $\mathbf{E}_1(t), \mathbf{E}_2(t)$ respectively.

It is easy to show that (3.17) holds if and only if $\mathbf{E}_1(t) = \mathbf{E}_2(t)$ and $\mathbf{E}^t = \mathbf{E}_1^t - \mathbf{E}_2^t$ satisfies

$$\int_\tau^\infty \dot{\mathbb{G}}(s)\mathbf{E}^{t+\tau}(s)\,\mathrm{d}s = \int_0^\infty \dot{\mathbb{G}}(s+\tau)\mathbf{E}^t(s)\,\mathrm{d}s = 0 \quad \forall\tau \geqslant 0. \tag{3.18}$$

In fact (3.17) is equivalent to

$$\begin{aligned}&\tilde{\mathbf{T}}(\mathbf{E}(t), (\mathbf{E}(t)^\dagger * \mathbf{E})^{t+\tau})\\ &\quad = \mathbb{G}(\tau)\mathbf{E}(t) + \int_0^\infty \dot{\mathbb{G}}(s+\tau)\mathbf{E}^t(s)\,\mathrm{d}s = 0,\end{aligned} \tag{3.19}$$

where

$$(\mathbf{E}(t)^\dagger * \mathbf{E})^{t+\tau}(s) = \begin{cases} \mathbf{E}(t) & s \leqslant \tau, \\ \mathbf{E}^t(s-\tau) & s > \tau. \end{cases}$$

We deduce that $\mathbf{E}(t)$ must vanish by recalling (3.13). According to the definition of the state σ given by Noll [29], two couples $(\mathbf{E}_1(t), \mathbf{E}_1^t)$ and $(\mathbf{E}_2(t), \mathbf{E}_2^t)$ that are equivalent in the sense of Definition 3.1 are represented by the same state $\sigma(t)$. In this sense, $\sigma(t)$ may be thought as the 'minimum' set of variables allowing a univocal relation between $\dot{\mathbf{E}}_P\colon [0,\tau) \to \mathrm{Sym}$ and the stress $\mathbf{T}(t+\tau) = \tilde{\mathbf{T}}(\mathbf{E}_P(\tau), (\mathbf{E}_P * \mathbf{E})^{t+\tau})$ for every $\tau \geqslant 0$.

Also, Equation (3.18) represents an equivalence relation between past histories, in the sense that $\mathbf{E}_1^t$ and $\mathbf{E}_2^t$ are said to be equivalent if their difference $\mathbf{E}^t = \mathbf{E}_1^t - \mathbf{E}_2^t$ satisfies (3.18).

Therefore [10, 22], denoting with Γ_0 the set of all the past histories of Γ satisfying (3.18), and by Γ/Γ_0 the usual quotient space, the state σ of a linear viscoelastic material is an element of

$$\Sigma := \mathrm{Sym} \times (\Gamma/\Gamma_0). \tag{3.20}$$

In particular the natural state σ_0 is the null element of Σ representing the equivalence class of any couple $(\mathbf{0}, \mathbf{h})$ with $\mathbf{h} \in \Gamma_0$. Henceforth, we also view a process as a function $P\colon \Sigma \to \Sigma$ which associates with an initial state $\sigma^i \in \Sigma$ a final state $P\sigma^i = \sigma^f \in \Sigma$.

It is apparent that, by virtue of (3.10) and Definition 3.1, the space of the states Σ depends on the properties of the material because the memory kernel $\dot{\mathbb{G}}$ occurs both in (3.10) and in the constitutive equation (3.1). This property distinguishes (3.20) from the usual fading memory spaces [4, 5]

4. Work and Recoverable Work

The work done on the material by the strain history $\mathbf{E}(\tau)$, $\tau \leqslant t$ is

$$\begin{aligned}\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t) &:= \int_{-\infty}^{t} \mathbf{T}(\tau)\cdot\dot{\mathbf{E}}(\tau)\,\mathrm{d}\tau \\ &= \tfrac{1}{2}\mathbb{G}_0\mathbf{E}(t)\cdot\mathbf{E}(t) + \int_{-\infty}^{t}\int_0^{\infty} \dot{\mathbb{G}}(s)\mathbf{E}^{\tau}(s)\cdot\dot{\mathbf{E}}(\tau)\,\mathrm{d}s\,\mathrm{d}\tau. \qquad (4.1)\end{aligned}$$

It will be clear from the representation of $\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t)$ in the frequency domain, given in Section 6, that it is a nonnegative quantity. We will restrict our considerations to histories such that $\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t) < \infty$.

Given the time t, the state $\sigma(t)$ and the process P of duration d so that $P\sigma$ is related to the strain $\mathbf{E}\colon (-\infty, t+d] \to \mathrm{Sym}$ according to (3.15), the work done from t to $t+d$ is given by

$$\begin{aligned}W(\sigma(t), P) &:= \int_t^{t+d} \mathbf{T}(\tau)\cdot\dot{\mathbf{E}}(\tau)\,\mathrm{d}\tau \\ &= \tfrac{1}{2}\mathbb{G}_0\mathbf{E}(t+d)\cdot\mathbf{E}(t+d) - \tfrac{1}{2}\mathbb{G}_0\mathbf{E}(t)\cdot\mathbf{E}(t) \\ &\quad + \int_t^{t+d}\int_0^{\infty} \dot{\mathbb{G}}(s)\mathbf{E}^{\tau}(s)\cdot\dot{\mathbf{E}}(\tau)\,\mathrm{d}s\,\mathrm{d}\tau. \qquad (4.2)\end{aligned}$$

The properties of the work have been extensively studied in [10]. In order to justify the definition of W as a function of the initial state σ and the process P, it is worth noting that two strains $(\mathbf{E}_1(t), \mathbf{E}_1^t)$ and $(\mathbf{E}_2(t), \mathbf{E}_2^t)$ are equivalent, in the sense of Definition 3, if and only if

$$\begin{aligned}&\int_t^{t+d} \tilde{\mathbf{T}}(\mathbf{E}_{P1}(\tau-t), (\mathbf{E}_{P1} * \mathbf{E}_1)^{\tau})\cdot\dot{\mathbf{E}}_P(\tau-t)\,\mathrm{d}\tau \\ &\quad = \int_t^{t+d} \tilde{\mathbf{T}}(\mathbf{E}_{P2}(\tau-t), (\mathbf{E}_{P2} * \mathbf{E}_2)^{\tau})\cdot\dot{\mathbf{E}}_P(\tau-t)\,\mathrm{d}\tau \qquad (4.3)\end{aligned}$$

holds for every $\mathbf{E}_P$: $(0, d] \to \mathrm{Sym}$ and for every $d > 0$. In fact (4.3) follows from (3.17). On the other hand, (4.3) may be rewritten as

$$\int_0^d \tilde{\mathbf{T}}(\mathbf{E}(t), (\mathbf{E}(t)^\dagger * \mathbf{E})^{t+\tau}) \cdot \dot{\mathbf{E}}_P(\tau)\,\mathrm{d}\tau = 0. \tag{4.4}$$

Since $\dot{\mathbf{E}}_P$ and $d > 0$ are arbitrary, (4.4) implies (3.19). The same conclusion has been reached in [10, Sec. 4].

We define the *maximum recoverable work* starting from the state σ as

$$W_R(\sigma) := \sup_{P \in \Pi}\{-W(\sigma, P)\}. \tag{4.5}$$

It follows from (4.3) that the quantity $W_R(\sigma)$ is a function of state in the sense that two different strain histories representing the same state σ admit the same maximum recoverable work.

REMARK 3.1. It has been shown [9, 10] that the work $\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t)$, defined in (4.1), is not a function of state unless the state σ is represented by the whole strain history i.e. $\Gamma_0 = \{\mathbf{0}^\dagger\}$.

We now prove the equivalence of certain alternative definitions of maximum recoverable work to that given by (4.5). Let $\Pi_0(\sigma)$ denote the following set

$$\Pi_0(\sigma) := \{P \in \Pi : P\sigma = (\mathbf{0}, \mathbf{h}) \quad \text{for any } \mathbf{h} \in \Gamma/\Gamma_0\}.$$

LEMMA 4.1. *Given a state $\sigma(t)$ related to the couple $(\mathbf{E}, \mathbf{E}^t)$, (for simplicity here and below $\mathbf{E} := \mathbf{E}(t)$) for every $\varepsilon > 0$ there exists two positive parameters a, r, such that the process $P_{a,r} \in \Pi_0(\sigma(t))$, of duration $a + r$ related to*

$$\mathbf{E}_P(\tau) = \begin{cases} \mathbf{E} & \text{for } \tau \in (0, a], \\ \mathbf{E} - (\mathbf{E}/r)(\tau - a) & \text{for } \tau \in (a, a + r], \end{cases} \tag{4.6}$$

yields work $W(\sigma(t), P_{a,r})$ satisfying

$$-W(\sigma(t), P_{a,r}) > \tfrac{1}{2}\mathbb{G}_\infty \mathbf{E} \cdot \mathbf{E} - \frac{\varepsilon}{2}. \tag{4.7}$$

Proof. We see from (4.2) and (3.15) that

$$\begin{aligned} &-W(\sigma(t), P_{a,r}) \\ &\quad = \tfrac{1}{2}\mathbb{G}_0 \mathbf{E} \cdot \mathbf{E} + \int_0^r \int_0^\infty \dot{\mathbb{G}}(s)\mathbf{E}(t + a + \tau - s)\,\mathrm{d}s\,\mathrm{d}\tau \cdot \frac{\mathbf{E}}{r} \\ &\quad = \tfrac{1}{2}\mathbb{G}_0 \mathbf{E} \cdot \mathbf{E} + \int_0^r \int_0^\tau \dot{\mathbb{G}}(s)\left[\mathbf{E} - \frac{\mathbf{E}}{r}(\tau - s)\right]\mathrm{d}s\,\mathrm{d}\tau \cdot \frac{\mathbf{E}}{r} \end{aligned}$$

$$+\int_0^r \int_\tau^{\tau+a} \dot{\mathbb{G}}(s)\,\mathrm{d}s\,\mathrm{d}\tau \mathbf{E}\cdot\frac{\mathbf{E}}{r}$$

$$+\int_0^r \int_{\tau+a}^{\infty} \dot{\mathbb{G}}(s)\mathbf{E}(t+a+\tau-s)\,\mathrm{d}s\,\mathrm{d}\tau\cdot\frac{\mathbf{E}}{r}. \tag{4.8}$$

Integrating terms of (4.8), making use of (3.3), we have

$$\begin{aligned}
&-W(\sigma(t), P_{a,r})\\
&\quad=\tfrac{1}{2}\mathbb{G}_0\mathbf{E}\cdot\mathbf{E}-\tfrac{1}{2}\check{\mathbb{G}}_0\mathbf{E}\cdot\mathbf{E}\int_0^r\int_0^\tau \check{\mathbb{G}}(s)\,\mathrm{d}s\,\mathrm{d}\tau\frac{\mathbf{E}}{r}\cdot\frac{\mathbf{E}}{r}\\
&\qquad+\int_0^r \check{\mathbb{G}}(\tau+a)\;\mathrm{d}\tau\mathbf{E}\cdot\frac{\mathbf{E}}{r}\\
&\qquad+\int_0^r\int_0^\infty \dot{\mathbb{G}}(s+\tau+a)\mathbf{E}^t(s)\,\mathrm{d}s\,\mathrm{d}\tau\cdot\frac{\mathbf{E}}{r},
\end{aligned}$$

where $\check{\mathbb{G}}_0=\mathbb{G}_0-\mathbb{G}_\infty$. By virtue of (3.4) the inequality

$$\left|\int_0^r\int_0^\tau \check{\mathbb{G}}(s)\,\mathrm{d}s\,\mathrm{d}\tau\frac{\mathbf{E}}{r}\cdot\frac{\mathbf{E}}{r}\int_0^r \check{\mathbb{G}}(\tau+a)\,\mathrm{d}\tau\mathbf{E}\cdot\frac{\mathbf{E}}{r}\right|<\frac{M}{r}$$

holds for some $M<\infty$. Moreover, from (3.13) we have

$$\left|\int_0^\infty \dot{\mathbb{G}}(s+\tau+a)\mathbf{E}^t(s)\,\mathrm{d}s\right|<\delta,\quad \forall(\tau+a)>a_\delta,$$

so that

$$\left|\int_0^r\int_0^\infty \dot{\mathbb{G}}(s+\tau+a)\mathbf{E}^t(s)\,\mathrm{d}s\,\mathrm{d}\tau\cdot\frac{\mathbf{E}}{r}\right|<\delta|\mathbf{E}|,\quad \forall a>a_\delta.$$

Choosing $r=4M/\varepsilon$ and $\delta=\varepsilon/4|\mathbf{E}|$, we obtain (4.7).

THEOREM 4.1. *For every state* σ *such that the set* $\{-W(\sigma,P),\ \ P\in\Pi\}$ *is bounded from above, we have*

$$W_R(\sigma)=\sup_{P\in\Pi_0(\sigma)}\{-W(\sigma,P)\}. \tag{4.9}$$

Proof. Obviously, since $\Pi_0(\sigma)\subset\Pi$, it follows that

$$W_R(\sigma)=\sup_{P\in\Pi}\{-W(\sigma,P)\}\geqslant\sup_{P\in\Pi_0(\sigma)}\{-W(\sigma,P)\}. \tag{4.10}$$

In view of our boundedness hypothesis, $W_R(\sigma) < \infty$, it follows that, for every $\varepsilon > 0$, there exists a process $P_\varepsilon \in \Pi$, of duration $d < \infty$, such that

$$W_R(\sigma) - \frac{\varepsilon}{2} < -W(\sigma, P_\varepsilon). \tag{4.11}$$

Suppose that $P_\varepsilon \sigma = (\mathbf{E}_\varepsilon, \mathbf{h})$ where $\mathbf{h} \in \Gamma / \Gamma_0$ and continue P_ε with a process $P_{a,r}$ described by (4.6). Then by, virtue of (4.11) and (4.7), there exists values of a and r such that

$$\begin{aligned} -W(\sigma, P_{a,r} * P_\varepsilon) &= -W(\sigma, P_\varepsilon) - W(P_\varepsilon \sigma, P_{a,r}) \\ &> W_R(\sigma) + \tfrac{1}{2}\mathbb{G}_\infty \mathbf{E}_\varepsilon \cdot \mathbf{E}_\varepsilon \varepsilon. \end{aligned} \tag{4.12}$$

Here the symbol $*$, according to [10], denotes the continuation of the strain associated to P_ε by the one related to $P_{a,r}$ as defined by (3.15). Finally, observe that $(P_{a,r} * P_\varepsilon) \in \Pi_0(\sigma)$ so that (4.12) and (3.8) imply

$$W_R(\sigma) \leqslant \sup_{P \in \Pi_0(\sigma)} \{-W(\sigma, P)\}. \tag{4.13}$$

Inequalities (4.10) and (4.13) ensure (4.9).

