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Toward a Field Theory for Elastic Bodies Undergoing Disarrangements

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Abstract. Structured deformations are used to refine the basic ingredients of continuum field theories and to derive a system of field equations for elastic bodies undergoing submacroscopically smooth geometrical changes as well as submacroscopically non-smooth geometrical changes (*disarrangements*). The constitutive assumptions employed in this derivation permit the body to store energy as well as to dissipate energy in smooth dynamical processes. Only one non-classical field G, the deformation without disarrangements, appears in the field equations, and a consistency relation based on a decomposition of the Piola–Kirchhoff stress circumvents the use of additional balance laws or phenomenological evolution laws to restrict G. The field equations are applied to an elastic body whose free energy depends only upon the volume fraction for the structured deformation. Existence is established of two universal phases, a spherical phase and an elongated phase, whose volume fractions are $(1-\gamma_0)^3$ and $(1-\gamma_0)$ respectively, with $\gamma_0 := (\sqrt{5}-1)/2$ the "golden mean".

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1. Introduction

The vast scope of elasticity as a continuum field theory includes the description at the macrolevel of the dynamical evolution of bodies that undergo large deformations, that respond to smooth changes in geometry by storing mechanical energy, and that experience internal dissipation in isothermal motions only during non-smooth macroscopic changes in geometry such as shock waves. The research described in this paper represents the first step in a program to employ structured deformations of continua to obtain a field theory capable of describing, in the context of dynamics and large isothermal deformations, the evolution of bodies that (i) undergo smooth deformations at the macroscopic length scale, that (ii) can experience piecewise smooth deformations at submacroscopic length scales, and that (iii) can not only store energy but can also dissipate energy during such multiscale geometrical changes.

The main goal of the present study is the derivation of the following field relations governing the smooth vector field χ that describes the macroscopic changes in geometry of a body and the smooth tensor field G that describes the contribution at the macrolevel of only the smooth part of the submacroscopic piecewise smooth geometrical changes experienced by the body. If we put $M := \nabla\chi - G$ and $K := (\nabla\chi)^{-1}G$, the desired relations are (10.1)–(10.5), which we record in the following detailed form:

$$\begin{aligned}
&\mathrm{div}_X(D_M\tilde{\Psi}(M(X,t),G(X,t)) + D_G\tilde{\Psi}(M(X,t),G(X,t)))\\
&\qquad + b_{\mathrm{ref}}(X,t) = \rho_{\mathrm{ref}}(X)\ddot{\chi}(X,t), &(1.1)\\
&D_G\tilde{\Psi}(M(X,t),G(X,t))(K(X,t)^{-T} - I)\\
&\qquad + D_M\tilde{\Psi}(M(X,t),G(X,t))K(X,t)^{-T} = 0, &(1.2)\\
&\mathrm{sk}(D_G\tilde{\Psi}(M(X,t),G(X,t))M(X,t)^T)\\
&\qquad + \mathrm{sk}(D_M\tilde{\Psi}(M(X,t),G(X,t))\,G(X,t)^T) = 0, &(1.3)\\
&D_G\tilde{\Psi}(M(X,t),G(X,t))\cdot\dot{M}(X,t)\\
&\qquad + D_M\tilde{\Psi}(M(X,t),G(X,t))\cdot\dot{G}(X,t) \geqslant 0, &(1.4)\\
&\det(G(X,t) + M(X,t)) \geqslant \det G(X,t) > m(t) > 0. &(1.5)
\end{aligned}$$

Here, $(M,G) \mapsto \tilde{\Psi}(M,G)$ is the response function that gives the Helmholtz free energy density $\tilde{\Psi}(M(X,t),G(X,t))$ at each point X in the reference configuration and each time t, $D_M\tilde{\Psi}$ and $D_G\tilde{\Psi}$ denote its partial derivatives, b_{ref} is the body force in the reference configuration, ρ_{ref} is the mass density in the reference configuration, $m(t)$ is a positive number depending on time, I is the identity tensor, sk denotes the skew part of a tensor, and superposed dots denote differentiation with respect to time. The balance of linear momentum (1.1), the "consistency relation" (1.2), and the frame-indifference relation (1.3) amount to 12 scalar equations that restrict the unknown fields χ and G representing 12 scalar fields in all. The Piola–Kirchhoff stress field S in the reference configuration is related constitutively to the fields G and M by the stress relation

$$S(X,t) = D_M\tilde{\Psi}(M(X,t),G(X,t)) + D_G\tilde{\Psi}(M(X,t),G(X,t)), \tag{1.6}$$

and the "mixed power" inequality (1.4) guarantees that the internal dissipation is non-negative on each dynamical process of the body. Finally, the inequality (1.5) guarantees that no interpenetration of matter occurs submacroscopically [1]. In addition to the frame-indifference relation (1.3), the free energy response function $\tilde{\Psi}$ is required to be frame-indifferent in the sense described in Section 9. There we show that these two conditions of frame-indifference imply that the law of balance of angular momentum is satisfied and, hence, need not be imposed directly.

The theory of structured deformations [1, 2] shows that the tensor field $M = \nabla\chi - G$ describes the contributions at the macrolevel of "disarrangements," i.e., of the non-smooth part of piecewise smooth submacroscopic geometrical changes, and we use the term "elasticity with disarrangements" to distinguish the nascent field theory described in (1.1)–(1.5) from that embodied in the now standard field theory of non-linear elasticity:

$$\mathrm{div}_X(D\Psi(\nabla\chi(X,t))) + b_{\mathrm{ref}}(X,t) = \rho_{\mathrm{ref}}(X)\ddot{\chi}(X,t), \tag{1.7}$$

$$\det\nabla\chi(X,t) > 0 \tag{1.8}$$

along with the stress relation

$$S(X,t) = D\Psi(\nabla\chi(X,t)). \tag{1.9}$$

We note that the balance of angular momentum need not be imposed explicitly if the response function $F \mapsto \Psi(F)$ is required to be frame-indifferent.

We describe in this paper how the multiscale geometry embodied in structured deformations affords not only the decomposition

$$\nabla\chi = G + M \tag{1.10}$$

of the macroscopic deformation gradient $\nabla\chi$ into a part M due to disarangements and a part G without disarrangements, but also the decomposition

$$(\det K)S = S_\backslash + S_d, \tag{1.11}$$

with $S_\backslash := (\det K)SK^{-T}$ the stress without disarrangements and $S_d := (\det K)S - S_\backslash$ the stress due to disarrangements. The decomposition of stress (1.11) is the basis for the consistency relation (3.6) which, in turn, yields the field equation (1.2), once constitutive assumptions are laid down. The two decompositions (1.10) and (1.11) are central to the "top-down" nature of our methodology, in which standard macroscopic fields, such as the stress power $S \cdot \nabla\dot{\chi}$ and the volume density of moments due to contact forces $\mathrm{sk}(S(\nabla\chi)^T)$, are refined and enriched by substitution of the decompositions (1.10) and (1.11) for the factors S and $\nabla\chi$:

$$(\det K)S \cdot \nabla\dot{\chi} = S_\backslash \cdot \dot{G} + S_d \cdot \dot{M} + S_\backslash \cdot \dot{M} + S_d \cdot \dot{G}, \tag{1.12}$$

$$\begin{aligned}&(\det K)\mathrm{sk}(SF^T)\\ &\quad = \mathrm{sk}(S_\backslash G^T) + \mathrm{sk}(S_d M^T) + \mathrm{sk}(S_\backslash M^T) + \mathrm{sk}(S_d G^T).\end{aligned} \tag{1.13}$$

We utilize in the sequel an "identification relation" $(2.2)_2$ for the field G, an identification relation (2.4) for M, and one for $\mathrm{div}S_\backslash$ (relation (A.3) in the Appendix), that are provided by the theory of structured deformations. These relations

(i) describe G, M and $\mathrm{div}S_\backslash$ as limits of geometrical or statical quantities calculated in terms of the piecewise smooth, injective deformations that approximate a structured deformation,
(ii) justify the attributes "without disarrangements" and due to "disarrangements", and
(iii) provide unambiguous interpretations for the terms in the decompositions (1.12) and (1.13).

This methodology is supplemented by factorizations of the type

$$(\chi, G) = (\chi, \nabla\chi) \circ (\pi, K) \tag{1.14}$$

in which the pair $(\chi, \nabla\chi)$ represents only classical geometrical changes and (π, K) represents purely submacroscopic geometrical changes. This factorization permits us to introduce a "virgin configuration," macroscopically identical to the reference configuration, and to interpret the stress without disarrangements $S_\backslash$ as a stress in the virgin configuration. In the case of invertible structured deformations, the virgin configuration also can serve as an intermediate configuration for theories based on classical deformations.

We are able with these tools to scrutinize and refine principal ingredients in continuum field theories, namely,

· geometry	· power	· material symmetry
· kinematics	· dissipation	· material frame indifference
· forces and moments	· constitutive relations	· material uniformity,

and to arrive at the field relations (1.1)–(1.5) for an elastic body undergoing disarrangements. The new relations derived here incorporate the effects of submacroscopic disarrangements, such as slips, separations, the formation of voids, and the switching or reorientation of submacroscopic units. They also cover submacroscopically smooth geometrical changes, such as the distortion of atomic lattices and of molecular networks at length scales large enough to justify the use of smooth fields to extrapolate the discrete geometrical changes of the lattice or network. However, relations (1.1)–(1.5) do not directly incorporate the effect of jumps in the gradients of approximating piecewise smooth deformations ("gradient disarrangements"), so that fine mixtures of phases are not captured. Moreover, the effects of time-like disarrangements, in which changes in position occur at very short time scales, are not incorporated, and macroscopic disarrangements such as fracture, shear bands, shock waves, and acceleration waves are formally excluded by our assumption that χ and G are smooth. Nevertheless, the inclusion of macroscopic disarrangements can be accomplished in a manner analogous to that used in the field theory based on (1.7)–(1.9). In addition, time-like disarrangements and gradient disarrangements are amenable to treatment via the concepts of "structured motions" [3, 4] and of "second-order structured deformations" [5] that go beyond the geometry and the kinematics in this paper. We note also that couple stresses and other multipolar entities, temperature variations, electromagnetic fields, and chemical reactions are left out of the present development.

It is evident from the constitutive expression $\tilde{\Psi}(M(X,t), G(X,t))$ for the volume density of the Helmholtz free energy that our theory permits energy to be stored by means of both smooth and non-smooth submacroscopic geometrical changes. For example, contribution to the energy both from the distortion of a crystalline lattice between slip bands and the relative translations of parts of the crystalline lattice across slip bands can be included here, because the former is captured by $G(X,t)$ and the latter by $M(X,t)$. Similarly, the macroscopic stretching produced when a polymer network deforms can be identified through $\nabla\chi(X,t) = G(X,t) + M(X,t)$ while the submacroscopic reorientations of attached nematic particles can be described by $G(X,t)$, so that $\tilde{\Psi}(M(X,t), G(X,t))$ can reflect the energetic contributions of both. Our development includes the possibility that $\tilde{\Psi}$ depends upon the material point X explicitly, and we discuss the concepts of material uniformity and homogeneity in Section 13.

The response function $(M,G) \mapsto \tilde{\Psi}(M,G)$ may or may not be obtained by means of a process of homogenization or relaxation from an "initial" response function describing the energy stored in piecewise smooth approximating deformations. In fact, it has been shown that such a relaxation procedure, starting from a standard form of the initial energy, does lead to response functions of the type $(M,G) \mapsto \tilde{\Psi}(M,G)$ [6], but also that the specific dependence on M and G obtained by such a relaxation may exclude response functions already found to be useful in applications ([3, 7–9]). We expect that response functions $\tilde{\Psi}$ identified in a variety of ways will play a role in applying the field relations (1.1)–(1.5) to specific bodies.

In the present paper we describe a class of response functions $\tilde{\Psi}$ restricted only by considerations of material frame indifference, material symmetry, or material uniformity as explained in Sections 9, 12, and 13. This choice provides a broad view of elasticity with disarrangements, but does not provide for the moment insights into specific solid or fluid bodies encountered in the laboratory. However, we do include a specific example, that of an "energetically nearsighted elastic body," in which the constitutive relation for the free energy takes the form

$$(M,G) \mapsto \tilde{\Psi}(M,G) = \bar{\psi}\left(\frac{\det G}{\det(G+M)}\right). \tag{1.15}$$

We note by (1.5) that $\det G/\det(G+M) = \det K$ takes values in (0, 1], and we can interpret $1-\det K$ as the "void fraction" created by the purely submacroscopic factor (i, K) in (1.14). Similarly, we call $\det K$ the "volume fraction" associated with (i, K). We show that the consistency relation (1.2), rewritten in terms of the variables $\nabla\chi$ and K, and the constitutive assumption (1.15) imply that such an elastic body can arise in two non-trivial "universal" phases: a spherical phase, in which $\det K = (1-\gamma_0)^3$, and an elongated phase, in which $\det K = 1-\gamma_0$, where $\gamma_0 := \frac{\sqrt{5}-1}{2}$ is the golden mean. The stress relation (1.6) for the spherical phase reduces to that of an ideal gas, so that the stress in the current configuration is a hydrostatic pressure that depends linearly on the density in the current configuration. For the elongated phase, a uniaxial stress in the direction of submacroscopic elongation is superposed on such a hydrostatic pressure, as an outcome of the stress relation (1.6).

We turn now to further introductory remarks on the nature of the field relations (1.1)–(1.5). Suppose that the response function $(M,G) \mapsto \tilde{\Psi}(M,G)$ is chosen to satisfy the condition

$$D_M\tilde{\Psi}(0,G) = 0, \tag{1.16}$$

for all tensors G with $\det G > 0$. If we consider a smooth deformation χ and put $G := \nabla\chi$, then the tensor field $M = \nabla\chi - G$ is identically zero and (1.2)–(1.4) are satisfied identically, the last with "$\leqslant$" replaced by "$=$". Consequently a classical motion satisfies (1.1)–(1.4) if and only if

$$\operatorname{div}_X(D_G\tilde{\Psi}(0, \nabla\chi(X,t))) + b_{\text{ref}}(X,t) = \rho_{\text{ref}}(X)\ddot{\chi}(X,t), \tag{1.17}$$

which is equivalent to (1.7), the balance of linear momentum for a non-linearly elastic body. The inequality (1.5) and the relation $G = \nabla\chi$ yield (1.8), and the stress relation (1.6) for an elastic body undergoing disarrangements reduces, by virtue of (1.16), to

$$S(X,t) = D_G\tilde{\Psi}(0, \nabla\chi(X,t)), \tag{1.18}$$

a relation equivalent to (1.9). Therefore, relation (1.16) implies that classical motions that satisfy the new relations (1.1)–(1.6) also satisfy the field relations (1.7)–(1.9) of non-linear elasticity. We note also that, in the example of energetically nearsighted elastic bodies treated in Section 14, the condition (1.16) is satisfied only in exceptional cases.

