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Mechanics of Materials 32 (2000) 389–403

MECHANICS OF MATERIALS

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A class of viscoelastoplastic constitutive models based on the maximum dissipation principle

L. Deseri [a,*], R. Mares [a,b]

[a] Dipartimento di Ingegneria, Università di Ferrara, Via G. Saragat 1, Ferrara 44100, Italy
[b] ENEA, Via don Fiammelli 2, Bologna 40127, Italy

Received 29 July 1998; received in revised form 7 January 2000

Abstract

A class of viscoelastoplastic constitutive models is derived from the maximum inelastic dissipation principle, in the framework of infinitesimal deformations, and in analogy to the elastoviscoplastic case examined in Simo and Honein (cf. Simo, J.C., Honein, T., 1990. J. Appl. Mech. 57, 488–497). Here the existence of the equilibrium response functional with respect to which the overstress is measured, and the existence of an instantaneous elastic response (Haupt, P., 1993. Acta Mech. 100, 129–154; Krempl, E., 1996. Unified Constitutive Laws of Plastic deformation. Academic Press, San Diego; Tsakmakis, Ch., 1996a. Acta Mech. 115, 179–202) are assumed. A broad set of overstress functions turns out to characterize the class of models derived herein. Both the flow rule for the viscoplastic deformation and the rate form of the constitutive equation for the class of models cited above are obtained, and the behavior of this equation under very slow strain rates and very high viscosity is investigated. A numerical simulation is also given by selecting two overstress functions available in the literature (Haupt, P., Lion, A., 1993. Continuum Mech. Thermodyn. 7, 73–90; Krempl, E., Yao, D., 1987. In: Rie, K.T. (Ed.), Low-Cycle Fatigue and Elasto-Plastic Behavior of Materials. Elsevier, New York, pp. 137–148). Loading conditions of repeated strain rate variation, monotonic strain rate with relaxation and cyclic loading at different strain rates are examined, and qualitative agreement is shown with the experimental observations done in Krempl and Kallianpur and Haupt and Lion (cf. Krempl, E., Kallianpur, V.V., 1984. J. Mech. Phys. Solids 32(4), 301–304; Haupt, P., Lion, A., 1993. Continuum Mech. Thermodyn. 7, 73–90) and references cited therein). © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Viscoelastoplasticity; Constitutive models; Maximum dissipation principle; Overstress; Asymptotic behavior

1. Introduction

Recent studies about inelastic behavior of metals have been motivated by the increase of applications of high strength alloys when severe mechanical loading and thermal cycling are involved. In these cases loading-rate sensitivity, creep, relaxation and cyclic hardening have been shown to be the most important features of the material behavior (Krempl, 1996 and references cited therein). Pioneering attempts to model rate effects in metals were related to plastic deformation processes only. To do this *elastoviscoplastic* constitutive theories have been employed (Perzyna, 1963, 1966; Chaboche, 1977, 1989, 1993). In the theories of this type it is customary to mean that the so-called viscoplastic response of the

* Corresponding author. Tel.: +39-0532-293800; fax: +39-0532-768602.
E-mail address: luca@ing.unife.it (L. Deseri).

0167-6636/00/$ - see front matter
PII: S0167-6636(00)00011-9

material becomes manifest as soon as some specified combinations of the stress exceeds a characteristic value, which is prescribed by a scalar yield function, in analogy to the usual plasticity theories. In this paper, we deal with materials that show rate dependence in both recoverable (elastic) and not recoverable (plastic) strains, so that we can consider *viscoelastoplastic* constitutive models for such a behavior. This terminology goes back to a paper of Naghdi and Murch (1963), and was recently employed by Haupt (1993).

The problem of describing viscoelastoplastic deformations of materials arises in modeling the behavior of structural elements under intensive loading and elevated temperature conditions, which in particular are characteristic of various industrial treatments, such as welding, forging, etc. (Carmignani et al., 1999; Dexter et al., 1991). Moreover, there is need to modify the current methodology to improve the capability for fatigue-life prediction of structural members, composed of materials that can be expected to exhibit rate dependent inelastic response under the given operating conditions. This is the case of loads experienced by aircrafts during landing and take-off.

Motivated by several experimental observations relative to the rate sensitivity of steels, in analogy with the procedure introduced by Simo and Honein (1990) for the elastoviscoplastic case, we follow a methodology based on the maximum dissipation principle taken as an axiom, in order to derive a class of viscoelastoplastic constitutive models.

To do this, a penalty version of the maximum dissipation principle is considered (Truesdell, 1984; Lubliner, 1991). The fairly general a priori restrictions on the *penalty function* allow for the description of a broad class of viscoelastoplastic models.

Besides the validity of the maximum dissipation principle, the extra constitutive assumptions required to derive the class of models mentioned above are: (i) the existence of an *equilibrium stress response functional*, which is a rate-independent functional of the strain in the sense of Pipkin and Rivlin (1965); (ii) an instantaneous elastic response (Haupt, 1993). Among other papers, motivations to support (i) are given in (Krempl, 1987 and references cited therein).