In view of (4.9), we see that the tensor $\mathbf{E}_\varepsilon$ in (4.12) must be small if ε is small.

It should be noted that this theorem does not imply that the limiting process belongs to $\Pi_0(\sigma)$ but rather to the closure of $\Pi_0(\sigma)$.

We conclude this section by giving a further equivalent formulation for the maximum recoverable work. Let Π_∞ be the set of all processes of finite duration $P \in \Pi$ continued with the null process P_0, so that the related strain is given by $\mathbf{E}\colon \mathcal{R}^+ \to \mathrm{Sym}$, with $\mathbf{E}(\tau) = \mathbf{E}(d)\ \forall \tau \geqslant d$ where d is the duration of the process. Obviously, each process $P \in \Pi$ yields the same work of the corresponding $P_0 * P \in \Pi_\infty$. Let $S(t)$ denote the following quantity

$$S(t) := \mathbf{T}(t) \cdot \mathbf{E}(t) - \tfrac{1}{2}\mathbb{G}_0 \mathbf{E}(t) \cdot \mathbf{E}(t), \tag{4.14}$$

so that putting $\mathbf{E}(\infty) = \mathbf{E}_\infty$, we have

$$\begin{aligned} S(\infty) &= \mathbb{G}_\infty \mathbf{E}_\infty \cdot \mathbf{E}_\infty - \tfrac{1}{2}\mathbb{G}_0 \mathbf{E}_\infty \cdot \mathbf{E}_\infty \\ &= \tfrac{1}{2}\mathbb{G}_\infty \mathbf{E}_\infty \cdot \mathbf{E}_\infty + \tfrac{1}{2}(\mathbb{G}_\infty - \mathbb{G}_0)\mathbf{E}_\infty \cdot \mathbf{E}_\infty. \end{aligned}$$

THEOREM 4.2. *The maximum recoverable work $W_R(\sigma(t))$ from the state $\sigma(t)$, defined by* (4.5)*, is given by*

$$W_R(\sigma(t)) = \sup_{P \in \Pi_\infty} \{S(\infty) - W(\sigma(t), P)\}. \tag{4.15}$$

Proof. First observe that $P \in \Pi_0(\sigma)$, of finite duration d, may be extended to Π_∞, by continuing P with the null process on $\mathcal{R}^+$. In such a case the recovered work $W(\sigma, P)$ does not change and the ensuing strain tensor vanishes at infinity, i.e. $\mathbf{E}_\infty = 0$, so that $S(\infty) = 0$. Therefore, denoting by $\mathcal{W}_0(\sigma)$ and $\mathcal{W}_\infty(\sigma)$ the following sets

$$\mathcal{W}_0(\sigma) := \{-W(\sigma, P);\ P \in \Pi_0(\sigma)\},$$

$$\mathcal{W}_\infty(\sigma) := \{S(\infty) - W(\sigma, P);\ P \in \Pi_\infty\},$$

we have

$$\mathcal{W}_0(\sigma) \subset \mathcal{W}_\infty(\sigma),$$

which implies that

$$\overline{\mathcal{W}_0(\sigma)} \subseteq \overline{\mathcal{W}_\infty(\sigma)}, \tag{4.16}$$

where the bar indicates closure in this context.

Now consider the strain $\mathbf{E}(\tau)$, $\tau \in \mathcal{R}$, related to $\sigma(t)$ continued with a process $P \in \Pi_\infty$ and such that $\lim_{\tau\to\infty} \mathbf{E}(\tau) = \mathbf{E}_\infty \neq 0$. Let $\mathbf{E}_d$ denote the truncation of $\mathbf{E}$ to the time $d > t$, continued with a discontinuity, followed by zero strain, yielding

$$\lim_{\tau\to d^-} \mathbf{E}_d(\tau) = \mathbf{E}(d), \qquad \mathbf{E}_d(\tau) = 0, \quad \tau \geqslant d$$

and let P_d be the related process. The work due to P_d is

$$\begin{aligned} W(\sigma(t), P_d) &= \int_t^d \mathbf{T}(\tau)\cdot\dot{\mathbf{E}}(\tau)\,\mathrm{d}\tau - \tfrac{1}{2}\left[\mathbf{T}(d^+) + \mathbf{T}(d^-)\right]\cdot\mathbf{E}(d) \\ &= \int_t^d \mathbf{T}(\tau)\cdot\dot{\mathbf{E}}(\tau)\,\mathrm{d}\tau - S(d). \end{aligned}$$

Since

$$\lim_{d\to\infty} W(\sigma(t), P_d) = W(\sigma(t), P) - S(\infty), \tag{4.17}$$

for every $P \in \Pi_\infty$ yielding $\mathbf{E}_\infty \neq 0$, we can find a family of processes $P_d \in \Pi_0(\sigma(t))$ such that (4.17) holds. Therefore

$$\mathcal{W}_\infty(\sigma(t)) \subset \overline{\mathcal{W}_0(\sigma(t))},$$

so that

$$\overline{\mathcal{W}_\infty(\sigma(t))} \subseteq \overline{\mathcal{W}_0(\sigma(t))}. \tag{4.18}$$

Equation (4.16) and (4.18) imply (4.15).

The maximum recoverable work in the form (4.15) arising from Theorem 4.2 was essentially the starting point of the approach used in [18] for the scalar case. Equation (4.9) in Theorem 4.1 will be our starting point in determining an explicit representation for W_R in the frequency domain. It is necessary in the first place to give a representation of the work $\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t)$, defined by (4.1), in the frequency domain. This has been obtained in [13] (see also [12]) in terms of integrals over $\mathcal{R}^+$. Using symmetry arguments, this representation can be expressed as an integral over $\mathcal{R}$ of the form

$$\begin{aligned}\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t) &= \tfrac{1}{2}\mathbb{G}_\infty \mathbf{E}(t) \cdot \mathbf{E}(t) \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{H}(\omega)\left(\mathbf{E}^t_+(\omega) - \frac{\mathbf{E}(t)}{i\omega}\right) \cdot \overline{\left(\mathbf{E}^t_+(\omega) - \frac{\mathbf{E}(t)}{i\omega}\right)}\,\mathrm{d}\omega \\ &= S(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{H}(\omega)\mathbf{E}^t_+(\omega) \cdot \overline{\mathbf{E}}^t_+(\omega)\,\mathrm{d}\omega, \end{aligned} \tag{4.19}$$

where for each given $\omega \in \mathcal{R}$, the fourth order tensor $\mathbb{H}(\omega) \in \mathrm{Lin(Sym)}$ is defined as

$$\mathbb{H}(\omega) := -\omega \dot{\mathbb{G}}_s(\omega) \tag{4.20}$$

and $S(t)$ is given by (4.14). The equivalence of the two forms of (4.19) follows from (3.7) and (3.12). They reduce to relations of Golden [18] in the scalar case.

5. Factorization of Positive Definite Tensors in the Frequency Domain

The function $\omega \mapsto \mathbb{H}(\omega)$ plays a key role in the determination of the explicit formula of W_R, as it was already recognized in [18] for the scalar case, where an appropriate factorization of the one-dimensional counterpart of $\mathbb{H}(\omega)$ was introduced to obtain an expression of W_R. Such a particular factorization does not apply to fourth order tensors, so that the extension of the result of [18] to the general case is not trivial.

In this section we show that $\omega \mapsto \mathbb{H}(\omega)$ always admits a factorization. To this aim, we make use of a result of Gohberg and Kreĭn [17], for matrix-valued functions. The results in [17] apply of course to matrices of arbitrary finite dimension but we state them for the case of 6×6 matrices. Given a nonsingular continuous tensor valued function $\mathbb{K}(\omega) \in \mathrm{Lin(Sym}(\Omega))$, $\omega \in \mathcal{R}$, we say that $\mathbb{K}$ has a left [right] factorization if it can be represented in the form

$$\mathbb{K}(\omega) = \mathbb{K}_{(+)}(\omega)\mathbb{K}_{(-)}(\omega), \qquad [\mathbb{K}(\omega) = \mathbb{K}_{(-)}(\omega)\mathbb{K}_{(+)}(\omega)], \tag{5.1}$$

where the matrix functions $\mathbb{K}_{(\pm)}$ admit analytic continuations, holomorphic in the interior and continuous up to the boundary of the corresponding complex half planes $\Omega^\pm$, and are such that

$$\det \mathbb{K}_{(\pm)}(\zeta) \neq 0, \quad \zeta \in \Omega^\pm.$$

In this relation the 6 × 6 matrices representation for $\mathbb{K}_{(\pm)}(\zeta)$ is employed.

We say that $\mathbb{K}$ belongs to $\Re_{6\times 6}$, $[\Re^{+}_{6\times 6}]$, $[[\Re^{-}_{6\times 6}]]$ if there exists a constant matrix $\mathbb{C}_0$ and a matrix function $\mathbb{F}(t)$ such that

$$\mathbb{K}(\omega) = \mathbb{C}_0 + \int_{-\infty}^{\infty} \mathbb{F}(t)\, \mathrm{e}^{i\omega t}\, \mathrm{d}t \tag{5.2}$$

$$\left[\mathbb{K}(\omega) = \mathbb{C}_0 + \int_{0}^{\infty} \mathbb{F}(t)\, \mathrm{e}^{i\omega t}\, \mathrm{d}t\right], \qquad \left[\left[\mathbb{K}(\omega) = \mathbb{C}_0 + \int_{-\infty}^{0} \mathbb{F}(t)\, \mathrm{e}^{i\omega t}\, \mathrm{d}t\right]\right].$$

Note that if $\mathbb{K} \in \Re^{\pm}_{6\times 6}$, it has the analytic properties ascribed to $\mathbb{K}_{(\pm)}$ above. The main result we use is Theorem 8.2 of [17], that can be stated in our context as follows

THEOREM 5.1 (Gohberg–Kreĭn). *In order that the nonsingular* (*Hermitian*) *matrix function* $\mathbb{K} \in \Re_{6\times 6}$ *possesses a representation of the form*

$$\mathbb{K}(\omega) = \mathbb{K}_{(+)}(\omega)\mathbb{K}^{*}_{(+)}(\omega), \tag{5.3}$$

in which the matrix function $\mathbb{K}_{(+)} \in \Re^{+}_{6\times 6}$ *and satisfies* $\det \mathbb{K}_{(+)}(\zeta) \neq 0$ *for* $\zeta \in \Omega^{+}$, *it is necessary and sufficient that* $\mathbb{K}(\omega)$ *be positive definite for every* $\omega \in \mathcal{R}$.

Observe that comparison of (5.3) with $(5.1)_1$ yields

$$\mathbb{K}_{(-)}(\omega) = \mathbb{K}^{*}_{(+)}(\omega).$$

Recalling that any fourth order symmetric tensor maps into a 6 × 6 matrix under the isomorphism discussed at the beginning of Section 2, let us consider for each given $\omega \in \mathcal{R}$ the fourth order tensor $\mathbb{H}(\omega) \in \mathrm{Lin(Sym)}$ defined by (4.20). By virtue of (3.5), the odd signature of $\omega \mapsto \dot{\mathbb{G}}_s(\omega)$ and the assumption that $\mathbb{G}(t)$ is symmetric, it follows that, for each fixed $\omega \in \mathcal{R} \setminus \{0\}$, $\mathbb{H}(\omega)$ is a real valued, symmetric, positive definite tensor. Moreover, since

$$\lim_{\omega\to 0} \frac{\dot{\mathbb{G}}_s(\omega)}{\omega} = \int_0^{\infty} s\dot{\mathbb{G}}(s)\, \mathrm{d}s, \qquad \lim_{\omega\to\infty} \omega\dot{\mathbb{G}}_s(\omega) = \dot{\mathbb{G}}(0),$$

it follows from (3.2) and (3.9) that

$$\lim_{\omega\to 0} \frac{\mathbb{H}(\omega)}{\omega^2} = \mathbb{H}_0, \qquad \lim_{\omega\to\infty} \mathbb{H}(\omega) = \mathbb{H}_{\infty},$$

where $\mathbb{H}_0$ and $\mathbb{H}_{\infty}$ are symmetric and positive definite. Consider now the tensor

$$\mathbb{K}(\omega) := \frac{1+\omega^2}{\omega^2}\mathbb{H}_{\mathrm{sr}}^{-1}\mathbb{H}(\omega)\mathbb{H}_{\mathrm{sr}}^{-1}, \tag{5.4}$$

where

$$\mathbb{H}_{\rm sr} := \mathbb{H}_\infty^{1/2}. \tag{5.5}$$

The tensor $\mathbb{K}(\omega)$ is symmetric and positive definite $\forall\omega \in \mathcal{R}$; moreover it is such that

$$\lim_{\omega\to 0}\mathbb{K}(\omega) = \mathbb{K}_0 = \mathbb{H}_{\rm sr}^{-1}\mathbb{H}_0\mathbb{H}_{\rm sr}^{-1}, \qquad \lim_{\omega\to\infty}\mathbb{K}(\omega) = \mathbb{I},$$

where $\mathbb{I}$ denotes the identity of Lin(Sym). In order to apply some results given in [17] we have to show that $\mathbb{K} \in \mathfrak{R}_{6\times 6}$, i.e. that the representation (5.2) applies to $\mathbb{K}$.

PROPOSITION 5.1. *If* $\check{\mathbb{G}}$ *and* $\ddot{\mathbb{G}}$ *are integrable tensor functions, and the tensor* $\dot{\mathbb{G}}(0)$ *is finite and nonsingular, the tensor valued function* $\mathbb{K}$, *related to* $\mathbb{G}$ *though* (4.20) *and* (5.4), *belongs to* $\mathfrak{R}_{6\times 6}$.