The statical quantities underlying the relations (1.1)–(1.6) all can be expressed in terms of the classical measure of stress S, and the only balance law directly imposed is the classical balance of linear momentum. This observation provides a point of contrast between elasticity with disarrangements and theories of "structured continua," in which additional geometrical fields are accompanied by additional statical quantitities and additional balance laws [10, 11]. Moreover, the presence of the non-classical geometrical field G in the present theory does not require that we impose constitutively an evolution law expressing $\dot{G}$ in terms of other geometrical and statical quantities, as is the case in theories of materials with "internal variables." Instead, the decomposition (1.11) for the stress leads to the consistency relation (1.2) that restricts G and $M = \nabla\chi - G$ in dynamical processes for the body. We note further that the field relations (1.1)–(1.5) are not obtained by imposing balance laws and constitutive relations at a submacroscopic length scale followed by an averaging procedure that leads to corresponding relations at the macrolevel. This fact distinguishes the present theory from multiscale approaches employed in the field of micromechanics that use homogenization or other systematic schemes of averaging.

The present study does not address initial-boundary value problems for the field relations (1.1)–(1.5). Nevertheless, we expect that the problem of existence and uniqueness locally in time of smooth solutions χ, G satisfying initial conditions on χ, $\dot{\chi}$, and G and boundary conditions on χ can be attacked by means of the energy methods described in Chapter III of the monograph [12]. Our expectation is based on preliminary calculations for one-dimensional versions of (1.1)–(1.5), expressed in the equivalent form (10.23)–(10.27). Key issues in confirming our expectation are the local solvability of the consistency relation (10.24) at the initial time and satisfaction of the mixed power inequality (10.26) with strict inequality at the initial time.

2. Structured Deformations

Specification of a structured deformation from a region $\mathcal{A}$ in a Euclidean space $\mathcal{E}$ with translation space $\mathcal{V}$ includes the specification of two fields $g\colon \mathcal{A} \to \mathcal{E}$ and $G\colon \mathcal{A} \to \operatorname{Lin}\mathcal{V}$ called the *macroscopic deformation* and the *deformation without disarrangements*, respectively. In the present study we assume that the fields g and G are smooth, although discontinuites are permitted in the piecewise-smooth approximating deformations f_n introduced below (2.1), as well as in ∇f_n. This assumption excludes slip and separation at the macroscopic level while permiting such discontinuities at submacroscopic levels. To avoid some technical issues, precise smoothness assumptions on these fields and on the region $\mathcal{A}$ will not be specified here, but sufficient smoothness requirements on the fields can be inferred from the context. Other than smoothness requirements, the only conditions imposed on the fields g and G are the injectivity of g and the existence of a positive number m such that the inequalities

$$m < \det G(X) \leqslant \det \nabla g(X) \tag{2.1}$$

hold for all $X \in \mathcal{A}$. The Approximation Theorem for structured deformations [1] assures that there is a sequence $n \mapsto f_n$ of piecewise smooth, injective deformations defined on $\mathcal{A}$ (a *determining sequence*) such that

$$g = \lim_{n\to\infty} f_n, \qquad G = \lim_{n\to\infty} \nabla f_n, \tag{2.2}$$

with the limits taken in the sense of essentially uniform convergence (i.e., L^∞-convergence). The spatial derivatives ∇f_n are taken in the classical sense, and the limit G of derivatives in $(2.2)_2$ need not equal the corresponding derivative ∇g of the macroscopic deformation (nor need G even be the gradient of some deformation). Specific quantitative information about the difference $M := \nabla g - G$ is provided in the first subsection below and justifies the terminology *deformation due to disarrangements* for M.

2.1. DECOMPOSITIONS AND IDENTIFICATION RELATIONS

The *additive decomposition*

$$\nabla g = G + M \tag{2.3}$$

for the macroscopic deformation gradient is given deeper significance by means of the following limit relation [2] for M:

$$M(X) = \lim_{\delta\to 0}\lim_{n\to\infty} \operatorname{vol}\mathcal{B}(X;\delta)^{-1}\int_{\Gamma(f_n)\cap\mathcal{B}(X;\delta)} [f_n](Y)\otimes\nu(Y)\,\mathrm{d}A_Y. \tag{2.4}$$

In this relation, $n \mapsto f_n$ is an arbitrary sequence of piecewise smooth deformations that satisfies the limit relations (2.2). The symbol $\mathcal{B}(X;\delta)$ denotes the ball of radius $\delta > 0$ centered at a point X in $\mathcal{A}$, and $\Gamma(f_n) \subset \mathcal{E}$, $[f_n](Y) \in \mathcal{V}$, and $\nu(Y) \in \mathcal{V}$ denote, respectively, the jump set of the piecewise smooth deformation f_n, the jump of f_n at a point $Y \in \Gamma(f_n)$, and the unit normal to the jump set $\Gamma(f_n)$ at the point Y. The integrand in (2.4) is the tensor product of $[f_n](Y)$ and $\nu(Y)$, both vectors in $\mathcal{V}$.

The precise interpretations now available for G and $M = \nabla\chi - G$ permit us to understand and interpret various features of structured deformations in the following subsections.

2.2. FACTORIZATIONS, VIRGIN CONFIGURATIONS, AND INTERMEDIATE CONFIGURATIONS

The definition of composition of two structured deformations [1] is provided in the formula:

$$(\tilde{g}, \tilde{G}) \circ (g, G) := (\tilde{g} \circ g, (\tilde{G} \circ g)G). \tag{2.5}$$

Here, the symbol "$\circ$" on the left-hand side denotes the composition of two structured deformations, while on the right-hand side it denotes the composition of two functions. In addition, $(\tilde{G} \circ g)G$ denotes the pointwise composition of the two tensor fields $\tilde{G} \circ g$ and G. This formula provides the following factorizations for a structured deformation (g, G):

$$(g, G) = (g, \nabla g) \circ (i, K), \tag{2.6}$$
$$(g, G) = (i, \tilde{H}) \circ (g, \nabla g), \tag{2.7}$$

where $i(X) := X$ for all $X \in \mathcal{A}$, $K := (\nabla g)^{-1}G$ and $\tilde{H} := (G \circ g^{-1})((\nabla g)^{-1} \circ g^{-1})$. The first factorization (2.6) represents the given structured deformation as the classical deformation $(g, \nabla g)$ *following* a "purely submacroscopic" structured deformation (i, K) that accomplishes all of the disarrangements associated with (g, G). Analogously, the second factorization (2.7) represents (g, G) as the same classical deformation *followed by* the purely submacroscopic structured deformation $(i, \tilde{H})$. We emphasize that all of the factors in the above representions are deformations of the entire body.

The factorization (2.6) provides a distinction between the body before and after it undergoes the purely submacroscopic deformation (i, K), a distinction that permits us to distinguish between the *reference configuration*, from which the classical deformation $(g, \nabla g)$ procedes, and the *virgin configuration*, from which both (i, K) and (g, G) procede. Similarly, we may distinguish by means of (2.7) between the *deformed configuration without disarrangements*, attained from the virgin configuration via the classical deformation $(g, \nabla g)$ alone, and the *deformed configuration*, attained from the deformed configuration without disarrangements via the purely submacroscopic deformation $(i, \tilde{H})$. Of course, all of the configurations mentioned are global configurations of the body.

We note that the inequality (2.1) implies the relations

$$0 < \det K = \det \tilde{H} \leqslant 1 \tag{2.8}$$

and permits us to call $\det K = \det \tilde{H} = \det G / \det \nabla g$ the *volume fraction* associated with the given structured deformation. The case $\det K < 1$ reflects creation of voids through the purely submacroscopic deformations (i, K) and $(i, \tilde{H})$. Of particular interest in applications such as crystalline plasticity are *invertible structured deformations* (g, G), i.e., structured deformations for which the volume fraction equals 1. The term "invertible" is appropriate, because the pair $(g^{-1}, G^{-1} \circ g^{-1})$ then is itself a structured deformation that is a two-sided inverse for (g, G) with respect to the composition in (2.5) and with (i, I) playing the role of the identity structured deformation (here $Iv = v$ for all $v \in \mathcal{V}$). In this case, the purely submacroscopic factor (i, K) also is an invertible structured deformation with inverse $(i, K)^{-1} = (i, K^{-1})$, and we have the following factorization

$$(g, \nabla g) = (g, G) \circ (i, K)^{-1} \tag{2.9}$$

of the classical deformation $(g, \nabla g)$. For the structured deformation (g, G), the purely submacroscopic deformation (i, K) carried the virgin configuration into the reference configuration; consequently, its inverse $(i, K)^{-1}$ carries the reference configuration into the virgin configuration. Consequently, *the virgin configuration for the invertible structured deformation* (g, G) *plays the role of a* (*global) intermediate configuration for the classical deformation* $(g, \nabla g)$. Local intermediate configurations play an important role in descriptions of single and polycrystalline materials and of polymers (see [13, 14] and references cited therein).

2.3. MOTIONS VIA FAMILIES OF STRUCTURED DEFORMATIONS; SPACE-LIKE DISARRANGEMENTS

The most immediate way of capturing the possibility that a body evolves in time while undergoing structured deformations at each instant is to consider a given positive number T and a pair of smooth mappings $\chi\colon \mathcal{A} \times (0, T) \to \mathcal{E}$ and $G\colon \mathcal{A} \times (0, T) \to \operatorname{Lin} \mathcal{V}$ such that the pair $(\chi(\cdot, t), G(\cdot, t))$ is a structured deformation for each $t \in (0, T)$. When the Approximation Theorem and the identification relation in Section 2.1 are invoked at each time t, the relations (2.2) and (2.4) become:

$$\chi(\cdot, t) = \lim_{n\to\infty} \chi_n(\cdot, t), \qquad G(\cdot, t) = \lim_{n\to\infty} \nabla \chi_n(\cdot, t) \tag{2.10}$$

and

$$M(X, t) = \lim_{\delta\to 0} \lim_{n\to\infty} \operatorname{vol} \mathcal{B}(X; \delta)^{-1} \int_{\Gamma(\chi_n(\cdot,t)) \cap \mathcal{B}(X;\delta)} [\chi_n(\cdot, t)](Y) \otimes \nu(Y)\, \mathrm{d}A_Y. \tag{2.11}$$

In this context, the disarrangements associated with the approximating motions χ_n that are captured in the tensor field $M\colon \mathcal{A} \times (0, T) \to \operatorname{Lin} \mathcal{V}$ are space-like, so that time-like jumps in χ_n do not affect the fields associated with the family $t \mapsto (\chi(\cdot, t), G(\cdot, t))$ of structured deformations. The more complete treatment of "structured motions" described in [3], Part 2, introduces not only a deformation without disarrangements G but also a velocity without disarrangements $\dot{\chi}_\backslash$ that permit both space-like and time-like jumps to be captured in two analogues of the identification relation (2.11).

We choose here to follow the more immediate route, bodies evolving through time-parameterized families of structured deformations, and our theory of elasticity with disarrangements more accurately can be entitled *elasticity with space-like disarrangements*. The inequality (2.1) becomes in the case of time-parameterized families of structured deformations:

$$0 < m(t) < \det G(X, t) \leqslant \det \nabla\chi(X, t). \tag{2.12}$$

3. Contact and Body Forces

3.1. DECOMPOSITIONS

Earlier studies of balance laws for bodies undergoing structured deformations ([3], Part 2, Section 1, and [15]) showed that the classical law of balance of forces in the reference configuration is equivalent to a "refined balance law" that may be written as:

$$\operatorname{div}(SK^*) + \operatorname{div}((\det K)S - SK^*) - S\nabla(\det K) + (\det K)b_{\mathrm{ref}} = 0. \tag{3.1}$$

Here, $S\colon \mathcal{A}\times(0, T) \to \operatorname{Lin}\mathcal{V}$ is the Piola–Kirchhoff stress field, $K := (\nabla\chi)^{-1}G$, b_{ref} is the body force per unit volume in the reference configuration, and $A^* := (\det A)A^{-T}$ for all invertible $A \in \operatorname{Lin}\mathcal{V}$. Moreover, the decomposition (A.1) and the identification relations (A.2), (A.3) derived in earlier studies ([3, 15]) and recorded in the Appendix, permit us to call

$$S_\backslash := SK^* = (\det K)SK^{-T} \tag{3.2}$$

the *stress without disarrangements*, $\operatorname{div}(SK^*)$ the volume density of contact forces without disarrangments, and $\operatorname{div}((\det K)S - SK^*) - S\nabla(\det K)$ the volume density of contact forces due to disarrangements. We call

$$S_d := S[(\det K)I - K^*] \tag{3.3}$$

the *stress due to disarrangements.* The availability through structured deformations of a both a virgin configuration and a reference configuration permits one to view (3.1) as balance of forces in the virgin configuration, differing from the reference configuration by a purely submacroscopic deformation as described in Section 2.2. Of course, the scalar field $\det K$ may be thought of as the volume fraction associated with the given time-parameterized family of structured deformations.

The considerations above lead us not only to the *decomposition*

$$F = G + M \tag{3.4}$$

of the macroscopic deformation gradient $F = \nabla\chi\colon \mathcal{A}\times(0,T) \to \operatorname{Lin}\mathcal{V}$ but also, upon adding relations (3.2) and (3.3), to the *decomposition of the stress*:

$$(\det K)S = S_{\backslash} + S_d. \tag{3.5}$$

The stress tensor $(\det K)S$ is an analogue of the "weighted Cauchy tensor" $(\det F)T$ discussed in [16], and equations (3.1) and (3.5) show that it is this weighted measure of stress that readily decomposes into a part without disarrangements plus a part due to disarrangements.

3.2. CONSISTENCY RELATION

If we use the defining relation (3.2) for the stress without disarrangements $S_{\backslash}$ to eliminate the Piola–Kirchhoff stress S from the decomposition (3.5), we obtain a consistency relation between the stresses due to and without disarrangements:

$$S_{\backslash}K^T = S_{\backslash} + S_d. \tag{3.6}$$

Roughly speaking, there is less freedom in the decomposition (3.5) of the weighted stress $(\det K)S$ into parts with and without disarrangements than in the decomposition (3.4) of the macroscopic deformation gradient into parts with and without disarrangements. Accordingly, we refer to (3.6) as the *consistency relation*. It will provide, through the constitutive assumptions for $S_{\backslash}$ and S_d made in Section 7, the restriction (1.2) on the dynamical processes that can occur in a given elastic body.

An equivalent form of the consistency relation,

$$S_{\backslash}M^T + S_dG^T + S_dM^T = 0, \tag{3.7}$$

follows from (3.6), after substitution of G^TF^{-T} for K^T, and implies that

$$\operatorname{sk}(S_{\backslash}M^T) + \operatorname{sk}(S_dG^T) + \operatorname{sk}(S_dM^T) = 0, \tag{3.8}$$

where $\operatorname{sk}A := (A - A^T)/2$ denotes the skew part of $A \in \operatorname{Lin}\mathcal{V}$. This relation plays a role in the analysis of moment densities in Section 5.