Furthermore, because only linear kinematics is considered herein, the classical additive decomposition of the (infinitesimal) total strain in an elastic and an inelastic contribution is assumed to hold.

The derivative of the penalty function with respect to its argument is the *overstress function* governing the inelastic flow, and the expression of this flow rule is obtained as the result of the *optimality condition* (Luenberger, 1984). Because of (ii), by taking into account the equation for the viscoplastic flow, the rate form of the response functionals belonging to the class of viscoelastoplastic models discussed in the present paper turns out to be also formed by two terms. The first term is the *equilibrium stress*, which is a functional of the strain, and the second term is the *overstress*, i.e., roughly speaking the excess of stress with respect to the equilibrium one (Hoheneuser and Prager, 1932; Krempl, 1987; Haupt and Lion, 1993; Tsakmakis, 1996a; Krempl, 1996). In order to measure the magnitude of this excess of stress a norm, which is 'phenomenologically motivated', in the stress space is chosen (see Section 2, item (iv)).

Further, we investigate the behavior of the response functionals mentioned above in the limiting cases of infinitely slow motions, as well as when the viscosity-like parameter of the model tends to zero. In particular, we show that in this case the constitutive relation between the strain and the stress reduces to the relation between the strain and the associated equilibrium stress, no matter what the particular choice of the penalization function could be.

In the theory proposed in this paper, the character of generality for the choice of the response functional at equilibrium can lead to a so-called hierarchical theory, in the sense that it may include classical metals plasticity formulations as a special case, as well as *a non-standard rate dependent theory without yield surface*, see e.g. Valanis (1971), Lubliner (1973, 1991 and references cited therein), Gurtin et al. (1980), Krempl and Kallianpur (1984). This fact, and the choice to measure the magnitude of the overstress with any Euclidean norm (for instance the one cited above), establish the main differences between the present formulation and the one given in Simo and Hughes (1998).

Two particular (one-dimensional) overstress functions introduced by Krempl and Yao (1987) and Haupt and Lion (1993) are compatible with the fairly general properties listed in Luenberger (1984) (see Section 3 of the present paper, formulas (3.4) and (3.5)). These properties will be discussed in Section 4.

It is worth noting that the present formulation has essentially two key advantages: (a) the existence of yield surfaces is not needed, because of the validity of (i), (ii) and the maximum dissipation principle in the version mentioned above; (b) unlike in Valanis (1971), no 'intrinsic time' is required.

In the sequel, we will refer to the following notation. We denote by Sym the set of all symmetric linear transformations of vectors defined on the ordinary Euclidean point space, and by LinSym the set of all linear transformations on Sym. The inner product in Sym is

$$\mathbf{A} \cdot \mathbf{B} := \mathrm{tr}\,(\mathbf{AB}) \tag{1.1}$$

and the induced norm is

$$\|\mathbf{A}\| := (\mathbf{A} \cdot \mathbf{A})^{1/2}. \tag{1.2}$$

The space LinSym will be normed by the following norm:

$$|||\boldsymbol{C}||| := \sup_{\mathbf{A}\in \mathrm{Sym}\setminus\{0\}} \frac{\|\boldsymbol{C}\mathbf{A}\|}{\|\mathbf{A}\|}. \tag{1.3}$$

2. Phenomenological motivations

The starting point of any attempt to model real behavior of materials consists of the experimental analysis of the phenomenological aspects of suitable material specimens. The obtained experimental results must be interpreted in order to recognize certain few key properties, that can be assumed to build the mathematical model for the description of the material behavior. With this aim, several stress–strain diagrams representing the mechanical behavior of stainless steels, like AISI 304 and AISI 316, ferritic A553B pressure vessel steel and a Ti–7Al–2Cb–1Ta alloy, tested in different conditions of loading, strain rate, relaxation, and creep time (Haupt and Lion, 1993; Krempl and Kallianpur, 1984) at room temperature, have been considered. A typical stress–strain illustrative diagram coming from tests of this type is shown in Fig. 1.