Proof. Observe that

$$\mathbb{K}(\omega) = -\mathbb{H}_{\rm sr}^{-1}\left[\omega\dot{\mathbb{G}}_s(\omega) + \frac{1}{\omega}\dot{\mathbb{G}}_s(\omega)\right]\mathbb{H}_{\rm sr}^{-1}. \tag{5.6}$$

Integration by parts yields

$$-\frac{1}{\omega}\dot{\mathbb{G}}_s(\omega) = \check{\mathbb{G}}_c(\omega), \qquad \omega\dot{\mathbb{G}}_s(\omega) = \dot{\mathbb{G}}(0) + \ddot{\mathbb{G}}_c(\omega),$$

so that (5.6) becomes

$$\mathbb{K}(\omega) = \mathbb{H}_{\rm sr}^{-1}[-\dot{\mathbb{G}}(0) - \ddot{\mathbb{G}}_c(\omega) + \check{\mathbb{G}}_c(\omega)]\mathbb{H}_{\rm sr}^{-1}. \tag{5.7}$$

Consider now the tensors

$$\mathbb{C}_0 = -\mathbb{H}_{\rm sr}^{-1}\dot{\mathbb{G}}(0)\mathbb{H}_{\rm sr}^{-1},$$

$$\mathbb{F}(t) = \tfrac{1}{2}\mathbb{H}_{\rm sr}^{-1}\left[\ddot{\mathbb{G}}(t) + \check{\mathbb{G}}(t)\right]\mathbb{H}_{\rm sr}^{-1}, \quad t \in \mathcal{R} \tag{5.8}$$

where $\check{\mathbb{G}}$ and $\ddot{\mathbb{G}}$ are extended on the real line as even functions, so that $\check{\mathbb{G}}_F = 2\check{\mathbb{G}}_c$ and $\ddot{\mathbb{G}}_F = 2\ddot{\mathbb{G}}_c$. Then, in view of (5.8), (5.7) is equivalent to (5.2) and the assertion is proved.

Observe that (3.4) and (3.9) ensure that $\mathbb{G}$ has two of the required properties.

Since $\mathbb{K}(\omega)$ is real, symmetric and positive definite for every $\omega \in \mathcal{R}$, it satisfies Theorem 5. In particular, it has a representation of the form (left factorization)

$$\mathbb{K}(\omega) = \mathbb{K}_{(+)}(\omega)\mathbb{K}^*_{(+)}(\omega), \tag{5.9}$$

with $\mathbb{K}_{(+)}(\omega) \in \mathfrak{R}^+_{6\times 6}$ and

$$\det\mathbb{K}_{(+)}(\zeta) \neq 0 \quad \text{for } \zeta \in \Omega^+.$$

Moreover such a factorization is unique up to a post-multiplication on the right by a constant unitary tensor (see Remark on page 253 of [17]).

Similarly, $\mathbb{K}$ has a right factorization of the type:

$$\mathbb{K}(\omega) = \mathbb{K}_{(-)}(\omega)\mathbb{K}^*_{(-)}(\omega), \tag{5.10}$$

with corresponding properties. In fact, since $\mathbb{K}(\omega)$ is an even function of ω, we can replace ω by $-\omega$ on the right of (5.9). Now, $\mathbb{K}_{(+)}(-\omega) \in \mathscr{R}^-_{6\times 6}$ with nonzero determinant in $\Omega^{(-)}$ so that $\mathbb{K}_{(-)}(\omega) = \mathbb{K}_{(+)}(-\omega)$.

By virtue of (5.4) and (5.10), $\mathbb{H}(\omega)$ can be factorized as follows

$$\mathbb{H}(\omega) = \mathbb{H}_+(\omega)\mathbb{H}_-(\omega), \tag{5.11}$$

where

$$\mathbb{H}_+(\omega) = \frac{\omega}{\omega - i}\mathbb{H}_{\mathrm{sr}}\mathbb{K}_{(-)}(\omega), \quad \mathbb{H}_-(\omega) = \frac{\omega}{\omega + i}\mathbb{K}^*_{(-)}(\omega)\mathbb{H}_{\mathrm{sr}}. \tag{5.12}$$

The notation for $\mathbb{H}_+(\omega)$ and $\mathbb{H}_-(\omega)$ follow the convention used in [18], i.e. the sign indicates the half plane where any singularities of the tensor and any zeros in the determinant of the corresponding matrix occur. Alternatively, the left factorization (5.9) may be used, though the right factorization is more convenient in the present context. Representation (5.12) gives that

$$\mathbb{H}_\pm(\omega) = \mathbb{H}^*_\mp(\omega). \tag{5.13}$$

6. Explicit Representation of the Maximum Recoverable Work

A natural procedure for obtaining an explicit expression for $W_R(\sigma(t))$ is to find the 'optimal' future continuation which is related, according to Theorem 4.2, to the process $P_o \in \Pi_\infty$ that realizes the supremum of $S(\infty) - W(\sigma(t), P)$, the work W being defined by (4.2). This procedure leads to a Wiener–Hopf equation in the time domain [1, 7]. However, the problem was solved directly for the scalar case, using this approach, by going into the frequency domain [18]. An alternative, somewhat simpler procedure is to use Theorem 4.1 and find the future continuation $\mathbf{E}_o$, related to the limiting process that realizes the supremum of $-W(\sigma(t), P)$, $P \in \Pi_0(\sigma)$ in the frequency domain. This method will be used here. For either approach, the crucial result that allows us to generalize this procedure is Theorem 5.1.

Observe that, for any state $\sigma(t)$ related to the strain $\mathbf{E}(\tau)$, $\tau \leqslant t$, and a process P of duration d related to the continuation $\mathbf{E}_P(\tau)$, $\tau \in (0, d]$ we have

$$W(\sigma(t), P) = \widetilde{W}(\mathbf{E}_P(d), (\mathbf{E}_P * \mathbf{E})^{t+d}) - \widetilde{W}(\mathbf{E}(t), \mathbf{E}^t). \tag{6.1}$$

Therefore, considering (6.1) for processes $P \in \Pi_0(\sigma)$ of finite and infinite duration, we have from (4.9)

$$W_R(\sigma(t)) - \widetilde{W}(\mathbf{E}(t), \mathbf{E}^t) = \sup_{P\in\Pi_0(\sigma)} \left\{ -\int_{-\infty}^{\infty} \mathbf{T}(\tau) \cdot \dot{\mathbf{E}}_f(\tau)\, \mathrm{d}\tau \right\}, \tag{6.2}$$

where $\mathbf{E}_f$ is related to P as specified by (3.15). Both the representation of $\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t)$, given by (4.19), and of the quantity in brackets on the right-hand side of (6.2) in the frequency domain, are now required. Using the method sketched in [18], it can be shown from (4.19) that

$$\int_{-\infty}^{\infty} \mathbf{T}(\tau) \cdot \dot{\mathbf{E}}(\tau)\,\mathrm{d}\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{H}(\omega)\left(\mathbf{E}_+^t(\omega) + \mathbf{E}_-^t(\omega)\right) \cdot \left(\overline{\mathbf{E}}_+^t(\omega) + \overline{\mathbf{E}}_-^t(\omega)\right)\mathrm{d}\omega, \tag{6.3}$$

where the subscript on $\dot{\mathbf{E}}_f$ has been dropped and the property $\mathbf{E}(\infty) = 0$ (because $P \in \Pi_0(\sigma)$) has been used.

Next we make use of (6.3) in order to find the 'optimal' future continuation $\mathbf{E}_o(\tau)$, $\tau > t$, (or, equivalently, $\mathbf{E}_o^t(s)$, $s < 0$) maximizing the recoverable work as given by (4.9). Recalling a comment after (4.13), we observe that, as the maximum recoverable work is the supremum of the set defined in (4.9), $\mathbf{E}_o$ does not necessarily belong to the set of continued strains related to $P \in \Pi_0(\sigma)$. It belongs to the closure of such a set. This fact allows $\mathbf{E}_o$ to have some discontinuities; in particular it allows $\mathbf{E}(t) \neq \lim_{s\to 0^-} \mathbf{E}_o^t(s)$. Also, in the limit, $\mathbf{E}_o(\infty)$ need not vanish.

Our aim is now to maximize the integral on the right of (6.3), using a variational technique. Let $\mathbf{E}_m^t$ denote the Fourier transform of the optimal future continuation $\mathbf{E}_o^t$ viz. $\mathbf{E}_m^t(\omega) = \mathbf{E}_{o-}^t(\omega)$. Put

$$\mathbf{E}_-^t(\omega) = \mathbf{E}_m^t(\omega) + \mathbf{k}_-(\omega),$$

where $\mathbf{k}_-(\omega)$ is arbitrary apart from the fact that it must have the same analytic properties as $\mathbf{E}_-^t(\omega)$, i.e. $\mathbf{k}_-(z)$ must be analytic in Ω^+, and vanish as z^{-1} at large z. Then (6.3) will be maximized by $\mathbf{E}_m^t$ if

$$\int_{-\infty}^{\infty} \Re_e\{\mathbb{H}(\omega)\left(\mathbf{E}_+^t(\omega) + \mathbf{E}_m^t(\omega)\right) \cdot \overline{\mathbf{k}}_-(\omega)\}\,\mathrm{d}\omega = 0. \tag{6.4}$$

The restriction to the real part of the integral may be removed by virtue of the symmetric range of integration. Using the factorization (5.11), we can rewrite this condition in the form

$$\int_{-\infty}^{\infty} \mathbb{H}(\omega)[\mathbf{E}_+^t(\omega) + \mathbf{E}_m^t(\omega)] \cdot \overline{\mathbf{k}}_-(\omega)\,\mathrm{d}\omega = \int_{-\infty}^{\infty} \mathbb{H}_+(\omega)[\mathbb{H}_-(\omega)\mathbf{E}_+^t(\omega) + \mathbb{H}_-(\omega)\mathbf{E}_m^t(\omega)] \cdot \overline{\mathbf{k}}_-(\omega)\,\mathrm{d}\omega = 0. \tag{6.5}$$

Consider now the second order symmetric tensor $\mathbb{H}_-(\omega)\mathbf{E}_+^t(\omega)$, the components of which are continuous (in fact analytic) on $\mathcal{R}$ by virtue of the assumed analyticity of $\mathbb{H}_-(\omega)$ and $\mathbf{E}_+^t(\omega)$. The Plemelj formulae [28] give that

$$\mathbf{Q}^t(\omega) := \mathbb{H}_-(\omega)\mathbf{E}_+^t(\omega) = \mathbf{q}_-^t(\omega) - \mathbf{q}_+^t(\omega), \tag{6.6}$$

where

$$\mathbf{q}^t(z) := \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbf{Q}^t(\omega)}{\omega - z}\,\mathrm{d}\omega, \qquad \mathbf{q}^t_{\pm}(\omega) := \lim_{\alpha\to 0^{\mp}}\mathbf{q}^t(\omega + i\alpha). \tag{6.7}$$

Moreover, $\mathbf{q}^t_+(z) = \mathbf{q}^t(z),\ z \in \Omega^{(-)}$ is analytic on $\Omega^{(-)}$ and $\mathbf{q}^t_-(z) = \mathbf{q}^t(z),\ z \in \Omega^{(+)}$ is analytic on $\Omega^{(+)}$. Both are analytic on the real axis by virtue of an argument given in [18]. They are defined on the entire complex plane by analytic continuation. Therefore (6.5) becomes

$$\int_{-\infty}^{\infty}\mathbb{H}_+(\omega)\left[\mathbf{q}^t_-(\omega) - \mathbf{q}^t_+(\omega) + \mathbb{H}_-(\omega)\mathbf{E}^t_m(\omega)\right]\cdot\overline{\mathbf{k}}_-(\omega)\,\mathrm{d}\omega = 0. \tag{6.8}$$

Note that the integral

$$\int_{-\infty}^{\infty}\mathbb{H}_+(\omega)\mathbf{q}^t_+(\omega)\cdot\overline{\mathbf{k}}_-(\omega)\,\mathrm{d}\omega \tag{6.9}$$

vanishes identically, because the integrand is analytic on $\Omega^{(-)}$ and vanishes as z^{-2} at large z. Therefore (6.8) becomes

$$\int_{-\infty}^{\infty}\mathbb{H}_+(\omega)\left[\mathbf{q}^t_-(\omega) + \mathbb{H}_-(\omega)\mathbf{E}^t_m(\omega)\right]\cdot\overline{\mathbf{k}}_-(\omega)\,\mathrm{d}\omega = 0.$$

This will be true for arbitrary $\overline{\mathbf{k}}_-(\omega)$ only if the expression in brackets is a function that is analytic in $\Omega^{(-)}$. However, $\mathbf{E}^t_m(\omega)$ must be analytic in Ω^+. Remembering that $\mathbf{q}^t_-(\omega)$ and $\mathbb{H}_-(\omega)$ are also analytic in Ω^+, we see that the expression in brackets must be analytic in both the upper and lower half planes and on the real axis. Thus, it is analytic over the entire complex plane. Now $\mathbf{q}^t_-(\omega)$ clearly vanishes as ω^{-1} at infinity, as also must $\mathbf{E}^t_m(\omega)$ if the strain function is to be finite at $s = 0$. Therefore, the function in brackets is analytic everywhere, zero at infinity and consequently must vanish everywhere by Liouville's theorem. Thus

$$\mathbf{q}^t_-(\omega) + \mathbb{H}_-(\omega)\mathbf{E}^t_m(\omega) = 0 \quad \forall\omega \in \mathcal{R},$$

giving

$$\mathbf{E}^t_m(\omega) = -\left(\mathbb{H}_-(\omega)\right)^{-1}\mathbf{q}^t_-(\omega). \tag{6.10}$$

Therefore the maximum recoverable work from a given state $\sigma(t)$, corresponding to the strain $\mathbf{E}(\tau)$, $\tau \leqslant t$, may be written by means of (6.2) and (4.19) in the form

$$\begin{aligned} W_R(\sigma(t)) \;=\; & -\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbb{H}(\omega)(\mathbf{E}^t_+(\omega) + \mathbf{E}^t_m(\omega))\cdot(\overline{\mathbf{E}}^t_+(\omega) + \overline{\mathbf{E}}^t_m(\omega))\,\mathrm{d}\omega \\ & + S(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbb{H}(\omega)\mathbf{E}^t_+(\omega)\cdot\overline{\mathbf{E}}^t_+(\omega)\,\mathrm{d}\omega. \end{aligned}$$

With the aid of (6.6) and (6.10), we obtain

$$\begin{aligned} W_R(\sigma(t)) &= S(t) - \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\mathbf{q}_+^t(\omega)\right|^2 \, \mathrm{d}\omega \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\mathbf{q}_-^t(\omega) - \mathbf{q}_+^t(\omega)\right|^2 \, \mathrm{d}\omega \\ &= S(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\mathbf{q}_-^t(\omega)\right|^2 \, \mathrm{d}\omega \\ &\quad - \frac{1}{\pi}\int_{-\infty}^{\infty} \Re_e \left\{\mathbf{q}_-^t(\omega) \cdot \overline{\mathbf{q}}_+^t(\omega)\right\} \, \mathrm{d}\omega. \end{aligned}$$

Remembering that $\mathbf{q}_\pm^t(z)$ is analytic in the same half-plane as $\overline{\mathbf{q}}_\mp^t(z)$, namely $z \in \Omega^{(\mp)}$ and goes to zero as z^{-1} at large z, we have

$$\begin{aligned} &2\int_{-\infty}^{\infty} \Re_e \left\{\mathbf{q}_-^t(\omega) \cdot \overline{\mathbf{q}}_+^t(\omega)\right\} \, \mathrm{d}\omega \\ &\quad = \int_{-\infty}^{\infty} \left[\mathbf{q}_-^t(\omega) \cdot \overline{\mathbf{q}}_+^t(\omega) + \mathbf{q}_+^t(\omega) \cdot \overline{\mathbf{q}}_-^t(\omega)\right] \mathrm{d}\omega = 0, \end{aligned} \tag{6.11}$$

so that

$$W_R(\sigma(t)) = S(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\mathbf{q}_-^t(\omega)\right|^2 \, \mathrm{d}\omega. \tag{6.12}$$

This is the first explicit formula for the maximum recoverable work in terms of the given strain in the full tensorial case.