4. Power Expended; Balance Laws

We postulate that in a family of structured deformations (χ, G) the power expended at time $t \in (0,T)$ on a subbody $\mathcal{S} \subset \mathcal{A}$ by its exterior is given by the classical formula

$$\begin{aligned} P(\mathcal{S},t) \; = \; & \int_{\mathrm{bdy}\,\mathcal{S}} S(X,t)\nu(X)\cdot\dot{\chi}(X,t)\,\mathrm{d}A_X \\ & + \int_{\mathcal{S}} b^*(X,t)\cdot\dot{\chi}(X,t)\,\mathrm{d}V_X. \end{aligned} \tag{4.1}$$

Here, beyond the quantities S and $\dot{\chi}$ introduced in Sections 2 and 3, $\nu(X)$ denotes the outward unit normal at the point $X \in \text{bdy}\,\mathcal{S}$, and $b^* := b_{\text{ref}} - \rho_{\text{ref}}\ddot{\chi}$ is the total body force, with b_{ref} the body force in the reference configuration and ρ_{ref} the mass density in the reference configuration. Our use of the classical formula (4.1) for the power allows us to preserve much of the structure of the standard field theory of non-linear elasticity. (See [4] for a derivation of balance laws that arise from a non-classical formula for the power.)

A standard argument [17, 18] based on invariance of the power expended under superposed rigid motions yields the classical laws of balance of linear and angular momentum:

$$\text{div}S + b_{\text{ref}} = \rho_{\text{ref}}\ddot{\chi}, \tag{4.2}$$

$$\text{sk}(SF^T) = 0. \tag{4.3}$$

The definition (4.1) of the power expended and the balance law (4.2) yield by means of the divergence theorem and a standard product rule the following reduced expression for the power expended

$$P(\mathcal{S}, t) = \int_{\mathcal{S}} S(X, t) \cdot \nabla\dot{\chi}\,(X, t)\,\mathrm{d}V_X. \tag{4.4}$$

We note that the formula $\nabla\dot{\chi} = (\nabla\chi)^{\cdot} = \dot{F}$ and the two basic decompositions (3.4) and (3.5) permit us to decompose $(\det K)S \cdot \nabla\dot{\chi}$, the *density of stress power in the virgin configuration*, in the following manner:

$$(\det K)S \cdot \nabla\dot{\chi} = S_\backslash \cdot \dot{G} + S_d \cdot \dot{M} + S_\backslash \cdot \dot{M} + S_d \cdot \dot{G}. \tag{4.5}$$

The contribution $S_\backslash \cdot \dot{G} + S_d \cdot \dot{M}$ to the stress power involves pairing like quantities (a stress without disarrangements and a rate of deformation without disarrangements, or corresponding quantities due to disarrangements). Because the contribution $S_\backslash \cdot \dot{M} + S_d \cdot \dot{G}$ mixes factors with and without disarrangements, we refer to it as the *mixed* (*stress*) *power*.

In a similar way, we may decompose the volume density of moments in the balance law (4.3):

$$(\det K)\text{sk}(SF^T) = \text{sk}(S_\backslash G^T) + \text{sk}(S_d M^T) + \text{sk}(S_\backslash M^T) + \text{sk}(S_d G^T), \tag{4.6}$$

and individual terms on the right-hand side may be interpreted as particular moment densities, as described in the next section. By the consistency relation (3.8), the last three moment densities $\text{sk}(S_d M^T)$, $\text{sk}(S_\backslash M^T)$, and $\text{sk}(S_d G^T)$ appearing on the right-hand side of (4.6) must add to zero. By the balance of angular momentum (4.3), by (4.6), and by (3.8), the first moment density $\text{sk}(S_\backslash G^T)$ on the right-hand side of (4.6) must vanish.

5. Offset Moments

The volume densities of moments $\mathrm{sk}(S_{\backslash}M^T)$, $\mathrm{sk}(S_d M^T)$, $\mathrm{sk}(S_d G^T)$ arose in Section 4 through the formula (4.6). In this section we identify the terms $\mathrm{sk}(S_{\backslash}M^T)$ and $\mathrm{sk}(S_d M^T)$ as volume densities of "offset moments."

The significance of the moment density $\mathrm{sk}(S_d M^T)$ is revealed by the following identification relation:

$$\mathrm{sk}(S_d(X,t)M^T(X,t)) = \lim_{\delta\to 0}\lim_{n\to\infty} \frac{\int_{\Gamma(\chi_n(\cdot,t))\cap\mathcal{B}(X;\delta)} S_d(X,t)\nu(Y)\times[\chi_n(\cdot,t)](Y)\,\mathrm{d}A_Y}{\mathrm{vol}\,\mathcal{B}(X;\delta)}, \tag{5.1}$$

which we verify below. The vector $S_d(X,t)\nu(Y)$ is the traction due to disarrangements at the point Y on a disarrangement site in the virgin configuration, computed using the stress due to disarrangements at the center X of the ball. The vector product $S_d(X,t)\nu(Y)\times[\chi_n(\cdot,t)](Y)$ is (minus) the moment per unit area produced by that traction acting against the *offset* $[\chi_n(\cdot,t)](Y)$ caused by disarrangements. An elementary instance of such moments would arise if a deck of cards, in equilibrium under a system of loads, is shifted near the middle card without changing either the shape of the individual cards or the applied loads. The moment arising from the change in geometry of the deck corresponds to the moment calculated on the right-hand side of (5.1). Consequently, we call $\mathrm{sk}(S_d M^T)$ a *volume density of offset moments.* Of course, replacing S_d by $S_{\backslash}$ in (5.1) permits us also to call $\mathrm{sk}(S_{\backslash}M^T)$ a *volume density of offset moments.* The identification relation (5.1) follows immediately if we substitute the right-hand side of the identification relation (2.11) into the left-hand side of (5.1) and if we identify the skew tensor $\mathrm{sk}(S_d(X,t)\nu(Y)\otimes[\chi_n(\cdot,t)](Y))$ with its axial vector.

Because G measures the deformation away from disarrangement sites, we interpret the moment density $\mathrm{sk}(S_d G^T)$ in (4.6), arising even in motions involving disarrangements, as an analogue of the moment density $\mathrm{sk}(SF^T)$ arising in the classical balance law. (Recall that in Section 4 we pointed out that the analogous density $\mathrm{sk}(S_{\backslash}G^T)$ vanishes, because it is a scalar multiple of $\mathrm{sk}(SF^T)$.) According to (3.8), the offset moment densities $\mathrm{sk}(S_d M^T)$, $\mathrm{sk}(S_{\backslash}M^T)$, and the moment density $\mathrm{sk}(S_d G^T)$ in (4.6) must add to zero. Actually, we show in Section 9 that material frame-indifference implies that $\mathrm{sk}(S_d M^T)$ vanishes and, therefore, that $\mathrm{sk}(S_{\backslash}M^T)$ and $\mathrm{sk}(S_d G^T)$ also must add to zero.

6. Dynamical Processes, Constitutive Classes, and the Dissipation Inequality

A dynamical process is specified here by giving a motion χ, the deformation without disarrangements G, the stress field S, the volume density ψ of the Helmholtz free energy in the reference configuration, and the mass density ρ_{ref} in the reference configuration. Of course, the stresses without and due to disarrangements $S_{\backslash}$ and S_d are determined by the Piola–Kirchhoff stress S, the motion χ, and the deformation without disarrangements G through the relations (3.2) and (3.3), and the body force b_{ref} also is determined by fields in our list from the balance of linear momentum (4.2). (We may guarantee that the balance of angular momentum (4.3) is satisfied on every dynamical process by imposing the condition $\text{sk}(SF^T) = 0$, but we refrain from doing so pending the discussion of frame-indifference in Section 9.) We will omit ρ_{ref} in the list above for the sake of conciseness.

The concept of a constitutive class is central to the specification of the particular material that is to be considered. Here, following Gurtin [19], a *constitutive class* $\mathcal{C}$ simply is a collection of dynamical processes. A particular choice of constitutive class limits the dynamical processes that are to be considered. In practice, a constitutive class is specified by giving a list of *response functions*: the constitutive class is the collection of those dynamical processes that satisfy the relations on the fields χ, G, S, ψ provided by the response functions. Of course, these relations may include inequalities as well as equations.

Another limitation on dynamical processes is provided by the second law of thermodynamics which, in the present context of isothermal processes, is the *dissipation inequality*:

$$\dot{\psi}(X,t) \leqslant S(X,t) \cdot \nabla\dot{\chi}(X,t), \tag{6.1}$$

asserting that *the rate of change of the density of the Helmholtz free energy does not exceed the stress power*. We denote by $\mathcal{D}$ the collection of all dynamical processes χ, G, S, ψ that satisfy the dissipation inequality. The dissipation inequality is imposed by means of the requirement

$$\mathcal{C} \subset \mathcal{D}. \tag{6.2}$$

In other words, every dynamical process for the given material must obey (6.1).

The dissipation inequality may be used to impose restrictions on the response functions that specify a constitutive class $\mathcal{C}$, as first described in the context of the Clausius–Duhem inequality by Coleman and Noll [20] and now widely followed in continuum thermodynamics. According to this procedure, one seeks necessary and sufficient conditions on the response functions that specify $\mathcal{C}$ in order that $\mathcal{C} \subset \mathcal{D}$. We indicate in the next section that, when the free energy and stresses depend only upon $F = \nabla\chi$ and G, the restrictions obtained from the procedure of Coleman and Noll include the vanishing of the internal dissipation $S(X,t) \cdot \nabla\dot{\chi}(X,t) - \dot{\psi}(X,t)$ on dynamical processes in $\mathcal{C}$. We shall maintain the premise that it is useful to identify and study constitutive classes that admit internal dissipation on a non-trivial class of dynamical processes. Consequently, instead of following the procedure of Coleman and Noll, we are led in the next section to impose sufficient conditions on the constitutive class in order that it be included in $\mathcal{D}$. To do so, we specify a particular constitutive class $\mathcal{E}_d$ and show directly the inclusion $\mathcal{E}_d \subset \mathcal{D}$. Although all of the fields in our description of a dynamical process are smooth, the present approach echoes the standard use of the second law of thermodynamics to limit the class of non-smooth processes that can occur in the presence of a shock wave (see, for example, [21]). In our context, the non-smoothness occurs at a submacroscopic level and is made explicit only through the piecewise-smooth motions χ_n that arise in the Approximation Theorem.

In spite of the present choice not to pursue the procedure of Coleman and Noll, the constitutive class $\mathcal{C}$ obtained via that procedure merits detailed study, because it admits the possibility that internal dissipation arises via small jumps between points on a constitutive manifold determined by the consistency relation. (See [7, 8] for elementary examples.)

7. A Constitutive Class for Elastic Bodies Undergoing Disarrangements

The constitutive data that we employ initially for the specification of an elastic body undergoing disarrangements are the smooth response functions $(F, G) \mapsto \Psi(F, G)$, $(F, G) \mapsto S_\backslash(F, G)$, and $(F, G) \mapsto S_d(F, G)$ for the free energy, stress without disarrangements, and stress due to disarrangements, all defined on pairs of invertible tensors (F, G) satisfying the inequalities

$$0 < \det G \leqslant \det F. \tag{7.1}$$

An equivalent description of these response functions entails the specification of the mappings $(M, G) \mapsto \tilde{\Psi}(M, G) := \Psi(M + G, G)(M, G) \mapsto \tilde{S}_\backslash(M, G) := S_\backslash(M + G, G)$, and $(M, G) \mapsto \tilde{S}_d(M, G) := S_d(M + G, G)$ defined on pairs of tensors (M, G) satisfying

$$0 < \det G \leqslant \det(M + G). \tag{7.2}$$

For future reference, we record here the relations

$$D_M \tilde{\Psi}(M, G) = D_F \Psi(M + G, G), \tag{7.3}$$

$$D_G \tilde{\Psi}(M, G) = D_F \Psi(M + G, G) + D_G \Psi(M + G, G). \tag{7.4}$$

We allow the free energy response function also to depend upon the material point X at which the free energy is to be computed, but we delay until Section 9 making explicit this dependence on X in the symbols $\Psi(F, G)$ and $\tilde{\Psi}(M, G)$.

The functions $\tilde{\Psi}$, $\tilde{S}_\backslash$, $\tilde{S}_d$ now permit us to define the class $\mathcal{C}$ of dynamical processes satisfying the constitutive relations

$$\psi(X, t) = \tilde{\Psi}(M(X, t), G(X, t)), \tag{7.5}$$

$$S_\backslash(X, t) = \tilde{S}_\backslash(M(X, t), G(X, t)), \tag{7.6}$$

and

$$S_d(X, t) = \tilde{S}_d(M(X, t), G(X, t)) \tag{7.7}$$

for all X, t. We now indicate how the requirement (6.2) imposed via the procedure of Coleman and Noll leads to a constitutive class in which no internal dissipation occurs. (To simplify the relations below, we omit the argument (X, t) throughout.) We multiply both sides of the dissipation inequality (6.1) by $\det K$, use the constitutive relations (7.5)–(7.7) and the formula (4.5) for the stress power in the virgin configuration, and we conclude that the internal dissipation

$$\begin{aligned}
&\det K(S \cdot \nabla\dot{\chi} - \dot{\psi}) \\
&\quad = (\tilde{S}_\backslash(M, G) + \tilde{S}_d(M, G) - (\det K) D_M \tilde{\Psi}(M, G)) \cdot \dot{M} \\
&\qquad + (\tilde{S}_\backslash(M, G) + \tilde{S}_d(M, G) - (\det K) D_G \tilde{\Psi}(M, G)) \cdot \dot{G} \qquad (7.8)
\end{aligned}$$

is not negative on each dynamical process in $\mathcal{C}$. In spite of the restrictions that the consistency relation (3.6) together with the constitutive relations (7.6), (7.7) place on $\dot{M}$ and $\dot{G}$, we may reverse any dynamical process in $\mathcal{C}$ with respect to its time-evolution and obtain another dynamical process in $\mathcal{C}$. Consequently, $\dot{M}$ and $\dot{G}$ may be replaced by $-\dot{M}$ and $-\dot{G}$ in (7.8), leaving all other quantities unchanged. Therefore, the internal dissipation as given in (7.8) must vanish for every dynamical process in $\mathcal{C}$, and the dissipation inequality (6.1) must be satisfied as an equality.