Different stress–strain curves at different strain rates are observed: the lowest is characterized by a strain rate near zero and is called *equilibrium curve*. The vertical lines between a stress–strain curve and the equilibrium curve represent relaxation tests (constant strain). We interpret the equilibrium curve as the set of all the 'totally relaxed' stress–strain pairs, i.e. points in the stress–strain space obtained by extreme retardation procedure of the response functional (see e.g. Del Piero and Deseri (1995) for the linear viscoelastic case, Tsakmakis (1996b) and Section 3 of the present paper). In other words, this is equivalent to take the limit for $\dot{\varepsilon} \to 0$ of the response functional (Haupt and Lion, 1993). To the extent of reproducing relaxation tests, we consider experiments at constant strain rate α^{-1}, where $(\mathbf{E}, \mathbf{T}_{\alpha}^{\mathrm{meas}})$ is a measured strain–stress pair and $(\mathbf{E}, \mathbf{T}_{\infty})$ is the pair on the (theoretical) equilibrium curve having the same strain. Roughly speaking, we say that if $\alpha_1^{-1} < \alpha_2^{-1}$ are two values of the strain rate and $(\mathbf{E}, \mathbf{T}_{\alpha_1}^{\mathrm{meas}})$, $(\mathbf{E}, \mathbf{T}_{\alpha_2}^{\mathrm{meas}})$ are the corresponding stress–strain pairs, each of these represents a relaxed pair if, for an arbitrary small fixed parameter p, there exists a constant α_p such that, for strain rate sufficiently small, i.e., such that if $\alpha_1^{-1} < \alpha_2^{-1} < \alpha_p^{-1}$, then $\|\mathbf{T}_{\alpha_2}^{\mathrm{meas}} - \mathbf{T}_{\alpha_1}^{\mathrm{meas}}\| < p$ and $\|\mathbf{T}_{\alpha_i}^{\mathrm{meas}} - \mathbf{T}_{\infty}\| < p$, $i = 1, 2$. Thus, the parameter p measures the precision threshold under which the relaxation test is performed.

It is experimental evidence (Krempl and Lu, 1984; Krempl, 1987; Haupt and Lion, 1993) that when steel samples are subjected to cyclic deformations (Hard devices) the yielding threshold does increase, irrespective to the strain rate. On the other hand, if the strain rate is increased by one order of magnitude after yielding, an increase of the stress (relative overstress) is observed. This increment does not depend on the number of cycles that the specimen experiences before each increase in strain rate after yielding. This suggests that the rate-dependent properties of these materials are not influenced by the strain history (see Fig. 2), but these properties are rather constitutive, as it is well known.

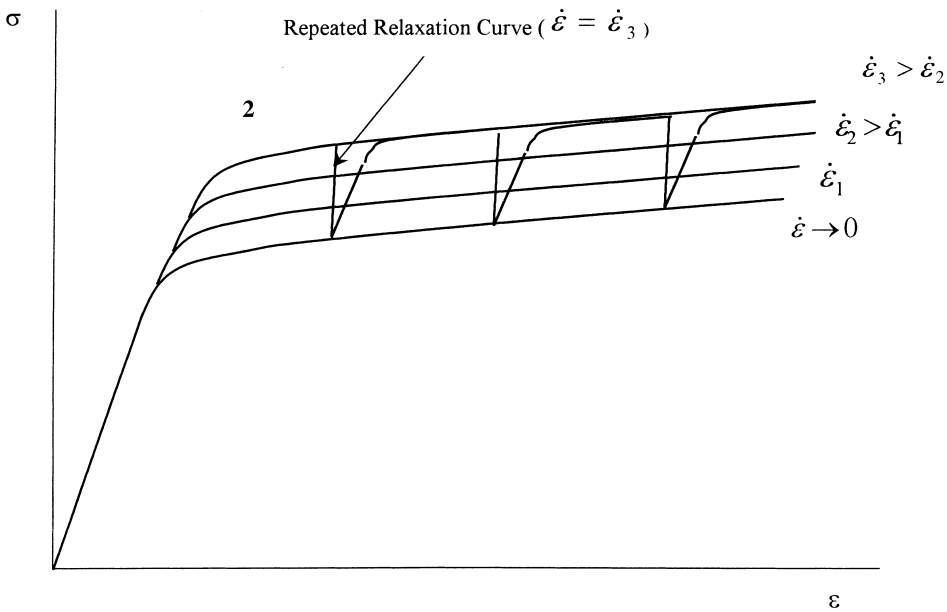

Fig. 1. Illustration of monotonic stress–strain curves at different strain rate and of repeated relaxation curve for stainless steel at room temperature. See e.g. Haupt and Lion (1993) or Krempl (1987).

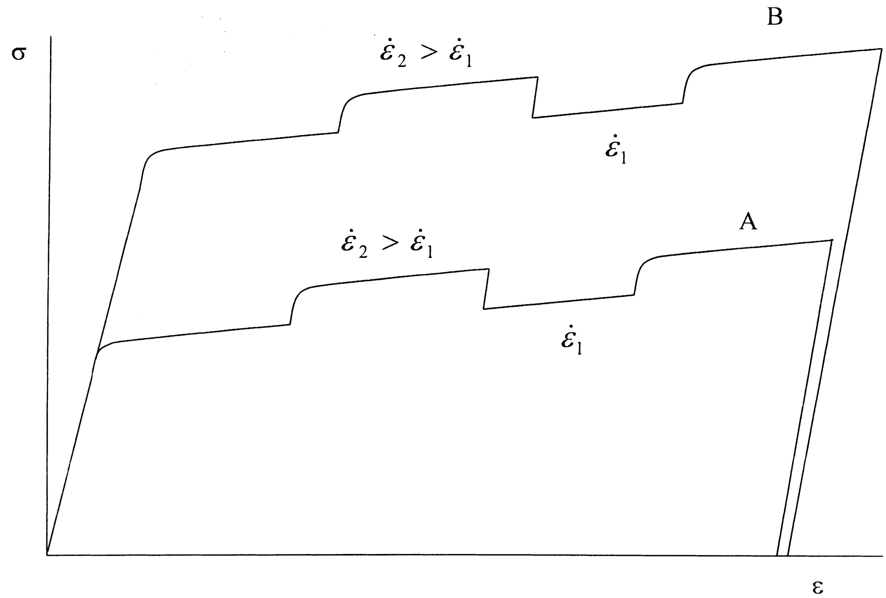

Fig. 2. Illustration of the stress–strain behavior of stainless steel for an abrupt variation in the strain rate. Curve A represents the response of a virgin material, curve B is obtained after prior cycling to saturation and subsequent unloading to zero stress and strain, see e.g. Krempl (1987).