Actually, it is worth noting that the dependence on the state $\sigma(t)$ of W_R is not explicit on the right-hand side of (6.12). This is due to the fact that it is not known how the state is characterized in the frequency domain, so that, for now, we abuse notation here by equating $W_R(\sigma(t))$ to an expression in which the dependence on $\sigma(t)$ is neither made explicit nor characterized. The issue of characterization of states in the frequency domain will be discussed in the next section.

Note that, by virtue of the dissipation property of the material, the work done to reach the strain related to the pair $(\mathbf{E}(t), \mathbf{E}^t)$ must be not less than the work recoverable from the state $\sigma(t)$ corresponding to $(\mathbf{E}(t), \mathbf{E}^t)$. This is satisfied because, by virtue of (4.19), (6.6) and (6.11), we find that

$$\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t) = S(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} [\left|\mathbf{q}_+^t(\omega)\right|^2 + \left|\mathbf{q}_-^t(\omega)\right|^2] \, \mathrm{d}\omega,$$

$$\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t) - W_R(\sigma(t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\mathbf{q}_+^t(\omega)\right|^2 \, \mathrm{d}\omega \geqslant 0. \tag{6.13}$$

7. Characterization of the State in the Frequency Domain

In this section we show that the tensor $\mathbf{q}^t_-(\omega)$ occurring in (6.12) represents an element of Γ/Γ_0 in the frequency domain, and also that the right-hand side of (6.12) is a function of the state. Two histories $\mathbf{E}^t_1, \mathbf{E}^t_2$ are equivalent if their difference $\mathbf{E}^t = \mathbf{E}^t_1 - \mathbf{E}^t_2$ satisfies (3.18)

$$\mathcal{F}^t(\tau) := \int_\tau^\infty \dot{\mathbb{G}}(s)\mathbf{E}^{t+\tau}(s)\,\mathrm{d}s = 0, \quad \forall\tau \geqslant 0. \tag{7.1}$$

Let $\mathbf{E}^{t+\tau}$ be equal to zero for $\tau > s$ and identify $\dot{\mathbb{G}}$ with its odd extension so that $\dot{\mathbb{G}}_F(\omega) = -2i\dot{\mathbb{G}}_s(\omega)$. Then $\mathcal{F}^t(\tau)$ can be rewritten in terms of Fourier transforms as in (3.12)

$$\mathcal{F}^t(\tau) = \int_{-\infty}^\infty \dot{\mathbb{G}}(s)\mathbf{E}^{t+\tau}(s)\,\mathrm{d}s = \frac{i}{\pi}\int_{-\infty}^\infty \dot{\mathbb{G}}_s(\omega)\mathbf{E}^{t+\tau}_+(\omega)\,\mathrm{d}\omega. \tag{7.2}$$

Moreover, we note that

$$\begin{aligned}\mathbf{E}^{t+\tau}_+(\omega) &= \int_0^\infty \mathbf{E}^{t+\tau}(s)\,\mathrm{e}^{-i\omega s}\,\mathrm{d}s \\ &= \int_\tau^\infty \mathbf{E}^t(s-\tau)\,\mathrm{e}^{-i\omega(s-\tau)}\,\mathrm{d}s\,\mathrm{e}^{-i\omega\tau} = \mathbf{E}^t_+(\omega)\,\mathrm{e}^{-i\omega\tau}.\end{aligned} \tag{7.3}$$

Substitution of (4.20) and (7.3) into (7.2) yields

$$\mathcal{F}^t(\tau) = -\frac{i}{\pi}\int_{-\infty}^\infty \frac{\mathbb{H}(\omega)}{\omega}\mathbf{E}^t_+(\omega)\,\mathrm{e}^{-i\omega\tau}\,\mathrm{d}\omega. \tag{7.4}$$

Remembering the factorization of $\mathbb{H}(\omega)$ given by (5.11) and (5.12), (7.4) can be rewritten as

$$\mathcal{F}^t(\tau) = -\frac{i}{\pi}\int_{-\infty}^\infty \frac{\mathbb{H}_+(\omega)}{\omega}\mathbb{H}_-(\omega)\mathbf{E}^t_+(\omega)\,\mathrm{e}^{-i\omega\tau}\,\mathrm{d}\omega \tag{7.5}$$

and the substitution of (6.6) in (7.5) yields

$$\mathcal{F}^t(\tau) = -\frac{i}{\pi}\int_{-\infty}^\infty \frac{\mathbb{H}_+(\omega)}{\omega}\left[\mathbf{q}^t_-(\omega) - \mathbf{q}^t_+(\omega)\right]\mathrm{e}^{-i\omega\tau}\,\mathrm{d}\omega. \tag{7.6}$$

According to (7.1) we consider $\mathcal{F}^t(\tau)$ for $\tau \geqslant 0$. Then observe that $\mathbb{H}_+(z)$, $\mathbf{q}^t_+(z)$ and $\mathrm{e}^{-iz\tau}$, $\tau \geqslant 0$ are analytic functions in the lower half plane $z \in \Omega^{(-)}$ so that by Cauchy's theorem, (7.6) reduces to

$$\mathcal{F}^t(\tau) = -\frac{i}{\pi}\int_{-\infty}^\infty \frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{q}^t_-(\omega)\,\mathrm{e}^{-i\omega\tau}\,\mathrm{d}\omega. \tag{7.7}$$

We extend the definition of $\mathcal{F}^t$ to $\mathcal{R}$ by considering (7.7) to hold for $\tau \in \mathcal{R}$. This form of $\mathcal{F}^t$ allows us to prove the following theorem characterizing the state (in the sense of Noll) of a viscoelastic material in the frequency domain.

THEOREM 6.1. *For every viscoelastic material with a symmetric relaxation function, a given strain history* $\mathbf{E}^t$ *is equivalent to the zero history* $\mathbf{0}^\dagger$ *in the sense of* (3.18) *if and only if the quantity* $\mathbf{q}^t_-$, *related to* $\mathbf{E}^t$ *by* (6.6–6.7), *is such that*

$$\mathbf{q}^t_-(\omega) = 0, \quad \forall \omega \in \mathcal{R}. \tag{7.8}$$

Proof. Observe that the theorem in effect states that

$$\mathcal{F}^t(\tau) = 0 \quad \forall \tau \geqslant 0 \iff \mathbf{q}^t_-(\omega) = 0 \quad \forall \omega \in \mathcal{R}. \tag{7.9}$$

The statement relating to the left arrow of (7.9) follows trivially from (7.7). In order to prove the statement relating to the right arrow, let us invert the Fourier transform in (7.7) to obtain

$$\mathbf{f}^t(\omega) = \frac{i\mathbb{H}_+(\omega)}{\pi\omega}\mathbf{q}^t_-(\omega) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{F}^t(\tau)\, \mathrm{e}^{iw\tau}\, \mathrm{d}\tau. \tag{7.10}$$

If $\mathcal{F}^t(\tau) = 0\ \forall \tau \geqslant 0$, it follows from (7.10) that $\mathbf{f}^t$ is analytic in $\Omega^{(-)}$. The zeros of $\mathbb{H}_+$ cannot cancel singularities of $\mathbf{q}^t_-$ since all such zeros are by construction in $\Omega^{(+)}$. Thus $\mathbf{q}^t_-$ must be analytic in $\Omega^{(-)}$ and therefore in Ω. Since it goes to zero at infinity, by Liouville's theorem, it must vanish and (7.8) is proved.

Theorem 7.1 makes exact the notion of equivalence of histories in the frequency domain, i.e. two different histories $\mathbf{E}^t_1$, $\mathbf{E}^t_2$, are equivalent in the sense of (3.18) if and only if $\mathbf{q}^t_{1-}(\omega) \equiv \mathbf{q}^t_{2-}(\omega)$, $\forall \omega \in \mathcal{R}$. As a consequence, the elements of Γ/Γ_0 now have an explicit representation in the frequency domain by means of $\mathbf{q}^t_-$, so that we need resort no more to a representative element $\mathbf{E}^t \in \Gamma$ of the equivalence class related to Γ_0. Thus the state in the sense of Noll may be written as

$$\sigma(t) = (\mathbf{E}(t), \mathbf{q}^t_-).$$

Therefore, to show that the right hand side of (6.12) is a function of the state $\sigma(t)$, we show that it can be written in terms of the pair $(\mathbf{E}(t), \mathbf{q}^t_-)$. Recalling the definition of $S(t)$ given by (4.14) and the manipulations leading to (7.7) (for $\tau = 0$), it is clear that (6.12) may be written as

$$\begin{aligned} W_R(\sigma(t)) &= \widehat{W}_R(\mathbf{E}(t), \mathbf{q}^t_-) \\ &= \tfrac{1}{2}\mathbb{G}_0\mathbf{E}(t)\cdot\mathbf{E}(t) - \mathbf{E}(t)\cdot\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{q}^t_-(\omega)\,\mathrm{d}\omega \\ &\quad +\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\mathbf{q}^t_-(\omega)\right|^2\,\mathrm{d}\omega, \end{aligned} \tag{7.11}$$

which ensures that the maximum recoverable work given by (6.12) is really a function of state as required in the previous section. Moreover it is a quadratic form of the variables $\mathbf{E}(t)$, $\mathbf{q}_-^t$, characterizing the state in the frequency domain.

8. The Maximum Recoverable Work as the Minimum Free Energy According to Graffi's Definition

The minimum free energy for a viscoelastic material in the state $\sigma(t)$ has been proved to be given by the maximum recoverable work from $\sigma(t)$, viz.

$$\psi_m(\sigma(t)) = W_R(\sigma(t)), \tag{8.1}$$

by several authors, in particular Theorems 7 and 8 of [13].

By virtue of (8.1) the explicit representations (6.12) and (7.11) of the maximum recoverable work W_R provide a closed expression for the minimum free energy ψ_m, viz.

$$\begin{aligned}\psi_m(\sigma(t)) &= \hat{\psi}_m(\mathbf{E}(t), \mathbf{q}_-^t) := S(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\mathbf{q}_-^t(\omega)\right|^2 \,\mathrm{d}\omega \\ &= \tfrac{1}{2}\mathbb{G}_0\mathbf{E}(t)\cdot\mathbf{E}(t) - \mathbf{E}(t)\cdot\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{q}_-^t(\omega)\,\mathrm{d}\omega \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\mathbf{q}_-^t(\omega)\right|^2\,\mathrm{d}\omega.\end{aligned} \tag{8.2}$$

This is the main result of the paper, since it is the first explicit expression for the minimum free energy for a viscoelastic material in the full tensorial case.