In order to obtain a theory that admits internal dissipation, we consider now a collection of dynamical processes different from $\mathcal{C}$. The constitutive class that we now specify is suggested by comparing the formula (4.5) for the stress power in the virgin configuration with the formula for $(\det K)\dot{\psi}$ obtained by differentiating both sides of (7.5) with respect to t:

$$(\det K) S \cdot \nabla\dot{\chi} = S_d \cdot \dot{M} + S_\backslash \cdot \dot{G} + S_d \cdot \dot{G} + S_\backslash \cdot \dot{M}, \qquad (7.9)$$

$$(\det K)\dot{\psi} = (\det K) D_M \tilde{\Psi}(M, G) \cdot \dot{M} + (\det K) D_G \tilde{\Psi}(M, G) \cdot \dot{G}. \qquad (7.10)$$

Our goal of specifying a material that can both store energy and dissipate energy in smooth processes can be achieved first by choosing some terms on the right-hand side of (7.9) to be set equal to the entire right-hand side of (7.10), thereby specifying the amount of work done on each time interval that will be stored by the body. In order to satisfy the dissipation inequality (6.1), the remaining terms on the right-hand side of (7.9) must be assumed to be non-negative. Accordingly, given a response function $\tilde{\Psi}$ with domain $\{(M, G) \mid 0 < \det G \leqslant \det(G + M)\}$, we consider the collection $\mathcal{E}_d$ of dynamical processes χ, G, S, ψ satisfying the constitutive relations

$$\psi(X, t) = \tilde{\Psi}(M(X, t), G(X, t)), \qquad (7.11)$$

$$S_d(X, t) = (\det K(X, t)) D_M \tilde{\Psi}(M(X, t), G(X, t)), \qquad (7.12)$$

$$S_\backslash(X, t) = (\det K(X, t)) D_G \tilde{\Psi}(M(X, t), G(X, t)), \qquad (7.13)$$

and the *mixed power inequality*

$$0 \leqslant S_\backslash(X, t) \cdot \dot{M}(X, t) + S_d(X, t) \cdot \dot{G}(X, t) \qquad (7.14)$$

for all X, t.

In making these choices we appeal to the idea that forces separated from a site of geometrical changes are unlikely to be able to maintain a metastable geometrical configuration at that site and, therefore, should be capable of contributing to dissipation. Thus, we take into account the separation of the points of applications of the contact forces due to disarrangements (produced by S_d) from the sites where the geometrical changes without disarrangements (that contribute to $\dot{G}$) occur. Analogous considerations can be made for the other term $S_\backslash \cdot \dot{M}$ in the mixed power. On the contrary, for the "pure" term $S_d \cdot \dot{M}$ the proximity of the points of application of the contact forces due to disarrangements to the sites where changes in the disarrangements occur enables the maintainance of metastability, and so also for the other "pure" term $S_\backslash \cdot \dot{G}$. The constitutive assumptions (7.12) and (7.13) embody the non-dissipative character of the terms $S_d \cdot \dot{M}$ and $S_\backslash \cdot \dot{G}$ in the stress power.

An important conclusion that can be drawn from the definition of the constitutive class $\mathcal{E}_d$ is that *the dissipation inequality is satisfied for every dynamical process in* $\mathcal{E}_d$, *i.e.,* $\mathcal{E}_d \subset \mathcal{D}$. Indeed, the constitutive relations (7.11)–(7.13), the mixed power inequality (7.14), and relations (3.4) and (3.5) tell us that

$$\begin{aligned}(\det K)\dot{\psi} &= (\det K)D_M\tilde{\Psi} \cdot \dot{M} + (\det K)D_G\tilde{\Psi} \cdot \dot{G} \\ &\leqslant (\det K)D_M\tilde{\Psi} \cdot \dot{M} + (\det K)D_G\tilde{\Psi} \cdot \dot{G} \\ &\quad + (\det K)D_M\tilde{\Psi} \cdot \dot{G} + (\det K)D_G\tilde{\Psi} \cdot \dot{M} \\ &= (\det K)S \cdot \dot{F},\end{aligned}$$

which is equivalent to the dissipation inequality (6.1).

It is also significant that *the consistency relation* (3.6)*, through the constitutive relations* (7.12) *and* (7.13)*, imposes a restriction on dynamical processes in* $\mathcal{E}_d$: for every dynamical process χ, G, S, ψ in $\mathcal{E}_d$ there holds

$$\begin{aligned}&D_G\tilde{\Psi}(M(X,t), G(X,t))K(X,t)^T \\ &\quad = D_G\tilde{\Psi}(M(X,t), G(X,t)) + D_M\tilde{\Psi}(M(X,t), G(X,t))\end{aligned} \tag{7.15}$$

for all (X, t). Consequently, the pairs $(M(X,t), G(X,t))$ available through dynamical processes in $\mathcal{E}_d$ lie in a submanifold of $\mathrm{Lin}\,\mathcal{V} \times \mathrm{Lin}\,\mathcal{V}$. In particular, for each (X, t), the pairs $(\dot{M}(X,t), \dot{G}(X,t))$ of time-derivatives available through dynamical processes in $\mathcal{E}_d$ lie in the tangent space of the submanifold at $(M(X,t), G(X,t))$ and, hence, cannot be arbitrary elements of $\mathrm{Lin}\,\mathcal{V} \times \mathrm{Lin}\,\mathcal{V}$. Similarly, *the mixed power inequality* (7.14) *imposes a restriction on the quantities* M, G, $\dot{M}$ *and* $\dot{G}$, *or, equivalently, on* F, G, $\dot{F}$ *and* $\dot{G}$, *that can arise for dynamical processes in the constitutive class* $\mathcal{E}_d$, and we shall discuss some of these restrictions in Section 8. Finally, for every *classical* dynamical process χ, $\nabla\chi$, S, ψ in $\mathcal{E}_d$, the consistency relation (7.15), and the fact that $K = I$ when $M = F - G = 0$, yield for all X, t:

$$D_M\tilde{\Psi}(0, \nabla\chi(X,t)) = 0, \tag{7.16}$$

and, equivalently, by (7.3),

$$D_F\Psi(\nabla\chi(X,t), \nabla\chi(X,t)) = 0, \tag{7.17}$$

a restriction on the classical dynamical processes for the given elastic body.

Our theory thus implies that a given choice of free-energy response function $\tilde{\Psi}$ restricts the dynamical processes available to a body through relations (7.11)–(7.15), and that choice also restricts the classical dynamical processes through relation (7.16). In contrast, our theory restricts the choice of $\tilde{\Psi}$, itself, only through the condition of frame-indifference (9.1).

8. Internal Dissipation

The *internal dissipation* in the reference configuration for a dynamical process χ, G, S, ψ in $\mathcal{E}_d$ is defined to be the excess of the stress-power over the rate of change of free energy: $S \cdot \nabla\dot{\chi} - \dot{\psi} = S \cdot \dot{F} - \dot{\psi}$. Because the dissipation inequality (6.1) is satisfied for every dynamical process in $\mathcal{E}_d$, the internal dissipation is non-negative, and we consider from now on

$$\Upsilon := (\det K)(S \cdot \dot{F} - \dot{\psi}) \geqslant 0, \tag{8.1}$$

the *internal dissipation in the virgin configuration*. It follows immediately from (4.5), (7.12), and (7.13) that *the internal dissipation in the virgin configuration equals the mixed stress power*:

$$\begin{aligned} \Upsilon &= S_\backslash \cdot \dot{M} + S_d \cdot \dot{G} \\ &= (\det K)(D_G\tilde{\Psi} \cdot \dot{M} + D_M\tilde{\Psi} \cdot \dot{G}) \\ &= (\det K)[(D_F\Psi + D_G\Psi) \cdot \dot{F} - D_G\Psi \cdot \dot{G}] \geqslant 0 \end{aligned} \tag{8.2}$$

for each dynamical process χ, G, S, ψ in the constitutive class $\mathcal{E}_d$. Our aim in this section is to relate the internal dissipation to familiar quantities in the literature by investigating the relative contributions of the two terms $S_\backslash \cdot \dot{M}$ and $S_d \cdot \dot{G}$ in (8.2).

An equivalent rewriting of (8.1) yields the relations

$$\begin{aligned} S \cdot \dot{F} &= (\det K)^{-1}(S_\backslash \cdot \dot{G} + S_d \cdot \dot{M} + S_\backslash \cdot \dot{M} + S_d \cdot \dot{G}) \\ &= D_G\tilde{\Psi} \cdot \dot{G} + D_M\tilde{\Psi} \cdot \dot{M} + (\det K)^{-1}\Upsilon \\ &= \dot{\psi} + (\det K)^{-1}\Upsilon, \end{aligned} \tag{8.3}$$

a *decomposition of the stress-power in the reference configuration into a non-dissipative part* $\dot{\psi}$ *and a dissipative part* $(\det K)^{-1}\Upsilon \geqslant 0$. Thus, by (8.2), *the dissipative part* $(\det K)^{-1}\Upsilon$ *of the stress-power equals the mixed stress-power in the reference configuration*. Moreover, for classical dynamical processes χ, $\nabla\chi$, S, ψ in the constitutive class $\mathcal{E}_d$, the internal dissipation vanishes, because $\dot{M} = S_d = 0$.

For a given stress S and for given deformation rates $\dot{M}$ and $\dot{G}$, the relative magnitudes of the terms $S_\backslash \cdot \dot{M}$ and $S_d \cdot \dot{G}$ can be altered by adjusting $K = F^{-1}G$, because of the formulas $S_\backslash = SK^*$ and $S_d = (\det K)S - SK^*$. In particular, for K close to the identity I, we have $S_\backslash = S + \mathrm{O}(K - I)$ and $S_d = \mathrm{O}(K - I)$, and we expect that the term $S_\backslash \cdot \dot{M}$ dominates the term $S_d \cdot \dot{G}$ in the expression (8.2) for Υ

as K tends to the identity I. (The symbol $\mathrm{O}(K-I)$ denotes a tensor whose norm is bounded above by a constant times the norm of $K-I$.) In order to understand this idea in more depth, it is enlightening to express the internal dissipation Υ in terms of the Cauchy stress T, the macroscopic deformation F and its time-derivative $\dot{F}$, and the deformation without disarrangements G and its derivative $\dot{G}$. In doing so, we employ (8.2) along with the the formulas $S_\backslash = SK^*$ and $S_d = (\det K)S - SK^*$, and we find it convenient to suppress the arguments X, t, and $\chi(X,t)$ for the sake of simplicity of notation. We record the result here, omitting its routine derivation:

$$(\det F)^{-1}\Upsilon = TH^* \cdot (\dot{F}F^{-1} - \dot{G}G^{-1}) + TH^* \cdot \dot{G}G^{-1}(H-I)^2, \qquad (8.4)$$

where $H := GF^{-1}$ is the referential version of the tensor field $\tilde{H}$ appearing in Section 2.2 and, as usual, $H^* = (\det H)H^{-T}$. We note that the expression $TH^* \cdot \dot{G}G^{-1}(H-I)^2$ on the right-hand side of (8.4) is quadratic in $H-I$, while the first term $TH^* \cdot (\dot{F}F^{-1} - \dot{G}G^{-1})$ equals $T \cdot (\dot{F}F^{-1} - \dot{G}G^{-1})$ plus a term linear in $H-I$. In other words, we conclude from (8.4) that

$$\begin{aligned}(\det F)^{-1}\Upsilon &= TH^* \cdot (\dot{F}F^{-1} - \dot{G}G^{-1}) + \mathrm{O}((H-I)^2) \\ &= T \cdot (\dot{F}F^{-1} - \dot{G}G^{-1}) + \mathrm{O}(H-I). \qquad (8.5)\end{aligned}$$

In order to relate the last formula for the internal dissipation to more familiar quantities, we note that the fields $L_G := \dot{G}G^{-1}$ and $L_M := \dot{F}F^{-1} - \dot{G}G^{-1}$ appeared in the study [13] of multiple slip in single crystals as the *relative rate of deformation without disarrangements* and the *relative rate of deformation due to disarrangements*, respectively. (In [13], the term "slip" replaced "disarrangement" because of the particular context of that study.) Moreover, the factorization (2.7) implies that the tensor field $T_\backslash := TH^*$ in (8.4) and (8.5) is analogous to $S_\backslash = SK^*$ and may be called the *stress in the current configuration without disarrangements*, a configuration macroscopically identical to the current configuration but containing none of the disarrangements associated with χ and G. Accordingly, $T_\backslash$ represents a stress without disarrangements. (In view of (3.3), the tensor field $T_d := (\det H)T - T_\backslash$ is the analogue of S_d and represents a stress due to disarrangements.) Therefore, (8.5) may now be recast in the form

$$\begin{aligned}(\det F)^{-1}\Upsilon &= T_\backslash \cdot L_M + \mathrm{O}((H-I)^2) \\ &= T \cdot L_M + \mathrm{O}(H-I). \qquad (8.6)\end{aligned}$$

The tensor $H-I$ measures the disarrangements from the current configuration without disarrangements to the current configuration, and the decomposition (8.6) tells us that *the quantities $T \cdot L_M$ and $T_\backslash \cdot L_M$ provide approximations to the internal dissipation to within, respectively, linear and quadratic terms in the disarrangements from the current configuration without disarrangements*. This result places in perspective with respect to the present theory the frequent identification of the internal dissipation as an expression of the form $T \cdot L_M$ (sometimes called "plastic power" in phenomenological theories of plasticity).

9. Material Frame-Indifference

We consider here the transformation properties of the kinematical quantities associated with dynamical processes under changes of observer. These transformation properties can be obtained by replacing the motion χ, and the approximating motions χ_n from the Approximation Theorem, by $(X,t) \mapsto r(\chi(X,t),t)$ and $(X,t) \mapsto r(\chi_n(X,t),t)$, where r denotes a rigid motion $(X,t) \mapsto x_0(t) + Q(t)(X - X_0)$ with $Q(t)$ a proper orthognal tensor. From this observation and the fact that $G = \lim_{n\to\infty} \nabla\chi_n$, we obtain the transformation rules

$$\begin{aligned} F &\to QF \\ G &\to QG \\ M &\to QM \\ K &\to K \\ \dot F &\to Q\dot F + \dot Q F \\ \dot G &\to Q\dot G + \dot Q G \\ \dot M &\to Q\dot M + \dot Q M. \end{aligned}$$

In the present context of an elastic body undergoing disarrangements, we say that the response function $\tilde\Psi$ is *frame-indifferent* if, for all proper orthogonal tensors Q and pairs (M, G) with $0 < \det G \leqslant \det(M + G)$, there holds

$$\tilde\Psi(QM, QG) = \tilde\Psi(M, G), \tag{9.1}$$

or, equivalently,

$$\Psi(QF, QG) = \Psi(F, G) \tag{9.2}$$

for all proper orthogonal tensors Q and pairs (F, G) with $0 < \det G \leqslant \det F$.