The value of the stress that can be reached for non-zero strain rate with respect to the equilibrium curve is called *overstress* (Hoheneuser and Prager, 1932; Perzyna, 1963; Chaboche, 1977; Haupt and Lion, 1993; Majors and Krempl, 1994; Krempl, 1995; Tsakmakis, 1996a; Krempl, 1996 and references listed therein), and its rate dependence is strongly non-linear.

We can list the most relevant facts extracted from the experimental results, that can form the basis for the derivation of a suitable class of constitutive models for the materials cited above as follows:

(i) there is no practical evidence of a yielding threshold;
(ii) the existence of an equilibrium curve is motivated (Haupt and Lion, 1993, among others);
(iii) non-linear dependence of the stress on strain rate is detected;
(iv) the response of the material during the raising of the strain rate appears to be equal to the response before yielding independently to the number of cycles (Krempl and Lu, 1984; Krempl, 1987; Haupt and Lion, 1993).

These observations, together with the principle of maximum inelastic dissipation, may provide a guideline to construct a constitutive model.

3. A class of viscoelastoplastic constitutive models

3.1. Derivation

As a general consequence of the experimental observations it is reasonable to assume for every class of viscoelastoplastic models an instantaneous elastic response governed by the following law (Haupt, 1993, Section 3):

$$\dot{\mathbf{T}} = \boldsymbol{G}_0 \dot{\mathbf{E}}_{\mathrm{e}} = \boldsymbol{G}_0 [\dot{\mathbf{E}} - \dot{\mathbf{E}}_{\mathrm{in}}], \tag{3.1}$$

where $\mathbf{T}$ is the stress tensor, $\boldsymbol{G}_0$ the (constant) fourth-order positive definite tensor of the instantaneous elastic moduli (see e.g., Haupt, 1993, Section 9, for the isotropic case), and $\dot{\mathbf{E}} = \dot{\mathbf{E}}_{\mathrm{e}} + \dot{\mathbf{E}}_{\mathrm{in}}$, where $\dot{\mathbf{E}}_{\mathrm{e}}$, $\dot{\mathbf{E}}_{\mathrm{in}}$ are the elastic and inelastic parts of the strain rate tensor $\dot{\mathbf{E}}$.

Moreover, we admit the existence of an equilibrium domain Σ in the stress–strain space, whose boundary will be denoted by $\partial\Sigma$.

One of the main features of the class of models derived in this paper is the rate dependence of the overstress (Haupt and Lion, 1993; Krempl, 1987; Tsakmakis, 1996a), or the fact that a state of stress can exist beyond the equilibrium domain. In order to measure the distance from a state of stress to the equilibrium domain equivalent norms can be chosen, since the space of stresses has finite dimension (see e.g., Halmos, 1958). By virtue of observation (iv) of Section 2, it seems natural to take the norm

$$\|\mathbf{T}\| = \sqrt{\mathbf{T} \cdot \boldsymbol{G}_0^{-1} \mathbf{T}}, \tag{3.2}$$

whose dimension is a square root of an energy per unit volume. Indeed, the choice of the norm just introduced is motivated by the fact that for abrupt jumps of the strain rate the response of the material appears to be elastic, and governed by the tensor $\boldsymbol{G}_0$ of the instantaneous moduli (see e.g. Fig. 2).

Here we do not make use of any concept related to intrinsic time like in Valanis' theory (Valanis, 1971), and despite the classical elastoviscoplasticity laws (Perzyna, 1963, 1966; Chaboche, 1977, 1989, 1993) we do not need initial definition of a yielding or loading surface.

We consider further the following penalty version of the maximum inelastic dissipation principle (Truesdell, 1984; Lubliner, 1991 and references cited therein) find $\mathbf{T} \in \mathrm{Sym}$ s.t.:

$$-D_\tau(\mathbf{T}, \dot{\mathbf{E}}_{\mathrm{in}}) = \min_{\mathbf{P} \in \mathrm{Sym}} [-D_\tau(\mathbf{P}; \dot{\mathbf{E}}_{\mathrm{in}})], \tag{3.3a}$$

$$D_\tau(\mathbf{P}, \dot{\mathbf{E}}_{\mathrm{in}}) := \mathbf{P} \cdot \dot{\mathbf{E}}_{\mathrm{in}} - \frac{1}{\tau} \varphi^+(\|\mathbf{T} - \mathbf{T}_\infty\|), \quad \mathbf{T}_\infty \in \partial\Sigma. \tag{3.3b}$$