However there are some criteria that a functional must fulfil in order to be considered a free energy. In particular, in this section we consider the requirements of Graffi's definition [20, 21] (also [27, 13]). Here the free energy must be a functional of the history and present value of the deformation, obeying certain properties that have been proved to hold by Coleman [3] for materials with fading memory as a consequence of the second law of thermodynamics. Recalling Remark 3.2 on the definition of processes, Graffi's definition of the free energy may be stated as follows [10, 13]:

DEFINITION 8.1. A functional $\tilde{\psi}\colon \Gamma \times \mathrm{Sym} \to \mathscr{R}$ is said to be a free energy for a viscoelastic material described by (3.11) if it satisfies the following properties:

(*P1*) (integrated dissipation inequality)

$$\begin{aligned}&\tilde{\psi}(\mathbf{E}_P(d), (\mathbf{E}_P * \mathbf{E})^{t+d}) - \tilde{\psi}(\mathbf{E}(t), \mathbf{E}^t) \\ &\quad\leqslant \int_0^d \tilde{\mathbf{T}}(\mathbf{E}_P(\tau), (\mathbf{E}_P * \mathbf{E})^{t+\tau})\cdot\dot{\mathbf{E}}_P(\tau)\,\mathrm{d}\tau,\end{aligned} \tag{8.3}$$

for every deformation history couple $(\mathbf{E}(t), \mathbf{E}^t)$ and for every process $\mathbf{E}_P(d)$ of arbitrary duration d with $\mathbf{E}_P(0) = \mathbf{E}(t)$;

($P2$) for every $(\mathbf{E}(t), \mathbf{E}^t)$, the derivative of $\tilde{\psi}(\cdot, \mathbf{E}^t)$ at $\mathbf{E}(t)$ is equal to the stress $\tilde{\mathbf{T}}(\mathbf{E}(t), \mathbf{E}^t)$;

($P3$) for every $(\mathbf{E}(t), \mathbf{E}^t)$,

$$\tilde{\psi}(\mathbf{E}(t), \mathbf{E}(t)^\dagger) \leqslant \tilde{\psi}(\mathbf{E}(t), \mathbf{E}^t),$$

where $\mathbf{E}(t)^\dagger$ is the static history with value $\mathbf{E}(t)$;

($P4$) for every deformation $\mathbf{E}(t)$,

$$\tilde{\psi}(\mathbf{E}(t), \mathbf{E}(t)^\dagger) - \tilde{\psi}(\mathbf{0}, \mathbf{0}^\dagger) = \phi(t),$$

where $\phi(t)$ is the elastic free energy defined by

$$\phi(t) = \tfrac{1}{2}\mathbb{G}_\infty \mathbf{E}(t) \cdot \mathbf{E}(t).$$

THEOREM 8.1. *The function* ψ_m *defined by* (8.1), (8.2) *is a free energy in the sense of Graffi* i.e. *it satisfies properties P1–P4.*

Proof. We can write (8.2) in alternative forms

$$\begin{aligned}
\psi_m(\sigma(t)) &= \tilde{\psi}_m(\mathbf{E}(t), \mathbf{E}^t) \\
&= S(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{H}(\omega)\mathbf{E}_m^t(\omega) \cdot \bar{\mathbf{E}}_m^t(\omega)\,\mathrm{d}\omega \\
&= \phi(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbb{H}(\omega)\left[\mathbf{E}_m^t(\omega) + \frac{\mathbf{E}(t)}{i\omega^+}\right] \\
&\qquad \cdot \overline{\left[\mathbf{E}_m^t(\omega) + \frac{\mathbf{E}(t)}{i\omega^+}\right]}\,\mathrm{d}\omega \\
&= \phi(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\mathbf{q}_-^t(\omega) - \frac{\mathbb{H}_-(\omega)\mathbf{E}(t)}{i\omega^+}\right|^2 \mathrm{d}\omega.
\end{aligned} \tag{8.4}$$

The dependence of $\tilde{\psi}_m$ on the past history $\mathbf{E}^t$ is made explicit in the last two forms of (8.4), in the latter form with the aid of (6.6–6.7). The last form of (8.4) follows from the second one by virtue of (5.11) and (6.10). The second form is equivalent to the first provided that

$$\int_{-\infty}^{\infty} \dot{\mathbb{G}}_s(\omega)\Im_m[\mathbf{E}_+^t(\omega) + \mathbf{E}_m^t(\omega)]\,\mathrm{d}\omega = 0, \tag{8.5}$$

as it may be shown by an argument similar to the demonstration that the two forms of (4.19) are equivalent. Condition (8.5) is a special case of (6.4) obtained by taking $\bar{\mathbf{k}}_{-}(\omega) = \mathbf{I}(i\omega^{-})^{-1}$ where $\mathbf{I}$ is the unit tensor in Sym. The restriction to the imaginary part may be removed by symmetry. The resulting relation may be proved directly with the aid of (6.6) and (6.10), by showing that the integrand has the form $\mathbb{H}_{+}(\omega)\mathbf{q}^{t}_{+}(\omega)/\omega$. Observe that (8.2) can be deduced without difficulty from the third relation of (8.4).

Property ($P2$) is clearly obeyed by $\tilde{\psi}_m$ given by the first form of (8.4).

From (2.7), we see that for a static history, $\mathbf{E}^{t}_{+}(\omega)$ is given by $\mathbf{E}(t)/(i\omega^{-})$. Substituting this form into (6.10), with $\mathbf{q}^{t}_{-}(\omega)$ defined by (6.7), and closing the contour over $\Omega^{(+)}$, we obtain for a static history

$$\mathbf{E}^{t}_{m}(\omega) = -\frac{\mathbf{E}(t)}{i\omega^{+}},$$

so that the integral in the second form of (8.4) vanishes for a static history. Thus, properties (*P3*) and (*P4*) are obeyed, provided that $\tilde{\psi}(\mathbf{0}, \mathbf{0}^{\dagger}) = 0$, which is in effect a normalization condition.

In order to prove (*P1*), let $t = 0$ be the initial time. For each given initial state σ_0 related to the couple $(\mathbf{E}(0), \mathbf{E}^{0})$ and process P of duration d related to the continuation $\mathbf{E}_P$, let $\mathcal{D}_m(\sigma_0, P)$ denote the following expression

$$\begin{aligned}\mathcal{D}_m(\sigma_0, P) \;:=\; & \int_0^d \tilde{\mathbf{T}}(\mathbf{E}_P(t), (\mathbf{E}_P * \mathbf{E})^{t}) \cdot \dot{\mathbf{E}}_P(t)\,\mathrm{d}t \\ & -\tilde{\psi}_m(\mathbf{E}_P(d), (\mathbf{E}_P * \mathbf{E})^{d}) + \tilde{\psi}_m(\mathbf{E}(0), \mathbf{E}^{0}).\end{aligned} \tag{8.6}$$

The integrated dissipation inequality ($P1$) becomes

$$\mathcal{D}_m(\sigma_0, P) \geqslant 0. \tag{8.7}$$

Formulae (6.2), (6.3), (6.6) and (6.10) allow us to make (8.6) explicit in the frequency domain, i.e.

$$\mathcal{D}_m(\sigma_0, P) = \frac{1}{2\pi}\int_{\infty}^{\infty} \left(\left|\mathbf{q}^{d}_{+}(\omega)\right|^2 - \left|\mathbf{q}^{0}_{+}(\omega)\right|^2\right) \mathrm{d}\omega.$$

In order to prove that (8.7) holds for every process and for every d it is sufficient to show that the function $t \mapsto \mathbf{q}^{t}_{+}(\omega)$ is differentiable almost everywhere in $(0, d)$ and the derivative of its squared modulus is defined a.e. and non-negative. To this aim we recall that, for each fixed time t, functions $\omega \mapsto \mathbf{q}^{t}_{\pm}(\omega)$, defined by (6.7), are analytic on the real axis and they may be written in the form

$$\begin{aligned}\mathbf{q}^{t}_{\pm}(\omega) \;=\; & \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathbb{H}_{-}(\omega')\left[\mathbf{E}^{t}_{+}(\omega') - \frac{\mathbf{E}(t)}{i\omega'^{-}}\right]}{\omega' - \omega^{\mp}}\,\mathrm{d}\omega' \\ & + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\mathbb{H}_{-}(\omega')}{(\omega' - \omega^{\mp})i\omega'^{-}}\,\mathrm{d}\omega'\mathbf{E}(t),\end{aligned} \tag{8.8}$$

where the last integral vanishes for $\mathbf{q}^t_+(\omega)$ and gives $\mathbb{H}_-(\omega)\mathbf{E}(t)/(i\omega)$ for $\mathbf{q}^t_-(\omega)$. It follows from (8.8) that the function $t \mapsto \mathbf{q}^t_+(\omega)$ is differentiable a.e. in $(0, d)$ if and only if $t \mapsto \mathbf{E}^t_+(\omega)$ has this property. This can be proved by virtue of definition $(2.4)_1$ and of $(A.2)_1$, to have the form

$$\mathbf{E}^t_+(\omega) = \mathbf{E}^{t,ac}_+(\omega) + \mathcal{J}\mathbf{E}^t_+(\omega),$$

where

$$\mathbf{E}^{t,ac}_+(\omega) := \int_0^\infty \mathbf{E}^{t,ac}(s)\, \mathrm{e}^{-i\omega s}\, \mathrm{d}s, \qquad \mathcal{J}\mathbf{E}^t_+(\omega) := \int_0^\infty \mathcal{J}\mathbf{E}^t(s)\, \mathrm{e}^{-i\omega s}\, \mathrm{d}s,$$

are the Fourier transform of the absolutely continuous part and of the jumps of $\mathbf{E}^t$ respectively and $\mathcal{J}\mathbf{E}^t(s)$ is defined by (A.1) with $[a, b] = [0, d]$. The differentiability a.e. of $t \mapsto \mathbf{E}^{t,ac}_+(\omega)$ on $(0, d)$ is ensured by the fact that $t \mapsto \mathbf{E}^{t,ac}$ is absolutely continuous. It remains to prove the same properties for $t \mapsto \mathcal{J}\mathbf{E}^t_+(\omega)$.

The substitution of (A.1) into $(2.4)_1$ and simple calculations yield

$$\mathcal{J}\mathbf{E}^t_+(\omega) = -\frac{i}{\omega^-} \sum_j \Delta\mathbf{E}^t_j (1 - \mathrm{e}^{-i\omega(t-s_j)}).$$

The convergence of $\Sigma_j |\Delta\mathbf{E}^t_j|$ ensures the differentiability of the function $t \mapsto \mathcal{J}\mathbf{E}^t_+(\omega)$ in $(0, d)$.

The differentiability a.e. of $t \mapsto \mathbf{q}^t_+(\omega)$ leads to the same property for $t \mapsto |\mathbf{q}^t_+(\omega)|^2$. This allows us to rewrite formula (7.12) as follows

$$\mathcal{D}_m(\sigma_0, P) = \frac{1}{2\pi} \int_\infty^\infty \int_0^d \frac{\mathrm{d}}{\mathrm{d}t} (|\mathbf{q}^t_+(\omega)|^2)\, \mathrm{d}t\, \mathrm{d}\omega. \tag{8.9}$$

Required derivatives will now be written down, understood to exist a.e. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}^t_+(\omega) = -i\omega\mathbf{E}^t_+(\omega) + \mathbf{E}(t),$$

obtained by differentiating the integral definition of $\mathbf{E}^t_+(\omega)$. Using this in (6.7) we find

$$\frac{\mathrm{d}\mathbf{q}^t_\pm(\omega)}{\mathrm{d}t} = -i\omega\mathbf{q}^t_\pm(\omega) - \mathbf{K}_0(t) + \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{\mathbb{H}_-(\omega')}{\omega' - \omega^\mp}\, \mathrm{d}\omega' \mathbf{E}(t),$$

$$\mathbf{K}_0(t) = \frac{1}{2\pi} \int_{-\infty}^\infty \mathbb{H}_-(\omega)\mathbf{E}^t_+(\omega) d\omega = \overline{\mathbf{K}}_0(t), \tag{8.10}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{q}^t_+(\omega) = -i\omega\mathbf{q}^t_+(\omega) - \mathbf{K}(t),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{q}^t_-(\omega) = -i\omega\mathbf{q}^t_-(\omega) - \mathbf{K}(t) + \mathbb{H}_-(\omega)\mathbf{E}(t),$$

$$\begin{aligned}\mathbf{K}(t) &= \mathbf{K}_0(t) + \tfrac{1}{2}\mathbb{H}_{\mathrm{sr}}\mathbf{E}(t)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbb{H}_-(\omega)\left[\mathbf{E}^t_+(\omega) - \frac{\mathbf{E}(t)}{i\omega^-}\right]\mathrm{d}\omega,\end{aligned} \tag{8.11}$$

where $\mathbb{H}_{\mathrm{sr}}$ is defined in (5.5). These results follow from contour integration of the integral in the first relation of (8.10) over $\Omega^{(+)}$. This integral and $\mathbf{K}_0(t)$ exist in the sense of limits or principal values. They are conveniently evaluated by closing the contour, remembering that there is a contribution from the infinite portion of the contour. For a static history, $\mathbf{K}(t)$ vanishes.

From (8.10), (8.11) and the properties of the functions $t \mapsto \mathbf{q}^t_+(\omega)$, $t \mapsto \mathbf{E}(t)$, it follows that $(\omega, t) \mapsto (\mathrm{d}/\mathrm{d}t)(|\mathbf{q}^t_+(\omega)|^2)$ is integrable on $\mathcal{R}^{++} \times (0, d)$, so that Fubini's theorem applies to interchange the order of integration in (8.9) giving

$$\mathcal{D}_m(\sigma_0, P) = \frac{1}{2\pi}\int_0^d\int_{-\infty}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}\left(|\mathbf{q}^t_+(\omega)|^2\right)\,\mathrm{d}\omega\,\mathrm{d}t. \tag{8.12}$$

From (8.8) and the comment immediately following, we deduce that

$$\begin{aligned}&\lim_{|\omega|\to\infty}\omega\mathbf{q}^t_+(\omega) = i\mathbf{K}(t),\\ &\lim_{|\omega|\to\infty}\omega\mathbf{q}^t_-(\omega) = i(\mathbf{K}(t) - \mathbb{H}_{\mathrm{sr}}\mathbf{E}(t)).\end{aligned} \tag{8.13}$$

Also, integrating over the appropriate contour, one obtains

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{q}^t_+(-\omega)\,\mathrm{d}\omega = -\tfrac{1}{2}\mathbf{K}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{q}^t_+(\omega)\,\mathrm{d}\omega,$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{q}^t_-(-\omega)\,\mathrm{d}\omega = \tfrac{1}{2}(\mathbf{K}(t) - \mathbb{H}_{\mathrm{sr}}\mathbf{E}(t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{q}^t_-(\omega)\,\mathrm{d}\omega.$$

The complex conjugates of relations (8.11) are easily established and one deduces from (8.12) that

$$\mathcal{D}_m(\sigma_0, P) = \int_0^d |\mathbf{K}(s)|^2\,\mathrm{d}s,$$

which is obviously nonnegative so that (*P1*) is satisfied by $\psi_m(t)$.

This quantity is clearly differentiable with respect to d. Relabelling d as the current time t, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}_m(\sigma_0, P) = |\mathbf{K}(t)|^2, \tag{8.14}$$

where now P is understood of duration t. Note that functions $\mathcal{D}(\sigma_0, P)$, defined by (8.6) where any free energy $\tilde{\psi}$ replaces the minimum one $\tilde{\psi}_m$, viz.

$$\mathcal{D}(\sigma_0, P) := W(\sigma_0, P) - \tilde{\psi}(\mathbf{E}_P(d), (\mathbf{E}_P * \mathbf{E})^d) + \tilde{\psi}(\mathbf{E}(0), \mathbf{E}^0), \tag{8.15}$$

cannot in general be interpreted as the dissipation in the material because of the non-uniqueness of the free energy functional. In particular, the work $\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t)$ up to time t, given by (4.19), obeys properties *P1–P4* of a free energy. For this choice, $\mathcal{D}(\sigma_0, P)$ vanishes identically even if the material is dissipative [26].