A useful characterization of this condition follows from the polar decompositions $F = R_F U_F$ and $G = R_G U_G$. Indeed, we may put $Q := R_G^T$ in (9.1) or (9.2), or $Q := R_F^T$ in (9.1) and use the relations $R_F^T = U_F^{-1} F^T$, $R_G^T = U_G^{-1} G^T$ to obtain for all pairs (M, G), with $0 < \det G \leqslant \det(M + G)$, the representations

$$\tilde\Psi(M, G) = \tilde\Psi(U_G^{-1} G^T M, U_G) = \overset{\leftrightarrow}{\Psi}(G^T M, C_G), \tag{9.3}$$

$$\Psi(F, G) = \Psi(U_G^{-1} G^T F, U_G) = \widehat\Psi(G^T F, C_G), \tag{9.4}$$

$$\Psi(F, G) = \Psi(U_F, U_F^{-1} F^T G) = \underline{\Psi}(C_F, F^T G), \tag{9.5}$$

where $C_F := F^T F$ and $C_G := G^T G$ are the right Cauchy–Green tensors for F and G, respectively. Each one of these representations is both a necessary and a sufficient condition for the frame-indifference of the response functions $\tilde\Psi$ and Ψ in the context of elastic bodies undergoing disarrangements.

A second characterization of the frame-indifference of the response function $\tilde\Psi$ follows by imposing (9.1) on smooth, time-parameterized families $t \mapsto Q(t)$

and $t \mapsto (M(t), G(t))$ and differentiating both sides of (9.1) with respect to t to conclude that

$$D_M\tilde{\Psi}(QM, QG) \cdot [\dot{Q}M + Q\dot{M}] + D_G\tilde{\Psi}(QM, QG) \cdot [\dot{Q}G + Q\dot{G}] \\ = D_M\tilde{\Psi}(M, G) \cdot \dot{M} + D_G\tilde{\Psi}(M, G) \cdot \dot{G}. \tag{9.6}$$

Because the restriction (9.1) applies throughout the domain of $\tilde{\Psi}$, we may vary $\dot{Q}$, $\dot{G}$, and $\dot{M}$ independently (subject to the constraints $\operatorname{sym}(\dot{Q}Q^T) = 0$ and $0 < \det G \leqslant \det(M + G)$) to conclude from the smoothness of $\tilde{\Psi}$ that

$$D_M\tilde{\Psi}(QM, QG) = QD_M\tilde{\Psi}(M, G), \tag{9.7}$$

$$D_G\tilde{\Psi}(QM, QG) = QD_G\tilde{\Psi}(M, G), \tag{9.8}$$

and

$$\operatorname{sk}(D_M\tilde{\Psi}(M, G)M^T + D_G\tilde{\Psi}(M, G)G^T) = 0 \tag{9.9}$$

for all proper orthogonal tensors Q and pairs (M, G) with $0 < \det G \leqslant \det(M + G)$. It is easy to verify that relations (9.7)–(9.9) imply that the response function $\tilde{\Psi}$ is frame-indifferent.

It is *crucial* to distinguish between, on the one hand, the smooth time-parameterized families $t \mapsto (M(t), G(t))$ used in establishing (9.6) and, on the other hand, the families

$$t \mapsto (M(X, t), G(X, t)) = (\nabla\chi(X, t) - G(X, t), G(X, t))$$

arising from dynamical processes in the constitutive class $\mathcal{E}_d$. In particular, the time derivatives of former pairs can be varied arbitrarily (when $\det G(t) < \det(M(t) + G(t))$), while those of the latter pairs cannot, as we observed near the end of Section 7.

We say that the mixed power $S_\backslash \cdot \dot{M} + S_d \cdot \dot{G} = (\det K)D_G\tilde{\Psi}(M, G) \cdot \dot{M} + (\det K)D_M\tilde{\Psi}(M, G) \cdot \dot{G}$ is *frame-indifferent* if, for all smooth, time-parameterized families $t \mapsto Q(t)$ and for all families $t \mapsto (M(X, t), G(X, t))$ arising from dynamical processes in $\mathcal{E}_d$, there holds

$$(\det K)D_G\tilde{\Psi}(M, G) \cdot \dot{M} + (\det K)D_M\tilde{\Psi}(M, G) \cdot \dot{G} \\ = \det(K)D_G\tilde{\Psi}(QM, QG) \cdot (QM)^{\cdot} \\ + \det(K)D_M\tilde{\Psi}(QM, QG) \cdot (QG)^{\cdot}. \tag{9.10}$$

This condition amounts to the assertion that the mixed power is invariant under superpositions of rigid motions on dynamical processes. We now show that, *given the frame-indifference of the response function $\tilde{\Psi}$, the mixed power is frame-indifferent if and only if*

$$\operatorname{sk}(D_G\tilde{\Psi}(M, G)M^T + D_M\tilde{\Psi}(M, G)G^T) = 0 \tag{9.11}$$

for all pairs (M, G) arising from dynamical processes in $\mathcal{E}_d$. In fact, expansion of the derivatives on the right-hand side of (9.10) tells us that (9.10) is equivalent to the relation

$$\begin{aligned}&(D_G\tilde{\Psi}(M,G) - Q^T D_G\tilde{\Psi}(QM,QG))\cdot\dot{M}\\ &\quad + (D_M\tilde{\Psi}(M,G) - Q^T D_M\tilde{\Psi}(QM,QG))\cdot\dot{G}\\ &= (D_G\tilde{\Psi}(QM,QG)M^T + D_M\tilde{\Psi}(QM,QG)G^T)\cdot\dot{Q}.\end{aligned}$$

Given the frame-indifference of $\tilde{\Psi}$, we conclude from (9.7) and (9.8) that the previous relation is equivalent to

$$0 = (D_G\tilde{\Psi}(M,G)M^T + D_M\tilde{\Psi}(M,G)G^T)\cdot Q^T\dot{Q},$$

and the relation $\operatorname{sym}(Q^T\dot{Q}) = 0$ along with the arbitrariness of $t \mapsto Q(t)$ provides the asserted characterization of frame-indifference of the mixed power.

Our main result on material frame-indifference is a generalization of a result of Noll [22] in classical elasticity: *if both the free energy response function and the mixed power are frame indifferent, then the balance of angular momentum* (4.3) *is satisfied for all dynamical processes in* $\mathcal{E}_d$. (Noll actually showed that the frame-indifference of the free energy response is equivalent to the law of balance of angular momentum in the classical context.) Indeed, if we add (9.11) to (9.9) we conclude that

$$\operatorname{sk}(D_M\tilde{\Psi}(M,G)F^T + D_G\tilde{\Psi}(M,G)F^T) = 0 \tag{9.12}$$

for all pairs (M, G) arising from dynamical processes in $\mathcal{E}_d$. For each dynamical process, the constitutive relations (7.12), (7.13), and the decomposition (3.5) may be applied to yield (4.3), the law of balance of angular momentum.

This main result permits us to impose the law of balance of angular momentum indirectly by requiring that *both the free energy response function and the mixed power be frame-indifferent*, a requirement that we impose from now on through the relations (9.1) and (9.11). In these considerations, it is important to remember that (9.11) is a restriction on dynamical processes, while (9.1) is a restriction on the free energy response function. Moreover, our main result and the imposition of (9.1) and (9.11) permit us to omit the law of balance of angular momentum among the field equations that we provide in the next section.

We note from relations (7.12) and (7.13), that (9.11) implies

$$\operatorname{sk}(S_\backslash M^T + S_d G^T) = 0 \tag{9.13}$$

on all dynamical processes in the constitutive class $\mathcal{E}_d$, and relation (3.8) then implies also that

$$\operatorname{sk}(S_d M^T) = 0. \tag{9.14}$$

We conclude from frame-indifference as realized in (9.1) and (9.11) that:

(i) $\operatorname{sk}(SF^T) = 0$,
(ii) $\operatorname{sk}(S_d M^T) = 0$,
(iii) $\operatorname{sk}(S_\backslash G^T) = 0$,
(iv) $\operatorname{sk}(S_\backslash M^T + S_d G^T) = 0$.

Thus, *each of the "pure" moment densities* $\operatorname{sk}(S_d M^T)$ *and* $\operatorname{sk}(S_\backslash G^T)$ *is self-equilibrated*, while *the mixed moment densities* $\operatorname{sk}(S_\backslash M^T)$ *and* $\operatorname{sk}(S_d G^T)$ *add to zero*. Moreover, by (ii) and by the identification relation (5.1), *the traction due to disarrangements* $S_d(X,t)\nu(Y)$ *and the geometrical offset* $[\chi_n](Y,t)$ *are colinear on average*.

It is useful to record the forms that relations (9.1) and (9.11) assume when the response function $(M, G) \mapsto \tilde{\Psi}(M, G)$ is replaced by $(F, G) \mapsto \Psi(F, G) = \tilde{\Psi}(F - G, G)$. Of course, (9.1) is replaced by (9.2), a restriction on the response function Ψ. In view of (3.4), (7.5), and (7.11), the relation (9.11) is equivalent to

$$\operatorname{sk}(D_F\Psi(F, G)F^T + D_G\Psi(F, G)(F^T - G^T)) = 0, \tag{9.15}$$

a restriction on dynamical processes. Henceforth, when using F and G as the arguments of the free energy, we assume that (9.2) and (9.15) are satisfied, the first throughout the domain of Ψ and the second on all dynamical processes in $\mathcal{E}_d$.

10. Field Relations

Our analysis in Sections 7–9 has led us to the specification of one response function $(M, G) \mapsto \tilde{\Psi}(M, G)$ satisfying relations (9.1) and (9.11), the former throughout the domain of $\tilde{\Psi}$ and the latter on all dynamical processes in $\mathcal{E}_d$. Given the body force field b_{ref} in the reference configuration, the remaining relations employed in deriving the field relations are restrictions on dynamical processes: the balance of linear momentum (4.2), the constitutive relations (7.11)–(7.13), the mixed power inequality (7.14), and the consistency relation (7.15). As demonstrated in Section 9, the law of balance of angular momentum is a consequence of the assumptions of frame-indifference (9.1) and (9.11).

We now are in a position to record and derive from (9.1), (9.11), (4.2), and (7.11)–(7.15) the *field relations for an elastic body undergoing disarrangements*:

$$\operatorname{div}(D_M\tilde{\Psi} + D_G\tilde{\Psi}) + b_{\text{ref}} = \rho_{\text{ref}}\ddot{\chi}, \tag{10.1}$$

$$D_G\tilde{\Psi}(K^{-T} - I) + D_M\tilde{\Psi}K^{-T} = 0, \tag{10.2}$$

$$\operatorname{sk}(D_G\tilde{\Psi}M^T + D_M\tilde{\Psi}G^T) = 0, \tag{10.3}$$

$$D_G\tilde{\Psi} \cdot \dot{M} + D_M\tilde{\Psi} \cdot \dot{G} \geqslant 0, \tag{10.4}$$

$$\det(G + M) \geqslant \det G > m > 0, \tag{10.5}$$

where, as in (3.4), $\nabla\chi = G + M$, and, as in (2.12), m is a positive number depending upon t alone. (These are relations (1.1)–(1.5), with arguments omitted for the sake of conciseness.) The law of balance of linear momentum (10.1), the consistency relation (10.2), and the frame-indifference of the mixed power (10.3) amount to $3 + 6 + 3 = 12$ scalar equations for the unknowns χ and G, having a total of 12 scalar components. (That the consistency relation amounts to only 6 scalar equations follows from the fact that both sides of the original consistency relation (3.6), when multiplied by F^T, are symmetric tensors.) The inequalities (10.4) and (10.5) further restrict the dynamical processes satisfying (10.1), (10.2), and (10.3). We emphasize that, *given the body force field* b_{ref}, *the field relations are restrictions on dynamical processes, while the relation* (9.1) *is a restriction on the response function* $\tilde{\Psi}$. We must also keep in mind that $\tilde{\Psi}$ and its derivatives depend not only upon $M(X,t)$ and $G(X,t)$ but may also upon the material point X itself, and we make this dependence explicit when needed for clarity. For example, the first term on the left-hand side of the equation of balance of linear momentum (10.1) is the field

$$(X,t) \mapsto \operatorname{div}_X[D_M\tilde{\Psi}(\nabla\chi(X,t) - G(X,t), G(X,t), X) \\ + D_G\tilde{\Psi}(\nabla\chi(X,t) - G(X,t), G(X,t), X)]. \tag{10.6}$$

The field relations follow readily from (9.1), (9.11), (4.2), (7.11)–(7.15), and (2.12). The law of balance of linear momentum is a consequence of its counterpart (4.2), of the constitutive relations (7.12) and (7.13) for the stresses without and with disarrangements, and of the decomposition (3.5); the consistency relation (10.2) is the relation (7.15), rewritten with trivial algebraic changes; (10.3) is (9.11), and the mixed power inequality (10.4) is (7.14) with $S_\backslash$ and S_d replaced by the expressions in the formulas (7.12) and (7.13). Moreover, by the decomposition (3.5) and the constitutive relations (7.12), (7.13), one has the *stress relation*

$$S(X,t) = D_M\tilde{\Psi}(M(X,t), G(X,t)) + D_G\tilde{\Psi}(M(X,t), G(X,t)) \tag{10.7}$$

valid for all dynamical processes in $\mathcal{E}_d$.

When the motion χ and G determine a classical motion, i.e., $G = \nabla\chi$, then the relations $M = 0$, $K = I$, and $\det K = 1$ tell us that the balance of linear momentum (10.1), the consistency relation (10.2), and the inequality (10.5) become

$$\operatorname{div} D_G\tilde{\Psi}(0, \nabla\chi) + b_{\text{ref}} = \rho_{\text{ref}}\ddot{\chi}, \tag{10.8}$$

$$D_M\tilde{\Psi}(0, \nabla\chi) = 0, \tag{10.9}$$

$$\det\nabla\chi > m > 0. \tag{10.10}$$

The remaining relations (10.3) and (10.4) are satisfied identically in view of (10.9).