Here, $\mathbf{T}_\infty \in \partial\Sigma$ is the equilibrium or relaxed stress tensor, whose values are given by any choice of the response functional at equilibrium as long as $\boldsymbol{G}_\infty$ is positive definite. Moreover, $\tau \in (0, \infty)$ is the parameter of penalization, which represents a viscosity-like parameter, and $\varphi^+ : \mathbb{R} \to \mathbb{R}^+$ is the penalization function with the properties listed in Luenberger (1984), i.e.

$$\begin{aligned} &\varphi^+(x) \geqslant 0 \quad \forall x \in \mathbb{R}, \\ &\varphi^+(x) = 0 \iff x \leqslant 0. \end{aligned} \tag{3.4}$$

As additional assumption, we take

$$\frac{\mathrm{d}\varphi^+}{\mathrm{d}x}(x) > 0 \quad \forall x \in \mathbb{R}^+/\{0\}. \tag{3.5}$$

It is worth noting that in the case of elastoplasticity with elastic range Σ (Owen, 1968) and hardening (Lubliner, 1991 and the references cited therein; Simo and Honein, 1990), the counterpart of relations (3.3) are

$$\begin{aligned} &D^{\mathrm{p}}(\mathbf{T}, \mathbf{t}; \dot{\mathbf{E}}^{\mathrm{p}}, \dot{\mathbf{h}}) := \mathbf{T} \cdot \dot{\mathbf{E}}^{\mathrm{p}} + \mathbf{t} \cdot \dot{\mathbf{h}}, \\ &D^{\mathrm{p}}(\mathbf{T}, \mathbf{t}; \dot{\mathbf{E}}^{\mathrm{p}}, \dot{\mathbf{h}}) = \max_{(\mathbf{P},\mathbf{z}) \in \Sigma} \{D^{\mathrm{p}}(\mathbf{P}, \mathbf{z}; \dot{\mathbf{E}}^{\mathrm{p}}, \dot{\mathbf{h}})\}, \end{aligned} \tag{3.6}$$

where $\mathbf{E}^{\mathrm{p}}$ denotes the plastic deformation, $\mathbf{h}$ the list of hardening internal variables, and $\mathbf{t}$ is the generalized stress associated with $\mathbf{h}$.

From the *optimality conditions* for the unconstrained optimization problem (3.3) (Luenberger, 1984) we can easily obtain the following expression for the inelastic flow:

$$\dot{\mathbf{E}}_{\mathrm{in}} = \frac{1}{\tau} g(\|\mathbf{T} - \mathbf{T}_\infty\|) \boldsymbol{G}_0^{-1}[\mathbf{T} - \mathbf{T}_\infty], \tag{3.7}$$

where $g(x) =: \mathrm{d}\varphi^+/\mathrm{d}x$. By substituting (3.7) in Eq. (3.1) we have the rate form of the constitutive equation

$$\dot{\mathbf{T}} = \boldsymbol{G}_0 \dot{\mathbf{E}} - \frac{1}{\tau} g(\|\mathbf{T} - \mathbf{T}_\infty\|)[\mathbf{T} - \mathbf{T}_\infty] \tag{3.8}$$

after using the identity $\boldsymbol{G}_0\, \boldsymbol{G}_0^{-1} = \boldsymbol{G}_0^{-1} \boldsymbol{G}_0 = \mathbf{I}$, where $\mathbf{I}$ is the fourth-order identity tensor.

We note that the model proposed by Haupt (1993) (Eq. (50.1)) belongs to the class of constitutive equations described by (3.8). It is worth recalling that $\mathbf{T}_\infty$ is a rate-independent functional of the past strain history (Haupt, 1993, Section 9). Because the aim of this paper is the derivation of a class of viscoelastoplastic models, besides the general requirements summarized below in this section, no detailed information is required about $\mathbf{T}_\infty$. On the other hand, several choices can be made for the expression of $\mathbf{T}_\infty$, such as the ones proposed by Krempl et al. (1986), Sutcu and Krempl (1989), Haupt (1993) and Krempl (1996). Essentially, $\mathbf{T}_\infty \in \Sigma \cup \partial\Sigma$ is implicitly defined by virtue of a first-order non-linear differential equation (growth law for $\mathbf{T}_\infty$, see e.g., Krempl et al., 1986); this might also involve the other internal variables describing the 'state' of the material, such as the so-called kinematic stress (Krempl, 1996, Section II, B). In Section 5 a suitable equation for $\mathbf{T}_\infty \in \partial\Sigma$ in the one-dimensional case is taken to describe the one-dimensional equilibrium response. Obviously, for $\mathbf{T}_\infty \in \Sigma$ the material behaves elastically and, if we set $\boldsymbol{G}_\infty = \partial\mathbf{T}_\infty/\partial\mathbf{E}$, the evolution equation for $\mathbf{T}_\infty$ becomes