However, the meaning of the dissipation arguably can be assigned to $\mathcal{D}_m(\sigma_0, P)$. In fact, from (8.1), (8.6), (8.7) and (8.15) we obtain

$$\mathcal{D}_m(\sigma_0, P) = [W_R(\sigma_0) + W(\sigma_0, P)] - W_R(\sigma) \geqslant 0. \tag{8.16}$$

Therefore $\mathcal{D}_m(\sigma_0, P)$ may be thought as the 'dissipation' due to the process P in the sense that of the amount of energy that cannot be utilized to do work. In fact the quantity in the square parenthesis of (8.16) represents the energy available to do work from σ_0 added to the energy stored because of the work done during the process P, whereas the last term of (8.16) is the energy available to do work from $\sigma = P\sigma_0$. Therefore the difference yields the part of the energy stored by the work $W(\sigma_0, P)$ that cannot be utilized to do work i.e. the energy dissipated by P. In view of this, we shall refer to the quantity defined by (8.14) as the rate of dissipation.

We conclude the section by drawing attention to certain properties of the optimal future continuation.

PROPOSITION 8.1. *The optimal future continuation* $\mathbf{E}_o^t\colon \mathcal{R}^- \to \mathrm{Sym}$ *maximizing the recoverable work as given by* (4.9) *and* (4.15) *has an initial discontinuity and does not go to zero at* $-\infty$.

Proof. From (6.10) and (8.13), it follows that

$$\mathbf{E}_m^t(\omega) \underset{\omega\to\infty}{\longrightarrow} -\frac{\mathbf{E}(t) - \mathbb{H}_{\mathrm{sr}}^1\mathbf{K}(t)}{i\omega^+}, \tag{8.17}$$

where $\mathbf{E}_m^t$ is given in terms of the optimal continuation $\mathbf{E}_o^t$ by

$$\mathbf{E}_m^t(\omega) = \int_{-\infty}^0 \mathbf{E}_o^t(s)\,\mathrm{e}^{-i\omega s}\,\mathrm{d}s. \tag{8.18}$$

We can determine the form of $\mathbf{E}_o^t(s)$, $s \in \mathcal{R}^{--}$ from $\mathbf{E}_m^t(\omega)$ by the formula

$$\mathbf{E}_o^t(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{E}_m^t(\omega)\,\mathrm{e}^{i\omega s}\,\mathrm{d}\omega, \quad s \in \mathcal{R}^-,$$

evaluated by closing on $\Omega^{(-)}$. For $s \in \mathcal{R}^{++}$, we close on $\Omega^{(+)}$ to get zero. It follows from (8.17) and (8.18) that

$$\lim_{s\to 0} \mathbf{E}^t_o(s) = \mathbf{E}(t) - \mathbb{H}_{\mathrm{sr}}^{-1}\mathbf{K}(t). \tag{8.19}$$

Thus, the optimal deformation involves a sudden discontinuity at time t. It is interesting from a physical point of view that the magnitude of the discontinuity is closely related to the rate of dissipation, given by (8.14).

Also, putting

$$\mathbb{K}_1(\omega) = \frac{1}{\omega}\mathbb{H}_-(\omega),$$

we have from (6.10)

$$\mathbf{E}^t_m(\omega) \underset{\omega\to 0}{\longrightarrow} -\frac{\mathbb{K}_1(0)^{-1}}{2\pi i\omega^+}\int_{-\infty}^{\infty} \mathbb{K}_1(\omega')\mathbf{E}^t_+(\omega')\,\mathrm{d}\omega'.$$

Note that the zero in $\mathbb{H}_-(\omega)$ at the origin is moved slightly into $\Omega^{(-)}$ to maintain consistency with the requirement that its zeros be in that half-plane. This relation gives, with the aid of (2.8)

$$\mathbf{E}^t_o(s) \underset{s\to -\infty}{\longrightarrow} \frac{\mathbb{K}_1(0)^{-1}}{2\pi}\int_{-\infty}^{\infty} \mathbb{K}_1(\omega')E^t_+(\omega')\,\mathrm{d}\omega',$$

which quantity is in general non-zero.

9. The Maximum Recoverable Work as the Minimum Free Energy of Coleman and Owen

The definition of free energy, introduced by Coleman and Owen [6] in their general theory of thermodynamics, may be stated for the case of linear viscoelasticity as recalled in [9, 10].

DEFINITION 9.1. A function of the state $\psi\colon \Sigma \to \mathcal{R}$ is termed a free energy for a viscoelastic material if it is a *lower potential* for the work W, namely if, for every $\varepsilon > 0$ and every $\sigma, \sigma' \in \Sigma$ there exists a $\delta > 0$ such that

$$\psi(\sigma') - \psi(\sigma) < W(\sigma, P) + \varepsilon, \tag{9.1}$$

for every process $P \in \Pi$ such that $\|P\sigma - \sigma'\| < \delta$, where $\|\cdot\|$ is a suitable norm on the space of the states.

For simplicity we add to such a definition the normalization condition $\psi(\sigma_0) = 0$.

Actually, in [9, 10], a particular norm for the space of states was adopted, whereas in the broader framework of [6], there was full freedom in the choice of topology. Here, we consider the norm used in [9, 10] and also two other choices.

It is worth recalling that the existence of lower potentials for the work done during processes has been proved in [10], for the choice of norm used in that work. Indeed, in that case, any particular state σ' can be approximated by subjecting an arbitrary state σ to a sequence of processes P_n. More precisely, for every fixed $\varepsilon > 0$ there exists an index n_ε such that all the states $P_n\sigma$ attained in the sequence for $n > n_\varepsilon$ lie in a neighbourhood of σ' of radius ε, and the sequence of processes converges [9, 10]. In other words, the set of all attainable states, starting from an arbitrary given one, is dense in the whole space of states Σ. Moreover, in [9] it has been shown that the intersection of the set of all functions of state and the set of all free energies in the sense of Graffi contains the set of free energies in the sense of Definition 9.1 when the norm introduced in [10] is used.

It is easy to check that the following proposition holds (see also Proposition 2.5 of [9])

PROPOSITION 9.1. *If a function $\psi\colon \Sigma \to \mathcal{R}$ is a function of the state which satisfies the integrated dissipation inequality* (*P1*) *and it is lower semicontinuous with respect to a certain norm $\|\cdot\|$ on the state of the spaces Σ, then ψ is a lower potential of the work.*

Proof. In fact, if ψ is lower semicontinuous with respect to $\|\cdot\|$, then, for any fixed $\sigma' \in \Sigma$ and any $\varepsilon > 0$ there exists a suitable $\delta > 0$ such that

$$\psi(\sigma') - \psi(\sigma_1) < \varepsilon, \tag{9.2}$$

for every σ_1 such that $\|\sigma_1 - \sigma'\| < \delta$. Combining (9.2) with inequality (8.3) namely

$$\psi(\sigma_1) - \psi(\sigma) \leqslant W(\sigma, P), \quad P\sigma = \sigma_1,$$

we obtain (9.1).

The only issue that arises in Definition 9.1 and Proposition 9.1 is the choice of the topology of Σ with respect to which the neighbourhoods are defined. There are many possible choices of nonequivalent topologies, though some may be more convenient than others. The expressions (6.12), (7.11), (8.4) and the characterization of the state by means of the couple $\sigma(t) = (\mathbf{E}(t), \mathbf{q}_-^t)$ suggest the use of an L^2-type norm $\|\cdot\|$ defined by

$$\|\sigma(t)\|^2 := |\mathbf{E}(t)|^2 + \int_{-\infty}^{\infty} \left|\mathbf{q}_-^t(\omega)\right|^2 \, \mathrm{d}\omega. \tag{9.3}$$

LEMMA 9.1. *The function ψ_m, defined by* (8.2)*, is continuous with respect to the norm $\|\cdot\|$ defined by* (9.3).

Proof. It is easy to check the following estimate

$$\begin{aligned}
&\left|\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{q}_-^t(\omega)\cdot\mathbf{E}(t)\,\mathrm{d}\omega\right| \\
&\quad\leqslant \frac{1}{\pi}\int_{-\infty}^{\infty}\left|\frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{E}(t)\right|\cdot|\mathbf{q}_-^t(\omega)|\,\mathrm{d}\omega \\
&\quad\leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{E}(t)\right|^2\mathrm{d}\omega+\frac{1}{2\pi}\int_{-\infty}^{\infty}|\mathbf{q}_-^t(\omega)|^2\,\mathrm{d}\omega
\end{aligned}$$

and the chain of equalities

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{E}(t)\right|^2\mathrm{d}\omega \\
&\quad= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathbb{H}_+(\omega)\mathbb{H}_+^*(\omega)}{\omega^2}\,\mathrm{d}\omega\mathbf{E}(t)\cdot\mathbf{E}(t) \\
&\quad= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathbb{H}(\omega)}{\omega^2}\,\mathrm{d}\omega\mathbf{E}(t)\cdot\mathbf{E}(t) \\
&\quad= \tfrac{1}{2}(\mathbb{G}_0-\mathbb{G}_\infty)\mathbf{E}(t)\cdot\mathbf{E}(t),
\end{aligned}\tag{9.4}$$

where the second equality is ensured by (5.13) and the third one is a consequence of (3.7). The positivity of the last term is ensured by (3.8), so that the expression (8.2) for ψ_m can be estimated as follows

$$\begin{aligned}
|\psi_m(\sigma(t))| &= |\hat{\psi}_m(\mathbf{E}(t),\mathbf{q}_-^t)|\leqslant\tfrac{1}{2}|\mathbb{G}_0\mathbf{E}(t)\cdot\mathbf{E}(t)| \\
&\quad+\frac{1}{\pi}\left|\mathbf{E}(t)\cdot\int_{-\infty}^{\infty}\frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{q}_-^t(\omega)\,\mathrm{d}\omega\right|+\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\mathbf{q}_-^t(\omega)\right|^2\mathrm{d}\omega \\
&\leqslant \tfrac{1}{2}|(2\mathbb{G}_0-\mathbb{G}_\infty)\mathbf{E}(t)\cdot\mathbf{E}(t)|+\frac{1}{\pi}\int_{-\infty}^{\infty}\left|\mathbf{q}_-^t(\omega)\right|^2\mathrm{d}\omega.
\end{aligned}$$

Therefore there exists a positive constant C_1 such that

$$|\psi_m(\sigma)|\leqslant C_1\|\sigma\|^2,\tag{9.5}$$

for every $\sigma\in\Sigma$. Now ψ_m is a quadratic, positive definite (see $(8.4)_3$) functional on the space of states. Therefore it induces a norm $\|\cdot\|_m$ defined as follows

$$\|\sigma\|_m^2 := \psi_m(\sigma).\tag{9.6}$$

Hence, fixing a state σ', for every state σ_1 we have

$$|\psi(\sigma')-\psi(\sigma_1)| \leqslant \left|\|\sigma'\|_m^2-\|\sigma_1\|_m^2\right|$$

$$\begin{aligned} &= \big(\|\sigma'\|_m + \|\sigma_1\|_m\big)\,\big|\|\sigma'\|_m - \|\sigma_1\|_m\big| \\ &\leqslant \big(2\|\sigma'\|_m + \|\sigma_1 - \sigma'\|_m\big)\,\|\sigma' - \sigma_1\|_m \\ &\leqslant C_1\big(2\|\sigma'\| + \|\sigma_1 - \sigma'\|\big)\,\|\sigma' - \sigma_1\|, \end{aligned}$$

so that the continuity of ψ_m with respect to the norm $\|\cdot\|$ follows.

Now we are able to prove the following

THEOREM 9.1. *The function* ψ_m *defined by* (8.2) *is a free energy in the sense of Definition* 9.1 *when the norm* $\|\cdot\|$ *of* (9.3) *is used.*

Proof. In fact we have already proved in Theorem 8.1 that ψ_m satisfies $(P1)$. Moreover, Lemma 9.1 ensures the continuity, and hence the lower semicontinuity, of ψ_m with respect to the norm $\|\cdot\|$ defined by (9.3). Thus, Definition 9.1 is fulfilled by ψ_m.

The choice (9.3) for the norm on the space of states allows ψ_m to be a lower potential for the work done during processes. Therefore, ψ_m is also the minimal element of the set of lower potentials for the work done during processes, that is contained in the intersection of the set of all the functions of state with the set of all free energies in the sense of Graffi.

Is is easy to prove the following property of $\|\cdot\|_m$, defined by (9.6):

PROPOSITION 9.2. *The norm* $\|\cdot\|_m$ *defined by* (9.6) *is equivalent to the norm* $\|\cdot\|$ *defined by* (9.3).

Proof. The two norms are equivalent if there exist two constants $C_2, C_3 > 0$ such that

$$C_2\|\sigma\|_m \leqslant \|\sigma\| \leqslant C_3\|\sigma\|_m. \tag{9.7}$$

By virtue of (9.5) and (9.6) the leftmost inequality is satisfied with $C_2 = C_1^{-1/2}$. Let

$$\mathbf{p}^t(\omega) := \mathbf{q}^t_-(\omega) - \frac{\mathbb{H}_-(\omega)\mathbf{E}(t)}{i\omega^+}$$

so that, as a consequence of (8.4), ψ_m may be rewritten as

$$\psi_m(\sigma(t)) = \hat{\psi}_m(\mathbf{E}(t), \mathbf{q}^t) = \tfrac{1}{2}\mathbb{G}_\infty\mathbf{E}(t)\cdot\mathbf{E}(t) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\big|\mathbf{p}^t(\omega)\big|^2\,\mathrm{d}\omega \tag{9.8}$$

and $\|\cdot\|^2$ assumes the form

$$\|\sigma(t)\|^2 \;=\; |\mathbf{E}(t)|^2 + \int_{-\infty}^{\infty}\left|\mathbf{p}^t(\omega) + \frac{\mathbb{H}_-(\omega)\mathbf{E}(t)}{i\omega^+}\right|^2\,\mathrm{d}\omega.$$

By virtue of the estimate

$$\frac{1}{2}\int_{-\infty}^{\infty}\left|\mathbf{p}^t(\omega)+\frac{\mathbb{H}_-(\omega)\mathbf{E}(t)}{i\omega^+}\right|^2 \,\mathrm{d}\omega$$

$$\leqslant \int_{-\infty}^{\infty}\left|\mathbf{p}^t(\omega)\right|^2\,\mathrm{d}\omega+\int_{-\infty}^{\infty}\left|\frac{\mathbb{H}_-(\omega)\mathbf{E}(t)}{\omega}\right|^2\,\mathrm{d}\omega$$

and equalities (9.4), it is easy to find a suitable positive constant C_3 such that the rightmost inequality of (9.7) is satisfied.