In some applications, it is easier to use the field relations when they are expressed in terms of the response function $(F,G) \mapsto \Psi(F,G) = \tilde{\Psi}(F - G, G)$. In this case, the response function Ψ is assumed to satisfy (9.2), and the field relations (10.1)–(10.5) become

$$\operatorname{div}(2D_F\Psi + D_G\Psi) + b_{\text{ref}} = \rho_{\text{ref}}\ddot{\chi}, \tag{10.11}$$

$$D_F\Psi(2K^{-T} - I) + D_G\Psi(K^{-T} - I) = 0, \tag{10.12}$$

$$\operatorname{sk}((D_F\Psi + D_G\Psi)F^T - D_G\Psi G^T) = 0, \tag{10.13}$$

$$(D_F\Psi + D_G\Psi)\cdot\dot{F} - D_G\Psi\cdot\dot{G} \geqslant 0, \tag{10.14}$$

$$\det F \geqslant \det G > m > 0, \tag{10.15}$$

respectively, while the stress relation (10.7) becomes

$$S(X,t) = 2D_F\Psi(F(X,t), G(X,t)) + D_G\Psi(F(X,t), G(X,t)). \tag{10.16}$$

Corresponding to the expression (10.6), the first term on the left-hand side of the equation of balance of linear momentum (10.11) is the field

$$\begin{aligned}(X,t) \;\mapsto\; & \operatorname{div}_X[2D_F\Psi(\nabla\chi(X,t), G(X,t), X)\\ & +D_G\Psi(\nabla\chi(X,t), G(X,t), X)].\end{aligned} \tag{10.17}$$

It is convenient to record for future use the following formulas for the stresses with and without disarrangements in terms of Ψ (omitting X and t for the sake of brevity):

$$S_{\backslash} = (\det K)(D_F\Psi(F,G) + D_G\Psi(F,G)), \tag{10.18}$$

$$S_d = (\det K)D_F\Psi(F,G). \tag{10.19}$$

In view of the significance of the purely submacroscopic factor (i, K) in (2.6), some applications become more accessible if one employs the response function

$$(F,K) \mapsto \overline{\Psi}(F,K) := \Psi(F, FK) \tag{10.20}$$

with domain the set of pairs (F,K) satisfying $0 < \det F$ and $0 < \det K \leqslant 1$. The relations

$$D_F\Psi(F, FK) = D_F\overline{\Psi}(F,K) - F^{-T}D_K\overline{\Psi}(F,K)K^T \tag{10.21}$$

and

$$D_G\Psi(F, FK) = F^{-T}D_K\overline{\Psi}(F,K) \tag{10.22}$$

permit us to express the field relations in terms of this choice of variables:

$$\operatorname{div}[2D_F\overline{\Psi} + F^{-T}D_K\overline{\Psi}(I - 2K^T)] + b_{\text{ref}} = \rho_{\text{ref}}\ddot{\chi}, \tag{10.23}$$

$$D_F\overline{\Psi}(2K^{-T} - I) + F^{-T}D_K\overline{\Psi}K^T(\{K^{-T}\}^2 - 3K^{-T} + I) = 0, \tag{10.24}$$

$$\operatorname{sk}(D_F\overline{\Psi}F^T + F^{-T}D_K\overline{\Psi}(I - 2K^T)F^T) = 0, \tag{10.25}$$

$$(D_F\overline{\Psi} + F^{-T}D_K\overline{\Psi}(I - 2K^T))\cdot\dot{F} - D_K\overline{\Psi}\cdot\dot{K} \geqslant 0, \tag{10.26}$$

$$1 \geqslant \det K > 0. \tag{10.27}$$

In addition, the stress relation (10.16) becomes

$$S(X,t) = 2D_F\overline{\Psi}(F(X,t),K(X,t)) + F^{-T}(X,t)D_K\overline{\Psi}(F(X,t),K(X,t))(I-2K^T(X,t)). \quad (10.28)$$

It also is convenient to record for future use the following counterparts of (10.18) and (10.19):

$$S_\backslash = (\det K)\big(D_F\overline{\Psi}(F,K) + F^{-T}D_K\overline{\Psi}(F,K)\big(I-K^T\big)\big), \quad (10.29)$$

$$S_d = (\det K)\big(D_F\overline{\Psi}(F,K) - F^{-T}D_K\overline{\Psi}(F,K)K^T\big). \quad (10.30)$$

11. Submacroscopic Stability

Suppose that, for a given tensor F_0, there is a tensor G_0 (with $0 < \det G_0 \leqslant \det F_0$) that provides a local minimum for $G \mapsto \Psi(F_0, G)$, the Helmholtz free energy at the given macroscopic deformation gradient F_0. In this case, we say that the pair (F_0, G_0) is *submacroscopically stable*. If the pair (F_0, G_0) is submacroscopically stable and, in addition, $\det G_0 < \det F_0$, then $D_G\Psi(F_0, G_0) = 0$. By (7.12), (7.13), (7.3), (7.4), and (3.5), there holds

$$S_\backslash(X,t) = S_d(X,t) = \frac{1}{2}(\det K(X,t))S(X,t) \quad (11.1)$$

for all dynamical processes in $\mathcal{E}_d$ and pairs (X,t) satisfying $F(X,t) = F_0$ and $G(X,t) = G_0$. By the first formula in (8.2), we may conclude that *dynamical processes in $\mathcal{E}_d$ through submacroscopically stable pairs* (F_0, G_0) *with* $\det G_0 < \det F_0$ *proceed with internal dissipation* Υ *given by*

$$\Upsilon(X,t) = \frac{1}{2}(\det K(X,t))S(X,t)\cdot \dot{F}(X,t), \quad (11.2)$$

one half the stress-power in the virgin configuration. The stress-power in such submacroscopically stable processes thus is partitioned equally between energy stored and dissipated.

This result on equipartition of the stress-power does not apply to classical dynamical processes, because the relation $F(X,t) = G(X,t)$ for classical processes means that the strict inequality $\det G_0 < \det F_0$, assumed in the derivation of (11.2), would be violated. We shall provide now information about arbitrary dynamical processes that encounter a submacroscopically stable pair (F_0, G_0) at (X,t). To this end, we observe from (2.12) that, for a submacroscopically stable pair (F_0, G_0), the tensor G_0 is a solution of the constrained minimization problem: minimize $G \mapsto \Psi(F_0, G)$ subject to the constraint $\det G \leqslant \det F_0$. The Kuhn–Tucker theorem ([23], p. 314) implies that, corresponding to the given solution G_0, there is a number $\lambda \geqslant 0$ for which

$$D_G\Psi(F_0, G_0) + \lambda(\det G_0)G_0^{-T} = 0. \quad (11.3)$$

Again, by (11.3), (7.12), (7.13), (7.3), (7.4), and (3.5), we have

$$\begin{aligned}
&-\lambda(\det G_0)(\det K(X,t))G_0^{-T}\\
&\quad= (\det K(X,t))D_G\Psi(F_0, G_0)\\
&\quad= S_\backslash(X,t) - S_d(X,t)\\
&\quad= (\det K(X,t))S(X,t)(2K^{-T}(X,t) - I),
\end{aligned} \tag{11.4}$$

for a dynamical process and a pair (X, t) satisfying $F(X,t) = F_0$ and $G(X,t) = G_0$. Because the Cauchy stress and the Piola–Kirchhoff stress are related by $T = (\det F)^{-1}SF^T$, relation (11.4) implies the formula

$$T(X,t)(2I - H_0^T) = -\lambda I, \tag{11.5}$$

with $H_0 := G_0F_0^{-1}$, corresponding to the field $\tilde{H}$ defined below (2.7). For the special case when the submacroscopically stable pair (F_0, G_0) corresponds to a classical deformation, i.e., $F_0 = G_0$, (11.5) reduces to the relation

$$T(X,t) = -\lambda I. \tag{11.6}$$

In other words, *a submacroscopically stable pair* (F_0, F_0) *can arise in a classical dynamical process in* $\mathcal{E}_d$ *only if the corresponding Cauchy stress is a hydrostatic pressure*.

For general submacroscopically stable pairs (F_0, G_0) encountered in dynamical processes, the relation (11.5) may be written in the equivalent form

$$T_\backslash - T_d = -\lambda H_0^* \tag{11.7}$$

with $T_\backslash$ and T_d the stresses without and due to disarrangements with respect to the current configuration defined in Section 8. This result applies even in cases where $\det G_0 < \det F_0$. An equivalent form of (11.7) is obtained from (11.4) and reads

$$S_\backslash - S_d = -\lambda(\det K_0)G_0^*, \tag{11.8}$$

where (F_0, G_0) is a submacroscopically stable pair and $K_0 = F_0^{-1}G_0$.

The results in the previous paragraphs show that submacroscopic stability leads to hydrostatic states of stress for classical dynamical processes in $\mathcal{E}_d$, but not necessarily for non-classical dynamical processes in $\mathcal{E}_d$, and describe the states of stress encountered at arbitrary submacroscopically stable pairs. In the elastostatics of crystals, the conclusion that equilibrium leads to hydrostatic states of stress has been verified in several contexts ([24–27]). We note that the conclusions of Ericksen [24], of Chipot and Kinderlehrer [25], and of Fonseca and Parry [26] were based in part on the symmetries of the crystals they considered, whereas the results of the previous paragraphs are independent of the notion of material symmetry, a concept that we study in the next section. For isotropic elastic bodies, Mizel's results on the energetics of fractured states [28] foreshadow our results on submacroscopic stability.

12. Material Symmetry

For each point X_0 in the region $\mathcal{A}$ undergoing a dynamical process, we consider the transformation properties of the kinematical quantities at that point under a change of virgin configuration determined by a given unimodular tensor $\mathbb{H}_0$. These transformation properties can be obtained by replacing the time-parameterized family of structured deformations (χ, G) by the composition

$$\begin{aligned}(X,t) \;\mapsto\; & ((\chi, G) \circ (\xi_{\mathbb{H}_0}, \mathbb{H}_0))(X,t) \\ & = (\chi(\xi_{\mathbb{H}_0}(X,t),\; t),\; G(\xi_{\mathbb{H}_0}(X,t),\; t)\mathbb{H}_0),\end{aligned} \tag{12.1}$$

where $\xi_{\mathbb{H}_0}$ denotes the homogeneous, time-independent deformation $(X,t) \mapsto X_0 + \mathbb{H}_0(X - X_0)$. From this observation we obtain the following *transformation rules*

$$\begin{aligned} F &\to F\mathbb{H}_0 \\ G &\to G\mathbb{H}_0 \\ M &\to M\mathbb{H}_0 \\ K &\to \mathbb{H}_0^{-1} K \mathbb{H}_0 \end{aligned} \tag{12.2}$$

under change of virgin configuration. In this display, if a quantity on the left is evaluated at (X,t), the corresponding quantity on the right is evaluated at $(X_0 + \mathbb{H}_0(X - X_0), t)$.

We say that $\mathbb{H}_0$ is a *symmetry transformation* at X_0 *with respect to changes of virgin configuration* for the elastic body undergoing disarrangements if the response function $(F,G) \mapsto \Psi(F, G; X_0)$ satisfies

$$\Psi(F\mathbb{H}_0, G\mathbb{H}_0, X_0) = \Psi(F, G, X_0) \tag{12.3}$$

for all (F,G) with $0 < \det G \leqslant \det F$ or, equivalently, if the response function $(M,G) \mapsto \tilde{\Psi}(M, G, X_0)$ satisfies

$$\tilde{\Psi}(M\mathbb{H}_0, G\mathbb{H}_0, X_0) = \tilde{\Psi}(M, G, X_0) \tag{12.4}$$

for all (M,G) with $0 < \det G \leqslant \det(M + G)$. As in elasticity without disarrangements, the symmetry transformations at X_0 form a group $\mathcal{G}_{X_0}^{\text{virgin}}$.

In the special case when $\mathcal{G}_{X_0}^{\text{virgin}}$ is the proper orthogonal group, it is easy to obtain necessary and sufficient conditions that (12.3) holds for all $\mathbb{H}_0 \in \mathcal{G}_{X_0}^{\text{virgin}}$. Indeed, we can choose $\mathbb{H}_0$ to be R_F^T or R_G^T, from the polar decompositions $F = V_F R_F$ and $G = V_G R_G$, to obtain

$$\begin{aligned}\Psi(F, G, X_0) &= \Psi(V_F R_F R_F^T, G R_F^T, X_0) = \Psi(V_F R_F R_F^T, G F^T V_F^{-1}, X_0) \\ &= \Psi(B_F^{1/2}, G F^T B_F^{-1/2}, X_0) = \hat{\Psi}(B_F, G F^T, X_0),\end{aligned} \tag{12.5}$$

or, alternatively,

$$\Psi(F, G, X_0) = \check{\Psi}(F G^T, B_G, X_0). \tag{12.6}$$

Here, $B_F = FF^T$ and $B_G = GG^T$ are the left Cauchy–Green tensors for F and G. The existence of a function $\hat{\Psi}$ such that (12.5) holds for all (F, G) with $0 < \det G \leqslant \det F$ (or, equivalently, of $\check{\Psi}$ such that (12.6) holds for all such pairs) is a necessary and sufficient condition that $\mathcal{G}_{X_0}^{\text{virgin}}$ be the proper orthogonal group. Similarly, the existence of a function $(M, G) \mapsto \Psi^{\#}(M, G, X_0)$ such that

$$\tilde{\Psi}(M, G, X_0) = \Psi^{\#}(MG^T, B_G, X_0) \tag{12.7}$$

for all (M, G) with $0 < \det G \leqslant \det(M + G)$ is both necessary and sufficient for $\mathcal{G}_{X_0}^{\text{virgin}}$ to be the proper orthogonal group.

In the case when $\mathcal{G}_{X_0}^{\text{virgin}}$ is the proper unimodular group, we may put $\mathbb{H}_0 = (\det F)^{1/3} F^{-1}$ or $\mathbb{H}_0 = (\det G)^{1/3} G^{-1}$ to conclude that the Helmholtz free energy can be expressed as a function of the pair $(\det F, H)$ or, equivalently, in terms of $(\det G, H)$ with, as usual, $H = GF^{-1}$. In Section 14, we will consider the special case where the Helmholtz free energy depends on the volume fraction $\det K = \det G / \det F = \det H$, alone.

Alternatively, a notion of material symmetry may be formulated in terms of invariance of response to changes in *reference configuration*. For each point X_0 in the region $\mathcal{A}$ undergoing a dynamical process, we consider the transformation properties of the kinematical quantities at that point obtained first by factoring (χ, G) via the notion of composition introduced in (2.5),

$$(\chi, G) = (\chi, \nabla\chi) \circ (\pi, \nabla\chi^{-1} G). \tag{12.8}$$

Here, $\pi(X, t) = X$ for all X and t. The factor $(\chi, \nabla\chi)$ on the right-hand side of (12.8) is a family of classical deformations, while $(\pi, \nabla\chi^{-1} G)$ involves only purely submacroscopic deformations, because π leaves each point fixed. We next replace the expression on the right-hand side of (12.8) by

$$((\chi, \nabla\chi) \circ (\xi_{\mathbb{H}_0}, \mathbb{H}_0)) \circ (\pi, \nabla\chi^{-1} G), \tag{12.9}$$

where, as above, $\xi_{\mathbb{H}_0}$ denotes the homogeneous, time-independent deformation $(X, t) \mapsto X_0 + \mathbb{H}_0(X - X_0)$. This replacement leaves the purely submacroscopic factor $(\pi, \nabla\chi^{-1} G)$ unchanged and changes only the classical factor $(\chi, \nabla\chi)$. From this replacement we obtain the following *transformation rules*

$$\begin{aligned} F &\to F\mathbb{H}_0 \\ G &\to (F\mathbb{H}_0 F^{-1})G \\ M &\to (F\mathbb{H}_0 F^{-1})M \\ K &\to K \end{aligned} \tag{12.10}$$

under change of reference configuration. In this display, if a quantity on the left is evaluated at (X, t), the quantity on the right is evaluated at $(X_0 + \mathbb{H}_0(X - X_0), t)$.