$$\dot{\mathbf{T}}_\infty = \boldsymbol{G}_\infty \dot{\mathbf{E}}, \tag{3.9}$$

where $\boldsymbol{G}_\infty$ is the so-called equilibrium elasticity tensor, and it may depend on the current strain $\mathbf{E}$. The only a priori restriction required on $\boldsymbol{G}_\infty$ is the positive definiteness. Explicit formulas for $\boldsymbol{G}_\infty$ might be used to describe specific behavior (see e.g., Simo and Hughes, 1998, Tsakmakis, 1996b and references cited therein). For instance, if $\mathbf{T}_\infty \in \partial\Sigma$ and an isotropic material with isotropic hardening is concerned, we have

$$\begin{aligned} \boldsymbol{G}_\infty = {} & 2\mu_\infty \mathbf{I} + \left(k_\infty - \frac{2}{3}\mu_\infty\right)\mathbf{I} \otimes \mathbf{I} \\ & - \frac{2\mu_\infty}{1 + (h_\infty/3\mu_\infty)} \mathbf{N} \otimes \mathbf{N}, \end{aligned} \tag{3.10}$$

where μ_∞, k_∞ and h_∞ are the shear, the bulk and the (isotropic) hardening moduli at equilibrium, respectively, and

$$\mathbf{N} = \frac{\mathbf{T}_\infty - \frac{1}{3}\,\mathrm{tr}\mathbf{T}_\infty \mathbf{I}}{\left\|\mathbf{T}_\infty - \frac{1}{3}\,\mathrm{tr}\mathbf{T}_\infty \mathbf{I}\right\|}$$

is the projector onto the tangent plane to the equilibrium surface $\partial\Sigma$.

To complete the analysis, an evolution equation for the hardening parameter also has to be specified, as well as the choice of the arch length variable (see e.g., Tsakmakis, 1996a, Section 3.2, Eqs. (19)–(31)).

Unlike the class of viscoelastoplastic constitutive models considered in this paper, that exhibit rate-independent behavior at equilibrium, other models that exhibit rate dependence of $\mathbf{T}_\infty$ have been considered in the past (Majors and Krempl, 1994).

The constitutive equation (3.6) exhibits a viscoelastoplastic behavior in analogy to the one discussed in (Lubliner, 1973; Gurtin et al., 1980): in particular, in this paper, it is pointed out that (3.8) may lead to the description of rate-type viscoelastic behavior for suitable form of g. For instance, if we choose

$$\frac{1}{\tau}g(\|\mathbf{T}-\mathbf{T}_\infty\|)=\text{const.}=\eta, \tag{3.11}$$

$\mathbf{T}(0)=\mathbf{0}$, $\mathbf{E}(0)=\mathbf{0}$, by substituting (3.11) in (3.8) we obtain

$$\dot{\mathbf{T}}+\eta\mathbf{T}=\boldsymbol{G}_0\dot{\mathbf{E}}+\eta\boldsymbol{G}_\infty\mathbf{E}, \tag{3.12}$$

which represents the standard (Kelvin–Voigt) linear solid. We can rewrite (3.12) in its integral form

$$\mathbf{T}(t)=\int_{-\infty}^{t}\boldsymbol{G}(t-s)\dot{\mathbf{E}}(s)\,\mathrm{d}s, \tag{3.13}$$

where $\boldsymbol{G}(r)=\boldsymbol{G}_\infty+(\boldsymbol{G}_0-\boldsymbol{G}_\infty)\exp(-\eta r)$ is the so-called *relaxation function* of exponential type according to the terminology introduced in (Del Piero and Deseri, 1995).

We note further that (3.8) can also be reformulated in implicit form by the following system of equations:

$$\mathbf{T}=\mathbf{T}_\infty+\boldsymbol{\Omega}, \tag{3.14a}$$

$$\dot{\boldsymbol{\Omega}}=\boldsymbol{G}_0\dot{\mathbf{E}}-\dot{\mathbf{T}}_\infty-\frac{1}{\tau}g(\|\boldsymbol{\Omega}\|)\boldsymbol{\Omega}, \tag{3.14b}$$

where we define as additional internal variable the overstress tensor $\boldsymbol{\Omega}=\mathbf{T}-\mathbf{T}_\infty$; thus, (3.14b) is the evolution equation for the overstress and it has the same structure of the viscoplastic equation investigated by Gurtin et al. (1980) and by Cernocky and Krempl (1979), and by Tsakmakis (1996b). It follows that the rate sensitivity is directly built in (3.14), that is to say throughout the evolution equation for the overstress.

It is worth noting that one of the differences between the present approach and the one presented in (Simo and Hughes, 1998) is the fact that here no definition of yielding or loading surface is needed for the derived class of viscoelastoplastic constitutive models.

The analysis of limiting cases of high viscosity and very slow motions can be investigated. In the following, these two cases will be called the *asymptotic material behavior* and *asymptotic physical behavior*, respectively.