The following norm was used in [9, 10]:

$$\|\sigma\|_\Sigma := |\mathbb{G}_0\mathbf{E}(t)| + \|\mathbf{E}^t\|_\Gamma, \tag{9.9}$$

where

$$\begin{aligned}\|\mathbf{E}^t\|_\Gamma &:= \sup_{\tau\geqslant 0}\left|\int_0^{+\infty}\dot{\mathbb{G}}(s+\tau)\mathbf{E}^t(s)\,\mathrm{d}s\right| \\ &= \sup_{\tau\geqslant 0}\left|\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{q}^t_-(\omega)\,\mathrm{e}^{-i\omega\tau}\,\mathrm{d}\omega\right|,\end{aligned} \tag{9.10}$$

the second equality holding in view of (7.7). In order to compare the norm $\|\cdot\|$ with $\|\cdot\|_\Sigma$, we consider the estimate

$$\sup_{\tau\geqslant 0}\left|\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{q}^t_-(\omega)\,\mathrm{e}^{-i\omega\tau}\,\mathrm{d}\omega\right| \leqslant \frac{1}{\pi}\int_{-\infty}^{\infty}\left|\frac{\mathbb{H}_+(\omega)}{\omega}\mathbf{q}^t_-(\omega)\right|\,\mathrm{d}\omega$$

and the chain of equalities

$$\begin{aligned}|\mathbb{H}_+(\omega)\mathbf{q}^t_-(\omega)|^2 &= \mathbb{H}^*_+(\omega)\mathbb{H}_+(\omega)\mathbf{q}^t_-(\omega)\cdot\overline{\mathbf{q}}^t_-(\omega) \\ &= \mathbb{H}(\omega)\mathbf{q}^t_-(\omega)\cdot\overline{\mathbf{q}}^t_-(\omega),\end{aligned}$$

that follows from (5.13). Recalling that $\mathbb{H}(\omega)$ is real and symmetric, we see that the rightmost form is real. Moreover, since $\mathbb{H}(\omega)$ is positive definite the following inequality holds

$$\sup_{\mathbf{v}\in\mathrm{Sym}(\Omega)}\mathbb{H}(\omega)\mathbf{v}\cdot\overline{\mathbf{v}} \leqslant \mathrm{tr}(\mathbb{H}(\omega))|\mathbf{v}|^2,$$

where the representation (2.2) for tensors of Lin(Sym(Ω)) has been used. Then $\|\cdot\|_\Gamma$ can be estimated as follows

$$\|\mathbf{E}^t\|_\Gamma \leqslant \left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{tr}(\mathbb{H}(\omega))}{\omega^2}\,\mathrm{d}\omega\right)^{1/2}\left(\frac{1}{\pi}\int_{-\infty}^{\infty}|\mathbf{q}^t_-(\omega)|^2\,\mathrm{d}\omega\right)^{1/2}$$

and using (3.7) we obtain

$$\|\sigma\|_\Sigma \leqslant |\mathbb{G}_0\mathbf{E}(t)| + [\mathrm{tr}(\mathbb{G}_0 - \mathbb{G}_\infty)]^{1/2}\left(\frac{1}{\pi}\int_{-\infty}^{\infty}|\mathbf{q}_-^t(\omega)|^2\,\mathrm{d}\omega\right)^{1/2}.$$

Therefore, with the aid of the inequality $(|a| + |b|)^2 \leqslant 2(|a|^2 + |b|^2)$ we see that there exists a constant $C_4 > 0$ such that

$$\|\sigma\|_\Sigma \leqslant C_4\|\sigma\|.$$

This shows that the norm $\|\cdot\|$ is finer than $\|\cdot\|_\Sigma$, so that all the results proved in [9, 10] with the aid of the latter norm remain valid. In particular, this is the case for the property that the set of all attainable states from a given $\sigma \in \Sigma$ is dense in the whole space Σ, as well as the fact that all lower potentials for the work are also free energies in the sense of Graffi.

We conclude the section by showing that the two norms $\|\cdot\|$ and $\|\cdot\|_m$ satisfy the fading memory principle. In other words, considering the state $\sigma(t + d)$ relate d to the couple $(\mathbf{0}, (\mathbf{0} * \mathbf{E})^{t+d})$, where

$$(\mathbf{0} * \mathbf{E})^{t+d}(s) = \begin{cases} \mathbf{0} & 0 < s \leqslant d, \\ \mathbf{E}^t(s - d) & s > d, \end{cases}$$

obtained by applying the null process of duration d to the initial state related to the pair $(\mathbf{0}, \mathbf{E}^t)$, we have that

$$\lim_{d\to\infty}\|\sigma(t + d)\| = 0. \tag{9.11}$$

In fact, we see that

$$\mathbf{q}_-^{t+d}(\omega) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\mathbb{H}_-(\omega')\mathbf{E}_+^{t+d}(\omega')}{\omega' - \omega^+}\,\mathrm{d}\omega', \tag{9.12}$$

where

$$\begin{aligned}\mathbf{E}_+^{t+d}(\omega) &= \int_0^\infty \mathbf{E}_+^{t+d}(s)\,\mathrm{e}^{-i\omega s}\,\mathrm{d}s \\ &= \int_d^\infty \mathbf{E}_+^t(s - d)\,\mathrm{e}^{-i\omega s}\,\mathrm{d}s = \mathbf{E}_+^t(\omega)\,\mathrm{e}^{-i\omega d}.\end{aligned} \tag{9.13}$$

Let us evaluate (9.12) by closing the contour on Ω^-. The infinite part gives no contribution since $\mathbf{E}_+^t(\omega)$ behaves as ω^{-1} at infinity. The singularities of $\mathbb{H}_-$ in Ω^-, of whatever nature, do not touch the real axis by assumption, and will therefore give results that are in all cases exponentially decaying because of the factor $\mathrm{e}^{-i\omega d}$ in the rightmost expression of (9.13). Thus $\lim_{d\to\infty}\mathbf{q}_-^{t+d}(\omega) = \mathbf{0}$ and (9.11) follows.

Note that, in view of (9.9–9.10) the norm $\|\cdot\|_\Sigma$ satisfies the fading memory property too, as it was recognized in [9]. This is not the case however for the norm induced by the work $\widetilde{W}(\mathbf{E}(t), \mathbf{E}^t)$, given by (6.13), on a suitable subspace of Sym $\times$ Γ [13]. This quantity is referred to as the maximum free energy [27]. We see from (6.13) that it also depends on $\mathbf{q}_+^t$. From the property

$$\mathbf{q}_+^{t+d}(\omega) = \mathbf{q}_-^{t+d}(\omega) - \mathbb{H}_-(\omega)\mathbf{E}_+^{t+d}(\omega) \underset{d\to\infty}{\longrightarrow} -\mathbb{H}_-(\omega)\mathbf{E}_+^t(\omega)\,\mathrm{e}^{-i\omega d}$$

we have, on using (6.13)

$$\lim_{d\to\infty} \widetilde{W}(\mathbf{E}(t+d), \mathbf{E}^{t+d}) = \frac{1}{2}\int_{\infty}^{\infty} \mathbb{H}(\omega)\mathbf{E}_+^t(\omega)\cdot\bar{\mathbf{E}}_+^t(\omega)\,\mathrm{d}\omega \neq 0.$$

10. Particular Cases

Equation (6.12) gives the explicit form of the minimum free energy once a factorization of $\mathbb{H}$ has been carried out, that is once the explicit forms of $\mathbb{H}_+(\omega)$ and $\mathbb{H}_-(\omega)$ are known.

A direct extension to the tensorial case of the method used in [18] for the scalar one, and a simple direct construction of $\mathbb{H}_+(\omega)$ and $\mathbb{H}_-(\omega)$ is possible if $\mathbb{H}$ admits a commutative factorization (i.e. where $\mathbb{H}_+(\omega)$ and $\mathbb{H}_-(\omega)$ commute and the right and left factorizations coincide). This is the case when the eigenspaces of $\mathbb{G}(t)$ do not depend on t. Under such conditions, we have

$$\begin{aligned}
&\mathbb{G}(t) = \sum_{k=1}^{6} G_k(t)\mathbb{B}^k,\\
&\mathbb{H}(\omega) = -\omega\dot{\mathbb{G}}_s(\omega) = \sum_{k=1}^{6} H_k(\omega)\mathbb{B}^k, \qquad (10.1)\\
&H_k(\omega) > 0; \quad \omega \in \mathcal{R}\setminus\{0\}, \quad k = 1, \dots, 6,
\end{aligned}$$

where $\mathbb{B}^k = \mathbf{B}^k \otimes \mathbf{B}^k$ $k = 1, \ldots, 6$ are the orthogonal projectors on the 6 (constant) eigenspaces of $\mathbb{H}$ and $\{\mathbf{B}^k\}$ are its normal eigenvectors, which constitute an orthonormal basis of Sym. This is a special case of (2.2). The individual quantities $G_k(t)$ satisfy the properties of relaxation functions, as listed in Section 2, interpreted for the scalar case. The related tensor $\mathbb{K}$, defined by (5.4), can be written as

$$\mathbb{K}(\omega) = \sum_{k=1}^{6} \mathcal{K}_k(\omega)\mathbb{B}^k, \quad \mathcal{K}_k(\omega) > 0; \quad \omega \in \mathcal{R}, \ k = 1, \dots, 6,$$

where, denoting $H_k(\infty)$ by $H_{k\infty}$

$$\mathcal{K}_k(\omega) = \frac{1+\omega^2}{\omega^2}\frac{H_k(\omega)}{H_{k\infty}} > 0, \quad \forall \omega \in \mathcal{R},$$

so that the functions

$$L_k(\omega) = \log[\mathcal{K}_k(\omega)]$$

are well-defined on $\mathcal{R}$. The Plemelj formulae give that

$$L_k(\omega) = M_k^+(\omega) - M_k^-(\omega),$$

where

$$M_k(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{L_k(\omega)}{\omega - z}\,\mathrm{d}\omega \quad z \in \Omega \setminus \mathcal{R},$$

$$M_k^{\pm}(\omega) = \lim_{\alpha\to 0^+} M_k(\omega \pm i\alpha)$$

and the factorization of $H_k(\omega) = H_{k+}(\omega)H_{k-}(\omega)$ is given by

$$H_{k\pm}(\omega) = \frac{\omega}{\omega \mp i}\,h_{k\infty}\,\mathrm{e}^{\mp M_k^{\mp}(\omega)}, \quad h_{k\infty} = H_{k\infty}^{1/2}. \tag{10.2}$$

Since the $\{\mathbb{B}^k\}$ are orthonormal projectors, we have

$$\mathbb{H}(\omega) = \sum_{k=1}^{6} H_{k+}(\omega)H_{k-}(\omega)\mathbb{B}^k = \mathbb{H}_+(\omega)\mathbb{H}_-(\omega),$$

where

$$\mathbb{H}_{\pm}(\omega) = \sum_{k=1}^{6} H_{k\pm}(\omega)\mathbb{B}^k.$$

It is clear that the factors $\mathbb{H}_+(\omega)$ and $\mathbb{H}_-(\omega)$ commute.

In the basis $\{\mathbf{B}^k\}$, the individual components of each of the relevant quantities obey precisely the relationships that hold in the scalar case, as developed in [18]. Let us expand all relevant quantities in this basis

$$\begin{aligned}\mathbf{E}(t) &= \sum_{k=1}^{6} E_k(t)\mathbf{B}^k, \qquad \mathbf{E}_+^t(\omega) = \sum_{k=1}^{6} E_{k+}^t(\omega)\mathbf{B}^k,\\ \mathbf{E}_m^t(\omega) &= \sum_{k=1}^{6} E_{mk}^t(\omega)\mathbf{B}^k, \qquad \mathbf{E}_o^t(s) = \sum_{k=1}^{6} E_{ok}^t(s)\mathbf{B}^k\end{aligned} \tag{10.3}$$

and

$$\mathbf{Q}^t(\omega) = \sum_{k=1}^{6} Q_k^t(\omega)\mathbf{B}^k, \qquad \mathbf{q}_\pm^t(\omega) = \sum_{k=1}^{6} q_{k\pm}^t(\omega)\mathbf{B}^k,$$

where $q_{k\pm}^t(\omega)$ is related to $Q_k^t(\omega) = H_{k-}(\omega)E_{k+}(\omega)$ in the same manner as $\mathbf{q}^t$ and $\mathbf{Q}^t$ defined by (6.6) and (6.7), namely

$$q_k^t(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{Q_k^t(\omega)}{\omega - z}\,\mathrm{d}\omega, \qquad q_{k\pm}^t(\omega) = \lim_{\alpha\to 0^\mp} q_k^t(\omega + i\alpha). \tag{10.4}$$

Finally, the quantity $\mathbf{K}(t)$, given by (8.11), has the form

$$\begin{aligned} \mathbf{K}(t) &= \sum_{k=1}^{6} K_k(t)\mathbf{B}^k, \\ K_k(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{k-}(\omega)\left[E_{k+}^t(\omega) - \frac{E_k(t)}{i\omega^+}\right]\mathrm{d}\omega. \end{aligned} \tag{10.5}$$