For a pair $(\mathbb{H}_0, K_0)$, with $0 < \det K_0 \leqslant 1 = \det \mathbb{H}_0$, we say that $\mathbb{H}_0$ is a *symmetry transformation at* X_0 for K_0 *with respect to changes of reference configuration* for the elastic body undergoing disarrangements if the response function $F \mapsto \Psi(F, FK_0, X_0)$ satisfies

$$\Psi(F\mathbb{H}_0, F\mathbb{H}_0 K_0, X_0) = \Psi(F, FK_0, X_0) \tag{12.11}$$

for all tensors F with $0 < \det F$. Equivalently, we may use the definition in Section 10, $\overline{\Psi}(F, K, X_0) := \Psi(F, FK, X_0)$ for all pairs (F, K) with $0 < \det F$ and $0 < \det K \leqslant 1$, to write (12.11) in the simpler form

$$\overline{\Psi}(F\mathbb{H}_0, K_0, X_0) = \overline{\Psi}(F, K_0, X_0) \tag{12.12}$$

for all tensors F with $0 < \det F$. We denote by $\mathcal{G}^{\mathrm{ref}}_{X_0,K_0}$ the group formed by the symmetry transformations at X_0 for K_0. The symmetry group $\mathcal{G}^{\mathrm{ref}}_{X_0,K_0}$ defined through (12.11) or (12.12) corresponds to the usual symmetry group of an elastic body undergoing only classical deformations, because the influence of disarrangements is removed by fixing the value of $K = F^{-1}G$ at K_0.

We remark that there is a notion of invariance dual to (12.12):

$$\overline{\Psi}(F, K\mathbb{P}_0, X_0) = \overline{\Psi}(F, K, X_0) \tag{12.13}$$

for all tensors F and K with $0 < \det F$ and $0 < \det K \leqslant 1$, with $\mathbb{P}_0$ a given unimodular tensor. This invariance arises by replacing the right-hand side of the factorization (12.8) by

$$(\chi, \nabla\chi) \circ ((\pi, \nabla\chi^{-1}G) \circ (\pi, \mathbb{P}_0)). \tag{12.14}$$

The purely submacroscopic factor $(\pi, \mathbb{P}_0)$ alters the given one $(\pi, \nabla\chi^{-1}G)$ without changing the classical factor $(\chi, \nabla\chi)$. The resulting symmetry group $\mathcal{G}^{\mathrm{submac}}_{X_0}$ resembles a notion introduced by Šilhavý and Kratochvíl [29], in the context of Noll's new theory of simple materials [30], and adapted by Bertram [31] to formulate and solve problems in the plasticity of materials undergoing large deformations. Moreover, $\mathcal{G}^{\mathrm{submac}}_{X_0}$ is obtained by means of non-classical changes $(\pi, \mathbb{P}_0)$ in configuration, as distinct from the groups $\mathcal{G}^{\mathrm{ref}}_{X_0,K_0}$ and $\mathcal{G}^{\mathrm{virgin}}_{X_0}$, obtained via the classical changes $(\xi_{\mathbb{H}_0}, \mathbb{H}_0)$.

A case that merits further study consists of the assumption that, during a dynamical process of an elastic body undergoing disarrangements, there holds

$$K(X, t) \in \mathcal{G}^{\mathrm{submac}}_{X} \tag{12.15}$$

for all (X, t). It is easy to show that

$$D_F\overline{\Psi}(F, K(X, t), X) = D_F\overline{\Psi}(F, I, X) \tag{12.16}$$

and

$$D_K\overline{\Psi}(F, K(X, t), X)K(X, t)^T = D_K\overline{\Psi}(F, I, X) \tag{12.17}$$

for all F with $\det F > 0$ and for all dynamical processes satisfying (12.15). These relations should prove to be useful, because they restrict substantially the manner in which the field $(X, t) \mapsto K(X, t)$ can appear in the field relations (10.23)–(10.26). In other words, *the field relations simplify significantly when the disarrangements embodied in K correspond to submacroscopic symmetries of the elastic body.*

In the next section, we study invariance properties of the response function $\overline{\Psi}$ in (12.12) under simultaneous changes in K_0 and in X_0.

13. Material Uniformity

We now have evidence that the factorization

$$(\chi, G) = (\chi, \nabla\chi) \circ (\pi, \nabla\chi^{-1}G), \tag{13.1}$$

with $(X, t) \mapsto \pi(X, t) := X$ the trivial motion of the body, provides a useful way to distinguish between, on the one hand, the virgin configuration from which the purely submacroscopic family $(\pi, \nabla\chi^{-1}G)$ proceeds while introducing all of the disarrangements associated with the given family (χ, G), and, on the other hand, the classical reference configuration, a macroscopically time-independent configuration from which the classical motion $(\chi, \nabla\chi)$ proceeds without introducing further disarrangements. With a view toward capturing the influence of the purely submacroscopic factor $(\pi, \nabla\chi^{-1}G)$ on the material response, for each (X, t) we put as usual $K(X, t) = \nabla\chi(X, t)^{-1}G(X, t)$ and consider the mapping

$$F \mapsto \Psi(F, FK(X, t), X) = \overline{\Psi}(F, K(X, t), X), \tag{13.2}$$

the *classical free-energy response induced at* (X, t) *by the purely submacroscopic motion* $(\pi, \nabla\chi^{-1}G)$. We may interpret this response as that of an elastic material element to local deformations in which the disarrangements are frozen at their values for the material point X at time t. Because $\nabla\chi^{-1}G$ may vary with material point and time, the classical response will vary with the pair (X, t). However, the response function Ψ also depends upon X, and it may happen for a given time t_0 that the dependence of Ψ on X compensates for the dependence of $K(X, t_0)$ on X, i.e., for all $X, Y \in \mathcal{A}$ and for every tensor F with $\det F > 0$:

$$\overline{\Psi}(F, K(X, t_0); X) = \overline{\Psi}(F, K(Y, t_0); Y). \tag{13.3}$$

The condition (13.3) embodies the idea of a *materially uniform* elastic body as described by Truesdell and Noll [32], Noll [33]. (In particular, see relation (27.4) of [32], in which the dependence of response on material point is compensated by the choice of local configuration.) Moreover, the mapping $X \mapsto K(X, t_0)$ corresponds to their concept of a *uniform reference* for the body at time t_0. It is possible that one of the uniform references $X \mapsto K(X, t_0) = \nabla\chi(X, t_0)^{-1}G(X, t_0)$ for the body at time t_0 is the gradient of a mapping on $\mathcal{A}$, and this embodies the notion of a *materially uniform*, *homogeneous elastic body* (Truesdell and Noll [32], Noll [33]). If none of the uniform references at time t_0 is a gradient, the induced classical response is described as that of a *materially uniform, inhomogeneous elastic body*.

These considerations lead us to define the collection $\mathcal{E}_d^{\text{unif}}$ of dynamical processes in $\mathcal{E}_d$ that satisfy the *material uniformity condition*

$$\overline{\Psi}(F, K(X, t); X) = \overline{\Psi}(F, K(Y, t); Y) \tag{13.4}$$

for all $X, Y \in \mathcal{A}$, for every tensor F with $\det F > 0$, and for every time t. The dynamical processes in $\mathcal{E}_d^{\text{unif}}$ have the property that, for every t, $K(\cdot, t)$ is a uniform reference for the body. The material uniformity condition then may be viewed as a restriction on the field $(X, t) \mapsto K(X, t) = \nabla\chi(X, t)^{-1}G(X, t)$, to be appended to the field relations for elastic bodies undergoing disarrangements derived in Section 10. Determining or characterizing explicitly the collection $\mathcal{E}_d^{\text{unif}}$ would entail characterizing the solutions of the field relations, augmented by the material uniformity condition.

If we impose the requirement that the collection $\mathcal{E}_d^{\text{unif}}$ be non-empty, then the material uniformity condition (13.4) places restrictions on the response function $\overline{\Psi}$. To make this observation more apparent, we consider a given classical dynamical process χ, $\nabla\chi$, S, ψ in $\mathcal{E}_d$, and we ask how $\overline{\Psi}$ would be restricted by requiring that this classical dynamical process be in $\mathcal{E}_d^{\text{unif}}$. An immediate answer follows from the fact that, for every classical dynamical process in $\mathcal{E}_d$, $K(X, t) = I$ for all X, t. Therefore, the material uniformity condition applied to the given classical dynamical process in $\mathcal{E}_d$ becomes the condition

$$\overline{\Psi}(F, I, X) = \overline{\Psi}(F, I, Y) \tag{13.5}$$

for all $X, Y \in \mathcal{A}$ and for every tensor F with $\det F > 0$. Evidently, this relation restricts the constitutive function $\overline{\Psi}$ used to define $\mathcal{E}_d$.

The material uniformity condition (13.4) may be differentiated with respect to F to obtain the relation

$$D_F\overline{\Psi}(F, K(X, t), X) = D_F\overline{\Psi}(F, K(Y, t), Y)$$

valid for all F, X, Y, and t. In the relation (10.28) between the stress and free energy, the material uniformity condition does not seem to imply a corresponding uniformity condition on the response functions that determine any of the stresses S, $S_\backslash$, and S_d. Of course, one directly can impose in place of (13.4) – or in addition to it – a material uniformity condition on one or more of the stress responses, for example on the response function from (10.28) for the Piola–Kirchhoff stress S:

$$\begin{aligned} &2D_F\overline{\Psi}(F, K(X, t), X) + F^{-T}D_K\overline{\Psi}(F, K(X, t), X)(I - 2K^T(X, t)) \\ &\quad = 2D_F\overline{\Psi}(F, K(Y, t), Y) + F^{-T}D_K\overline{\Psi}(F, K(Y, t), Y)(I - 2K^T(Y, t)) \end{aligned}$$

for all F, X, Y, and t. Although this choice would provide a different restriction on the response function $\overline{\Psi}$, it can be interpreted and studied along the lines outlined for (13.4).

14. Energetically Nearsighted Elastic Bodies

The field relations obtained in Section 10, together with special properties of a body such as material uniformity and material symmetry, provide the setting for understanding the scope and range of applicability of elasticity with disarrangements. In this section we take a preliminary step in this direction by considering elastic bodies that are "energetically nearsighted" in the sense that only the purely submacroscopic factor $(\pi, (\nabla\chi)^{-1}G) = (\pi, K)$ in the factorization (13.1) affects the free energy. Thus, submacroscopic slips or formation of voids would permit the body to change its free energy, while the classical deformations embodied in the factor $(\chi, \nabla\chi)$ would not, and we consider now elastic bodies for which the free energy response $\overline{\Psi}$ in (10.20) does not depend upon the macroscopic deformation F and, therefore, satisfies $D_F\overline{\Psi}(F, K, X) = 0$ for all triples F, K, X in the domain of $\overline{\Psi}$. Accordingly, the field relations (10.23)–(10.27) and the stress relation (10.28) take the form

$$\operatorname{div}\big[F^{-T}\overline{\Psi}'(K)\big(I - 2K^T\big)\big] + b_{\text{ref}} = \rho_{\text{ref}}\ddot{\chi}, \tag{14.1}$$

$$\overline{\Psi}'(K)K^T(\{K^{-T}\}^2 - 3K^{-T} + I) = 0, \tag{14.2}$$

$$\operatorname{sk}(F^{-T}\overline{\Psi}'(K)(I - 2K^T)F^T) = 0, \tag{14.3}$$

$$(F^{-T}\overline{\Psi}'(K)(I - 2K^T)) \cdot \dot{F} - \overline{\Psi}'(K) \cdot \dot{K} \geqslant 0, \tag{14.4}$$

$$1 \geqslant \det K > 0, \tag{14.5}$$

$$S = F^{-T}\overline{\Psi}'(K)(I - 2K^T), \tag{14.6}$$

where we have written $\overline{\Psi}'$ in place of $D_K\overline{\Psi}$ to simplify notation. In some of the considerations below, it is helpful to use the field $H := GF^{-1} = FKF^{-1}$ associated with the factorization (2.7), and we note the relation

$$H^{-T} = F^{-T}K^{-T}F^T = (F^T)^{-1}K^{-T}F^T. \tag{14.7}$$

14.1. UNIVERSAL PHASES AND THE GOLDEN MEAN

The consistency relation in the form (14.2) provides a restriction on the field K, and the form of this restriction depends in general upon the response function $\overline{\Psi}$. We observe, however, that there are solutions K of the consistency relation that do not depend upon $\overline{\Psi}$, because the expression $\{K^{-T}\}^2 - 3K^{-T} + I$ occurs multiplicatively in (14.2). Consequently, each tensor K with $0 < \det K \leqslant 1$ for which K^{-T} is a solution of the quadratic, tensor equation

$$X^2 - 3X + I = 0, \quad X \in \operatorname{Lin}\mathcal{V}, \tag{14.8}$$

determines a solution of the consistency relation (14.2). It is easy to see that K^{-T} is a solution of the quadratic equation (14.8) if and only K itself is a solution. In turn, this is equivalent to the assertion that the tensor $H = FKF^{-1}$ is a solution of (14.8). We call solutions K (with $0 < \det K \leqslant 1$) of the consistency relation *universal* if they are solutions of (14.8), because they do not depend upon the free energy response function of the nearsighted elastic body. We also refer to the corresponding tensor $H = FKF^{-1}$as a universal solution of the consistency relation.

A necessary condition for H to be universal is the inclusion of the spectrum of H in the solution set $\{(3 + \sqrt{5})/2, (3 - \sqrt{5})/2\} = \{2 + \gamma_0, 1 - \gamma_0\}$ of the scalar quadratic equation $x^2 - 3x + 1 = 0$. Here, $\gamma_0 := (\sqrt{5} - 1)/2 \approx 0.618$ is the "golden mean," the positive number satisfying the relation $1/x = x/(1 - x)$. By elementary linear algebra, H is universal if and only if it is diagonalizable over the reals with diagonal entries given up to permutations by one of the two triples: $(1-\gamma_0, 1-\gamma_0, 1-\gamma_0)$, $(1-\gamma_0, 1-\gamma_0, 2+\gamma_0)$. (The possibilites $(1-\gamma_0, 2+\gamma_0, 2+\gamma_0)$ and $(2 + \gamma_0, 2 + \gamma, 2 + \gamma_0)$ are ruled out by the restriction $\det H \in (0, 1]$.)

Of course, the first triple $(1 - \gamma_0, 1 - \gamma_0, 1 - \gamma_0)$ determines the tensor

$$H_{\mathrm{sph}} := (1 - \gamma_0)I \tag{14.9}$$

that, in turn, determines the purely submacroscopic structured deformation $(i, (1-\gamma_0)I)$ to follow a classical deformation $(\chi(\cdot, t), \nabla\chi(\cdot, t))$. A piecewise smooth approximation h_n for $(i, (1-\gamma_0)I)$ takes a body in its current configuration without disarrangements, partitioned into congruent cubic cells of side $1/n$, and replaces each cell by one with the same center but now of side $(1 - \gamma_0)/n$. The simultaneous shrinking of each cell creates voids, and the resulting structured deformation has volume fraction $\det H_{\mathrm{sph}} = \det K_{\mathrm{sph}} = (1 - \gamma_0)^3 \approx 0.056$. This change of submacroscopic geometry determines the (*universal*) *spherical phase* of the energetically nearsighted elastic body.