3.2. Asymptotic material behavior

For high values of the viscosity a rate-independent behavior is expected from every viscoelastoplastic constitutive equation. For the class of models proposed in this paper, the rate-independent behavior is a consequence of the fact that the tensor $\mathbf{T}\in\text{Sym}$ which maximizes the inelastic dissipation is attained in the limit when $1/\tau$ tends to infinity, i.e. at $\mathbf{T}_\infty\in\partial\Sigma$. This result is proved in the following theorem.

Theorem. *Let $n\mapsto\lambda_n$, $\lambda_n:=1/\tau_n$, be a sequence of reals, and let $n\mapsto\mathbf{T}_n$ be the minimizer of* (3.3) *for $1/\tau=\lambda_n$. Assume that the function $\Gamma(\mathbf{T},\mathbf{T}_\infty):=\varphi^+(\|\mathbf{T}-\mathbf{T}_\infty\|)$ be continuous with respect to the variable $\mathbf{T}$ and note that $\bar{D}(\mathbf{T},\dot{\mathbf{E}}_{\text{in}}):=-\mathbf{T}\cdot\dot{\mathbf{E}}_{\text{in}}$ is continuous with respect to $\mathbf{T}$. Then, we have*

$$\lim_{n\to\infty}\mathbf{T}_n=\mathbf{T}_\infty\in\Sigma\cup\partial\Sigma. \tag{3.15}$$

Proof. Consider an unbounded monotonic increasing sequence $n\mapsto\lambda_n$ and let $n\mapsto\mathbf{T}_n$ be the associated sequence of minimizers of (3.3), where $\tau=1/\lambda_n$ and $\|\mathbf{T}_n\|<+\infty$. Thus, there exists a subsequence of $n\mapsto\mathbf{T}_n$ which is convergent to a limit $\tilde{\mathbf{T}}$. With abuse of notation, we will denote with $n\mapsto\mathbf{T}_n$ such a subsequence. The continuity of $\bar{D}(\cdot,\dot{\mathbf{E}}_{\text{in}})$ implies that

$$\lim_{n\to\infty}\bar{D}(\mathbf{T}_n,\dot{\mathbf{E}}_{\text{in}})=\bar{D}(\tilde{\mathbf{T}},\dot{\mathbf{E}}_{\text{in}}), \tag{3.16}$$

and if we consider the (n+1)th term of the sequence $n\mapsto\lambda_n$, and we recall (3.3b) and the definitions of $\bar{D}$ and Γ, we get

$$\begin{aligned}D_{1/\lambda_{n+1}}&=\bar{D}(\mathbf{T}_{n+1},\dot{\mathbf{E}}_{\text{in}})+\lambda_{n+1}\Gamma(\mathbf{T}_{n+1},\mathbf{T}_\infty)\\&\geqslant\bar{D}(\mathbf{T}_{n+1},\dot{\mathbf{E}}_{\text{in}})+\lambda_n\Gamma(\mathbf{T}_{n+1},\mathbf{T}_\infty),\end{aligned} \tag{3.17}$$

because $n\mapsto\lambda_n$ is monotonic non-decreasing by assumption. For every fixed λ_n, because the corresponding minimizer of (3.3a) and (3.3b) is $\mathbf{T}_n$, the following inequality holds:

$$\bar{D}(\mathbf{T}_{n+1}, \dot{\mathbf{E}}_{\text{in}}) + \lambda_n \Gamma(\mathbf{T}_{n+1}, \mathbf{T}_\infty) \geqslant \bar{D}(\mathbf{T}_n, \dot{\mathbf{E}}_{\text{in}}) + \lambda_n \Gamma(\mathbf{T}_n, \mathbf{T}_\infty) = D_{1/\lambda_n} \tag{3.18}$$

so that $D_{1/\lambda_{n+1}} \geqslant D_{1/\lambda_n}$. Let us consider the constrained minimization problem associated with the unconstrained problem (3.3a) and (3.3b): find $\hat{\mathbf{T}} \in \text{Sym}$ s.t.

$$\bar{D}(\hat{\mathbf{T}}, \dot{\mathbf{E}}_{\text{in}}) = \min_{\mathbf{T}\in\text{Sym}} \bar{D}(\mathbf{T}, \dot{\mathbf{E}}_{\text{in}}) \quad \text{and} \quad \|\mathbf{T} - \mathbf{T}_\infty\| = 0. \tag{3.19}$$

Let $\hat{\mathbf{T}} \in \text{Sym}$ be the solution of (3.19) and let $\hat{D}$ be the optimal value of the inelastic dissipation associated to $\hat{\mathbf{T}}$, i.e. $\hat{D} := \bar{D}(\hat{\mathbf{T}}, \dot{\mathbf{E}}_{\text{in}})$. Then, for every fixed λ_n we have

$$D_{1/\lambda_n} \leqslant \hat{D}_{1/\lambda_n} = \hat{D}, \tag{3.20}$$

because $\Gamma(\hat{\mathbf{T}}, \mathbf{T}_\infty) = 0$, so that the sequence $n \mapsto D_{1/\lambda_n}$ is bounded from above by $\hat{D}$ and non-decreasing. Thus,

$$\lim_{n\to\infty} D_{1/\lambda_n} = L, \tag{3.21}$$

where $L \leqslant \hat{D}$. By subtracting (3.16) from (3.21), and recalling the definitions of $\bar{D}(\mathbf{T}, \dot{\mathbf{E}}_{\text{in}})$ and of $\Gamma(\mathbf{T}, \mathbf{T}_\infty)$, we get

$$\lim_{n\to\infty} \lambda_n \Gamma(\mathbf{T}_n, \mathbf{T}_\infty) = L - \bar{D}(\tilde{\mathbf{T}}, \dot{\mathbf{E}}_{\text{in}}). \tag{3.22}$$