The minimum free energy defined by (8.1) and (6.12) becomes

$$\psi_m(t) = S(t) + \frac{1}{2\pi}\sum_{k=1}^{6}\int_{-\infty}^{\infty} \left|q_{k-}^t(\omega)\right|^2 \mathrm{d}\omega, \tag{10.6}$$

where

$$\begin{aligned} S(t) &= \sum_{k=1}^{6} S_k(t), \\ S_k(t) &= T_k(t)E_k(t) - \tfrac{1}{2}G_k(0)E_k^2(t), \\ T_k(t) &= G_k(0)E_k(t) + \int_0^{\infty} \dot{G}_k(s)E_k^t(s)\,\mathrm{d}s, \end{aligned} \tag{10.7}$$

while the rate of dissipation, defined by (8.14), is given by

$$D_m(t) = \sum_{k=1}^{6} K_k^2(t). \tag{10.8}$$

A nontrivial extension to the scalar case, satisfying the requirement of time independent eigenspaces, is when the relaxation function is a finite sum of exponentials of the form

$$\mathbb{G}(t) = \mathbb{G}_\infty + \sum_{i=1}^{n} \mathbb{G}_i e^{-\mathbb{D}_i t},$$

where n is a positive integer and $\mathbb{G}_\infty, \mathbb{G}_i, \mathbb{D}_i, i = 1, \ldots, n$ are constant, symmetric positive definite tensors which commute. In this case, there exists a orthonormal basis of Sym, independent of t, which will be denoted as before by $\{\mathbf{B}^k\}, k = 1, \ldots, 6$, in which all the tensors $\mathbb{G}_\infty, \mathbb{G}_i, \mathbb{D}_i$, are diagonal. The quantities $G_k(t)$, defined by (10.1), have the form

$$G_k(t) = G_{k\infty} + \sum_{i=1}^{n} G_{ik}\,\mathrm{e}^{-\alpha_{ik}t},$$

where the inverse decay times $\alpha_{ik} \in R^+, i = 1, 2, \ldots n$ and the coefficients G_{ik} are also generally assumed to be positive. Labelling is arranged so that $\alpha_{1k} < \alpha_{2k} < \alpha_{3k} \ldots$. We have

$$G_k'(t) = \sum_{i=1}^{n} g_{ik}\,\mathrm{e}^{-\alpha_{ik}t}, \quad g_{ik} = -\alpha_{ik}G_{ik} < 0 \tag{10.9}$$

and [18]

$$H_k(\omega) = -\omega^2 \sum_{i=1}^{n} \frac{g_{ik}}{\alpha_{ik}^2 + \omega^2} \geqslant 0.$$

Observe that $f(z) = H_k(\omega), z = -\omega^2$ has simple poles at $\alpha_{ik}^2, i = 1, 2, \ldots, n$. It will therefore have zeros at $\gamma_{ik}^2, i = 2, 3, \ldots, n$ where

$$\alpha_{1k}^2 < \gamma_{2k}^2 < \alpha_{2k}^2 < \gamma_{3k}^2 \ldots.$$

The function $f(z)$ also vanishes at $\gamma_{1k} = 0$. Therefore

$$H_k(\omega) = H_{k\infty} \prod_{i=1}^{n} \left\{ \frac{\gamma_{ik}^2 + \omega^2}{\alpha_{ik}^2 + \omega^2} \right\}$$

and either by inspection or by applying the general formula (10.2), one can show that

$$\begin{aligned} H_{k+}(\omega) &= h_{k\infty} \prod_{i=1}^{n} \left\{ \frac{\omega - i\gamma_{ik}}{\omega - i\alpha_{ik}} \right\}, \\ H_{k-}(\omega) &= h_{k\infty} \prod_{i=1}^{n} \left\{ \frac{\omega + i\gamma_{ik}}{\omega + i\alpha_{ik}} \right\}. \end{aligned} \tag{10.10}$$

We have

$$\begin{aligned} H_{k-}(\omega) &= h_{k\infty} \left[1 + i \sum_{i=1}^{n} \frac{R_{ik}}{\omega + i\alpha_{ik}} \right], \quad H_{k+}(\omega) = \overline{H}_{k-}(\omega), \\ R_{ik} &= (\gamma_{ik} - \alpha_{ik}) \prod_{\substack{j=1 \\ j \neq i}}^{n} \left\{ \frac{\gamma_{jk} - \alpha_{ik}}{\alpha_{jk} - \alpha_{ik}} \right\}. \end{aligned} \tag{10.11}$$

The quantities $q^t_{k-}(\omega)$, defined by (10.4), may be evaluated by closing on $\Omega^{(-)}$, giving

$$q^t_{k-}(\omega) = ih_{k\infty}\sum_{i=1}^{n}\frac{R_{ik}E^t_{k+}(-i\alpha_{ik})}{\omega + i\alpha_{ik}}. \tag{10.12}$$

Also

$$\begin{aligned} q^t_{k+}(\omega) &= q^t_{k-}(\omega) - H_{k-}(\omega)E^t_{k+}(\omega) \\ &= ih_{k\infty}\sum_{i=1}^{n}\frac{R_{ik}\left[E^t_{k+}(-i\alpha_{ik}) - E^t_{k+}(\omega)\right]}{\omega + i\alpha_{ik}} - h_{k\infty}E^t_{k+}(\omega), \end{aligned}$$

which has singularities at those of $E^t_{k+}(\omega)$ in $\Omega^{(+)}$. From (6.10), (10.10) and (10.12)

$$E^t_{mk}(\omega) = -i\sum_{i=1}^{n}J_{ik}(\omega)R_{ik}E^t_{k+}(-i\alpha_{ik}),$$

$$J_{ik}(\omega) = \frac{\prod_{j=1\,j\neq i}^{n}(\omega + i\alpha_{jk})}{\prod_{j=1}^{n}(\omega + i\gamma_{jk})} = \sum_{l=1}^{n}\frac{K_{ilk}}{\omega + i\gamma_{lk}},$$

where

$$K_{ilk} = \frac{\prod_{j=1\,j\neq i}^{n}(\gamma_{lk} - \alpha_{jk})}{\prod_{j=1\,j\neq l}^{n}(\gamma_{lk} - \gamma_{jk})},$$

so that

$$E^t_{mk}(\omega) = -i\sum_{l=1}^{n}\frac{B^t_{lk}}{\omega + i\gamma_{lk}},$$

$$B^t_{lk} = \sum_{i=1}^{n}R_{ik}K_{ilk}E^t_{k+}(-i\alpha_{ik}).$$

We conclude that the optimal deformation components, as defined in (10.3), have the form

$$\begin{aligned} E^t_{ok}(s) &= -\sum_{l=1}^{n}B^t_{lk}\,\mathrm{e}^{\gamma_{lk}s}, \quad s < 0 \\ &= B^t_{1k} - \sum_{l=2}^{n}B^t_{lk}\,\mathrm{e}^{\gamma_{lk}s}. \end{aligned}$$

Note that

$$E^t_{ok}(-\infty) = -B^t_{1k},$$

which is in general nonzero. By considering $\omega J_{ik}(\omega)$ for large ω, it can be deduced that

$$\sum_{l=1}^{n} K_{ilk} = 1,$$

so that

$$\begin{aligned}\sum_{l=1}^{n} B^t_{lk} &= \sum_{i=1}^{n} R_{ik} E^t_{k+}(-i\alpha_{ik}) \\ &= -E^t_{ok}(0).\end{aligned}$$

From (8.11), (10.5) and (10.11) we have

$$K_k(t) = h_{k\infty}\left[\sum_{i=1}^{n} R_{ik} E^t_{k+}(-i\alpha_{ik}) + E_k(t)\right],$$

so that the contribution to the discontinuity from the kth component at time t is $K_k(t)/h_{k\infty}$, which, on summation, agrees with the general result (8.19). From (10.11) and the fact that $H_{k-}(0) = 0$, we conclude that

$$\sum_{i=1}^{n} \frac{R_{ik}}{\alpha_{ik}} = -1$$

giving

$$K_k(t) = h_{k\infty} \sum_{i=1}^{n} \frac{R_{ik}}{\alpha_{ik}} \left[\alpha_{ik} E^t_{k+}(-i\alpha_{ik}) - E_k(t)\right],$$

which also follows from the last relation of (8.11).

We deduce from (10.12) that

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega \left|q^t_{k-}(\omega)\right|^2 \\ &\quad = H_{k\infty} \sum_{i,j=1}^{n} \frac{R_{ik}R_{jk}}{\alpha_{ik}+\alpha_{jk}} E^t_{k+}(-i\alpha_{ik}) E^t_{k+}(-i\alpha_{jk}) \\ &\quad = \frac{1}{2}\int_0^{\infty}\int_0^{\infty} E^t_k(s_1) F(s_1, s_2) E^t_k(s_2)\, \mathrm{d}s_1\, \mathrm{d}s_2,\end{aligned}$$

$$F(s_1, s_2) = 2H_{k\infty} \sum_{i,j=1}^{n} \frac{R_{ik}R_{jk}}{\alpha_{ik}+\alpha_{jk}} \mathrm{e}^{-\alpha_{ik}s_1 - \alpha_{jk}s_2}$$

and $S(t)$ is easily evaluated from (10.8) and (10.9). The minimum free energy $\psi_m(t)$ is given by (10.6). Also

$$\begin{aligned} K_k^2(t) &= H_{k\infty}\left\{\sum_{i=1}^{n}\frac{R_{ik}}{\alpha_{ik}}\left[\alpha_{ik}E_{k+}^t(-i\alpha_{ik}) - E(t)\right]\right\}^2 \\ &= H_{k\infty}\left\{\int_0^\infty \mathrm{d}s\sum_{i=1}^{n} R_{ik}\,\mathrm{e}^{-\alpha_{ik}s}[E_k^t(s) - E_k(t)]\right\}^2 \end{aligned}$$

and $D_m(t)$ is given by (10.8).

Appendix. Functions of Special Bounded Variation

The restriction of $\mathbf{E}^t\colon \mathcal{R}^{++} \to \mathrm{Sym}$ to the bounded interval $[a,b]$, $0 < a < b < +\infty$ is denoted by $\mathbf{E}^t_{[a,b]}$. We assume that $\mathbf{E}^t_{[a,b]}$ is continuous from the left and belongs to SBV($[a,b]$; Sym)), where SBV denotes the space of functions of special bounded variation, namely, of functions that are of bounded variation but with null cantorian part.

The meaning of this terminology is recalled here, making use of the Lebesgue decomposition for functions of bounded variation ([25], p. 341). This decomposition entails the representation formula

$$\mathbf{E}^t_{[a,b]}(s) = \mathbf{E}^{t,c}_{[a,b]}(s) + \mathcal{J}\mathbf{E}^t_{[a,b]}(s),$$

where $\mathbf{E}^{t,c}_{[a,b]}$ is a continuous function of bounded variation and $\mathcal{J}\mathbf{E}^t_{[a,b]}$ is a jump function, which can be written as

$$\mathcal{J}\mathbf{E}^t_{[a,b]}(s) = \sum_{s_j > s}\Delta\mathbf{E}^t_j\ , \tag{A.1}$$

where s_j are at most countably many discontinuity points in $\mathbf{E}^t_{[a,b]}$ and the quantities $\Delta\mathbf{E}^t_j$ are the discontinuities of $\mathbf{E}^t$ at s_j. We assume that $\Sigma_i|\Delta\mathbf{E}^t_i| < +\infty$. On the other hand, $\mathbf{E}^{t,c}_{[a,b]}$ can be decomposed in the sum $\mathbf{E}^{t,ac}_{[a,b]} + \mathbf{E}^{t,sc}_{[a,b]}$, where

$$\mathbf{E}^{t,ac}_{[a,b]}(s) = \mathbf{E}^{t,ac}_{[a,b]}(a) + \int_a^s \dot{\mathbf{E}}^{t,c}_{[a,b]}(r)\,\mathrm{d}r$$

is an absolutely continuous function, and $\mathbf{E}^{t,sc}_{[a,b]} = \mathbf{E}^{t,c}_{[a,b]} - \mathbf{E}^{t,ac}_{[a,b]}$ has the property that $\dot{\mathbf{E}}^{t,sc}_{[a,b]} = 0$ almost everywhere in $[a,b]$ and for this reason is called the singular or cantorian part of $\mathbf{E}^{t,c}_{[a,b]}$. Because functions of SBV($[a,b]$, Sym) are of bounded variation on $[a,b]$ with no cantorian part, the formula

$$\begin{aligned} \mathbf{E}^t_{[a,b]}(s) &= \mathbf{E}^{t,ac}_{[a,b]}(s) + \mathcal{J}\mathbf{E}^t_{[a,b]}(s) \\ &= \mathbf{E}^{t,ac}_{[a,b]}(a) + \int_a^s \dot{\mathbf{E}}^{t,c}_{[a,b]}(r)\,\mathrm{d}r + \mathcal{J}\mathbf{E}^t_{[a,b]}(s) \end{aligned} \tag{A.2}$$

holds. For simplicity, throughout this paper, $\dot{\mathbf{E}}^t_{[a,b]}$ denotes the distribution such that

$$\int_a^s \dot{\mathbf{E}}^t_{[a,b]}(r)\,\mathrm{d}r = \int_a^s \dot{\mathbf{E}}^{t,c}_{[a,b]}(r)\,\mathrm{d}r + \mathcal{J}\mathbf{E}^t_{[a,b]}(s). \tag{A.3}$$

Roughly speaking $\dot{\mathbf{E}}^t_{[a,b]}$ is the sum of an L^1-function and some isolated Dirac's δ-distributions corresponding to the jumps. Then (A.2) becomes

$$\mathbf{E}^t_{[a,b]}(s) = \mathbf{E}^{t,ac}_{[a,b]}(a) + \int_a^s \dot{\mathbf{E}}^t_{[a,b]}(r)\,\mathrm{d}r.$$

As a consequence, by the definition $\mathbf{E}^t(s) := \mathbf{E}(t-s)$, the strain $\mathbf{E}\colon (\infty, t] \to \mathrm{Sym}$ is continuous from the right and, for every fixed interval $[c,d] \subset (\infty, t]$, $\mathbf{E}\colon [c,d] \to \mathrm{Sym}$ can be represented by

$$\mathbf{E}(\tau) = \mathbf{E}(c) + \int_c^\tau \dot{\mathbf{E}}(r)\,\mathrm{d}r, \tag{A.4}$$

where the notation $\dot{\mathbf{E}}$ is analogous to $\dot{\mathbf{E}}^t_{[a,b]}$ defined as (A.3), viz.

$$\int_c^\tau \dot{\mathbf{E}}(r)\,\mathrm{d}r = \int_c^\tau \dot{\mathbf{E}}(r)\,\mathrm{d}r + \mathcal{J}\mathbf{E}(\tau), \quad \mathcal{J}\mathbf{E}(\tau) = \sum_{\tau_j<\tau} \Delta\mathbf{E}_j. \tag{A.5}$$

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