For each choice of basis $(d_{(1)}, d_{(2)}, d_{(3)})$, with corresponding reciprocal basis $(d^{(1)}, d^{(2)}, d^{(3)})$ satisfying $d_{(i)} \cdot d^{(j)} = \delta_i^j$, the second triple $(1 - \gamma_0, 1 - \gamma_0, 2 + \gamma_0)$ of diagonal entries determines a tensor

$$\begin{aligned} H_{\mathrm{long}} &:= (1 - \gamma_0)d_{(1)} \otimes d^{(1)} + (1 - \gamma_0)d_{(2)} \otimes d^{(2)} + (2 + \gamma_0)d_{(3)} \otimes d^{(3)} \\ &= H_{\mathrm{sph}} + (1 + 2\gamma_0)d_{(3)} \otimes d^{(3)} \end{aligned} \tag{14.10}$$

that, in turn, determines a purely submacroscopic deformation (i, H_{long}) to follow a classical deformation $(\chi(\cdot, t), \nabla\chi(\cdot, t))$ that we refer to as the (*universal*) *elongated phase* of the body. A piecewise smooth approximation h_n for (i, H_{long}) takes each of the basic cells of side $1/n$, with its edges now parallel to $d_{(1)}$, $d_{(2)}$, $d_{(3)}$, respectively, and stretches the "$d_{(3)}$" edge to the length $(2 + \gamma_0)/n \approx 2.618/n$ while shrinking the other two edges to the length $(1 - \gamma_0)/n \approx 0.382/n$. The simultaneous elongation of each cell creates voids, and the resulting structured deformation (i, H_{long}) has volume fraction

$$\det H_{\mathrm{long}} = \det K_{\mathrm{long}} = (1 - \gamma_0)^2(2 + \gamma_0) = (1 - \gamma_0) \approx 0.382. \tag{14.11}$$

We have used the fact that $(1 - \gamma_0)$ and $(2 + \gamma_0)$ are reciprocals in the last calculation. Moreover, after elongation, the cells may have to be translated slightly in order to avoid interpenetration of neighboring cells, because the piecewise smooth approximations h_n are required to be injective. While the universal solution H_{sph} does not vary in space and time, the universal solution H_{long} may vary through dependence of the dyad $d_{(3)} \otimes d^{(3)}$ on position and time. In addition, the basis $(d_{(1)}, d_{(2)}, d_{(3)})$ need not be orthogonal, so that the approximating deformations h_n map unit cubes into possibly non-rectangular parallelepipeds. For a particular class of energetically nearsighted elastic bodies, the basis $(d_{(1)}, d_{(2)}, d_{(3)})$ must be orthonormal, as we demonstrate below, and we have $d^{(i)} = d_{(i)}$ for $i = 1, 2, 3$.

14.2. FIELD RELATIONS FOR A CLASS OF NEARSIGHTED ELASTIC BODIES

We now specialize the discussion above to the case

$$\overline{\Psi}(F, K) = \bar{\psi}(\det K) = \bar{\psi}(\det H) \tag{14.12}$$

in which only the volume fraction $f := \det K = \det H \in (0, 1]$ produced by the purely submacroscopic deformations (i, H) and (i, K) affects the free energy. We shall restrict our attention to universal phases of the energetically nearsighted elastic material under consideration, so that the consistency relation need not be considered further, and the formula

$$\overline{\Psi}'(K) = f\bar{\psi}'(f)K^{-T} \tag{14.13}$$

along with the fact that $f \in \{1 - \gamma_0, (1 - \gamma_0)^3\}$ is a constant for each phase yields after some computations the following forms of the remaining field relations (14.4), (14.3), and (14.1), as well as the stress relation (14.6):

$$f\bar{\psi}'(f)\mathrm{div}((H^{-1} - 2I)F^{-T}) + b_{\mathrm{ref}} = \rho_{\mathrm{ref}}\ddot{\chi}, \tag{14.14}$$

$$\mathrm{sk}H = 0, \tag{14.15}$$

$$\bar{\psi}'(f)(H^{-1} - 2I) \cdot \dot{F}F^{-1} - \bar{\psi}'(f)\mathrm{tr}(\dot{H}H^{-1}) \geqslant 0, \tag{14.16}$$

$$S = f\bar{\psi}'(f)(H^{-1} - 2I)F^{-T}. \tag{14.17}$$

We have used the frame-indifference of the mixed power (14.15) to replace H^{-T} by H^{-1} in the balance law (14.14) and in the mixed power inequality (14.16), and we have assumed that $\bar{\psi}'(f) \neq 0$ for $f \in \{1 - \gamma_0, (1 - \gamma_0)^3\}$. The symmetry of H implies that we may write

$$H_{\mathrm{long}} = H_{\mathrm{sph}} + (1 + 2\gamma_0)d \otimes d, \tag{14.18}$$

with $d := d_{(3)} = d^{(3)}$. An easy calculation provides the formulas

$$H_{\mathrm{sph}}^{-1} = (2 + \gamma_0)I, \qquad H_{\mathrm{long}}^{-1} = (2 + \gamma_0)I - (1 + 2\gamma_0)d \otimes d, \tag{14.19}$$

and, using the constancy of H_{sph}, we obtain the specific forms for the balance of linear momentum, the mixed power inequality, and the stress relation *in the spherical phase*

$$\gamma_0(1 - \gamma_0)^3\bar{\psi}'((1 - \gamma_0)^3)\mathrm{div}\, F^{-T} + b_{\mathrm{ref}} = \rho_{\mathrm{ref}}\ddot{\chi}, \tag{14.20}$$

$$\bar{\psi}'((1 - \gamma_0)^3)\mathrm{tr}(\dot{F}F^{-1}) \geqslant 0, \tag{14.21}$$

$$S = \gamma_0(1-\gamma_0)^3\bar{\psi}'((1-\gamma_0)^3)F^{-T}. \tag{14.22}$$

The stress relation (14.22) implies that the Cauchy stress in the spherical phase is given by

$$\begin{aligned} T &= SF^T/\det F = \rho SF^T/\rho \det F \\ &= C_0\rho I, \end{aligned} \tag{14.23}$$

where $C_0 := \gamma_0(1-\gamma_0)^3\bar{\psi}'((1-\gamma_0)^3)/\rho_{\text{ref}}$ has the same sign as $\bar{\psi}'((1-\gamma_0)^3)$ and ρ denotes the density in the current configuration. Thus, in the spherical phase, the energetically nearsighted elastic body experiences a hydrostatic stress that is proportional to the density in the current configuration. If $\bar{\psi}'((1-\gamma_0)^3) < 0$, then the stress is a hydrostatic pressure, again proportional to the density, as in the case of an ideal gas. Thus, *if* $\bar{\psi}'((1-\gamma_0)^3) < 0$, *the equation of state of the energetically nearsighted elastic body in the spherical phase is that of an ideal gas undergoing isothermal dynamical processes.* Of course, the balance of linear momentum then takes the standard form in the current configuration for gas dynamics:

$$C_0 \operatorname{grad} \rho + b = \rho\dot{v}, \tag{14.24}$$

where ρ and b now denote the density and body force in the current configuration. However, the mixed power inequality now requires that

$$\operatorname{div} v \leqslant 0, \tag{14.25}$$

which tells us that, *when* $\bar{\psi}'((1-\gamma_0)^3) < 0$*, the spherical phase can arise only when the elastic material is not expanding.*

By employing relations (14.18), (14.19), and the formulas

$$\dot{H}_{\text{long}} = (1+2\gamma_0)(\dot{d}\otimes d + d\otimes\dot{d}), \quad d\cdot d = 1$$

we obtain in a similar way specific forms for the balance of linear momentum, the mixed power inequality, and the stress relation *in the elongated phase*

$$\begin{aligned} &(1-\gamma_0)\bar{\psi}'((1-\gamma_0))\operatorname{div}[(\gamma_0 I - (1+2\gamma_0)d\otimes d)F^{-T}] + b_{\text{ref}} \\ &\quad = \rho_{\text{ref}}\ddot{\chi}, \end{aligned} \tag{14.26}$$

$$(1-\gamma_0)\bar{\psi}'((1-\gamma_0))[\gamma_0 I - (1+2\gamma_0)d\otimes d]\cdot \dot{F}F^{-1} \geqslant 0, \tag{14.27}$$

$$S = (1-\gamma_0)\bar{\psi}'((1-\gamma_0))[\gamma_0 I - (1+2\gamma_0)d\otimes d]F^{-T}. \tag{14.28}$$

(In relations (14.26)–(14.28), the symbol d denotes the field introduced in (14.18) referred to the reference configuration.) The stress relation (14.28) implies that the Cauchy stress in the elongated phase is given by

$$T = C_1\rho[I - (3+\gamma_0)d\otimes d] \tag{14.29}$$

with $C_1 := \gamma_0(1-\gamma_0)\bar{\psi}'((1-\gamma_0))/\rho_{\text{ref}}$ having the same sign as $\bar{\psi}'((1-\gamma_0))$, and the relations (14.24) and (14.25) become in the elongated phase:

$$C_1 \operatorname{grad}\rho - C_1(3+\gamma_0)\operatorname{div}[\rho d \otimes d] + b = \rho\dot{v} \tag{14.30}$$

and

$$\operatorname{div} v \leqslant (3+\gamma_0)(\operatorname{grad} v)d \cdot d \tag{14.31}$$

the latter when $\bar{\psi}'((1-\gamma_0)) < 0$. We conclude that the elongated phase can persist when $\operatorname{div} v$ is positive, as long as the stretching $(\operatorname{grad} v)d \cdot d$ in the direction of d is at least $\operatorname{div} v/(3+\gamma_0)$. Consequently, *when* $\bar{\psi}'((1-\gamma_0)) < 0$*, the material in the elongated phase may expand or contract as long as the direction of submacroscopic elongation d is strongly aligned with directions of stretching in the macroscopic motion.*

Appendix: Decomposition of Flux Densities

We record here for the sake of completeness the principal relations obtained in [3, 15], concerning the decomposition of flux densities arising through a structured deformation, with g the macroscopic deformation and G the deformation without disarrangements. If $w\colon \mathcal{A} \to \mathcal{V}$ is a smooth vector field and $K = (\nabla g)^{-1}G$, then the identity

$$\begin{aligned}(\det K)&\operatorname{div} w \\ &= \operatorname{div}(K^{*T}w) + \operatorname{div}((\det K)w - K^{*T}w) - w \cdot \nabla(\det K)\end{aligned} \tag{A.1}$$

is an immediate consequence of the product rule

$$\operatorname{div}(\varphi w) = \nabla\varphi \cdot w + \varphi \operatorname{div} w,$$

where φ is an arbitrary smooth scalar field. By an appropriate choice of determining sequence $n \mapsto h_n$ for the purely submacroscopic structured deformation (i, K), one can derive the following identification relations for $(\det K)\operatorname{div} w$ and $\operatorname{div}(K^{*T}w)$ at each point $X \in \mathcal{A}$:

$$(\det K)\operatorname{div} w|_X = \lim_{r\to 0}\lim_{n\to\infty} r^{-3} \sum_{\mathcal{C}\in\mathbf{C}_n} \int_{h_n(\operatorname{bdy}(C_r(X)\cap\mathcal{C}))} w(Y)\cdot\nu(Y)\,\mathrm{d}A_Y, \tag{A.2}$$

$$\operatorname{div}(K^{*T}w)|_X = \lim_{r\to 0}\lim_{n\to\infty} r^{-3} \sum_{\mathcal{C}\in\mathbf{C}_n} \int_{h_n(\operatorname{bdy}(C_r(X))\cap\mathcal{C})} w(Y)\cdot\nu(Y)\,\mathrm{d}A_Y. \tag{A.3}$$

In (A.2) and (A.3), for each positive integer n, $\mathbf{C}_n$ is a collection of closed cubes $\mathcal{C}$ that cover the region $\mathcal{A}$ and whose faces together include the jump-sites of h_n

and of ∇h_n, and $C_r(X)$ is a cube centered at X of side r whose faces intersect the jump-sites of all the functions h_n and ∇h_n in a set of area zero.

The surface integral in (A.2) is taken over the image under h_n of all the faces of the parallelepiped $C_r(X) \cap \mathcal{C}$, so that the sum in (A.2) represents the total flux of w across the image of the faces of $C_r(X)$ *and* across the image of the faces of cubes $\mathcal{C}$ in $\mathbf{C}_n$ containing the jump-sites of h_n and ∇h_n inside $C_r(X)$. Therefore, the limit on the right-hand side of (A.2) and, hence, the left-hand side $(\det K)\mathrm{div}\, w|_X$, represents the *volume density of the total flux of* w.

The surface integral in (A.3) is taken instead over the image under h_n of only those faces of the parallelepiped $C_r(X) \cap \mathcal{C}$ that belong to the boundary of $C_r(X)$ and *not* to the images of faces of cubes $\mathcal{C}$ in $\mathbf{C}_n$ containing the jump-sites of h_n and ∇h_n inside $C_r(X)$. Therefore, the limit on the right-hand side of (A.3) and, hence, the left-hand side $\mathrm{div}(K^{*T} w)|_X$, represents the *volume density of the flux of* w *without disarrangements*. Consequently, the remaining terms $(\mathrm{div}((\det K)w - K^{*T} w) - w \cdot \nabla(\det K))|_X$ on the right-hand side of (A.1) represent the *volume density of the flux of* w *due to disarrangements*.

The identification relation (A.3) also permits us to call the vector field

$$w_{\backslash} := K^{*T} w \tag{A.4}$$

the *portion of* w *without disarrangements*. Moreover, from (A.2) the divergence of the vector field

$$w_d := (\det K)w - K^{*T} w \tag{A.5}$$

together with the term $-w \cdot \nabla(\det K)$, account for all the volume density of flux due to disarrangements, and we call w_d the *portion of* w *due to disarrangements*.

By (A.5) we may write

$$(\det K)w = w_{\backslash} + w_d \tag{A.6}$$

as an additive decomposition of the vector field $(\det K)w$ into the portion of w without disarrangements and the portion of w due to disarrangements. Relations (A.4)–(A.6) yield the *consistency relation*

$$K^T w_{\backslash} = w_{\backslash} + w_d. \tag{A.7}$$

We note that for each fixed vector $a \in \mathcal{V}$, we may set $w = S^T a$ in (A.1)–(A.7) and, in view of the arbitrariness of a, recover the relations (3.2), (3.3), (3.5) and (3.6).

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