Since $\Gamma(\mathbf{T}_n, \mathbf{T}_\infty) \geqslant 0$ by assumption (3.3a), and $\lambda_n \to \infty$ for $n \to \infty$, (3.22) implies

$$\lim_{n\to\infty} \Gamma(\mathbf{T}_n, \mathbf{T}_\infty) = 0 \tag{3.23}$$

and the assumed continuity of Γ implies $\Gamma(\tilde{\mathbf{T}}, \mathbf{T}_\infty) = 0$. By definition of Γ, we obtain $\lim_{n\to\infty} \mathbf{T}_n = \tilde{\mathbf{T}} = \mathbf{T}_\infty \in \partial\Sigma$, which completes the proof of the theorem. □

3.3. Asymptotic physical behavior

The rate-independent behavior is also expected when the material is subjected to very slow processes. In order to investigate the material response to such processes, the time dependence of the overstress function $\hat{g}(\mathbf{\Omega}) := g(\|\mathbf{\Omega}\|)$, $\mathbf{\Omega} \in \text{Sym}$, must be made explicit. It is worth noting that (3.5) and the definitions of g ensure that $\hat{g}(\mathbf{\Omega}) = 0 \iff \mathbf{\Omega} = \mathbf{0}$, $\mathbf{0}$ being the null element of Sym. In other words, because $\mathbf{\Omega}$ is meant to be equal to the difference between every stress tensor $\mathbf{T}$ and $\mathbf{T}_\infty \in \partial\Sigma$, the so-called overstress $\mathbf{\Omega}$ turns out to be zero if and only if $\mathbf{T} \in \partial\Sigma$. The explicit dependence of the overstress function with respect to time can be described by defining the function $\tilde{g}(\cdot)$ such that $\tilde{g}(t) := \hat{g}(\mathbf{\Omega}(t))$, $\mathbf{\Omega}(t) := \mathbf{T}(t) - \mathbf{T}_\infty(t)$. It turns out that $\tilde{g}(\cdot)$ satisfies the following differential equation:

$$\dot{\tilde{g}}(t) + \frac{1}{\tau} K(t) \tilde{g}(t) = \left(\boldsymbol{G}_0 - \frac{\partial \mathbf{T}_\infty}{\partial \mathbf{E}} \right) \dot{\mathbf{E}} \cdot \frac{\partial \hat{g}}{\partial \mathbf{\Omega}}, \tag{3.24}$$

where $K(t) = (\partial\hat{g}/\partial\mathbf{\Omega}) \cdot \mathbf{\Omega}(t)$, obtained by the differentiation of $t \mapsto \tilde{g}(t)$, $t \mapsto \mathbf{\Omega}(t)$, of (3.6), and taking into account that $\dot{\mathbf{T}}_\infty = (\partial\mathbf{T}_\infty/\partial\mathbf{E})\dot{\mathbf{E}}$. By integrating (3.24) in the time interval $[a, b]$ with $\tilde{g}(a) = 0$, we obtain

$$g(t) = \int_a^t \mathbf{L}(t, t') \cdot \dot{\mathbf{E}}(t')\, dt', \tag{3.25}$$

where

$$\mathbf{L}(t, t') = \left[\exp\left(-\frac{1}{\tau} \int_t^{t'} K(\theta)\, d\theta \right) \right] \times \left(\boldsymbol{G}_0 - \frac{\partial \mathbf{T}_\infty}{\partial \mathbf{E}} \right) \frac{\partial \hat{g}}{\partial \mathbf{\Omega}}(t'), \tag{3.26}$$

for $t' \in [a, t)$. It is worth noting that Eq. (3.24) is formally analogous to the evolution equation for the stress in aging viscoelastic materials (see e.g., Rabotnov, 1980, p. 201 and references cited therein). The (formal) solution (3.25) of (3.24) is not time-translation invariant, because the scalar term $K(\theta)$ is generally non-constant: the constant case leads to the standard linear solid.

Since we are interested to study the influence of slow deformation processes on the material behavior, we may follow a time-rescaling procedure introduced in Del Piero and Deseri (1995), even though an equivalent formulation can be found in Tsakmakis (1996b). Let $[a, b]$ be a given time interval. We seek for the function which maps the interval $[a, b]$ into a rescaled interval of length $\alpha(b - a)$, with a uniform contraction if $\alpha < 1$ and a uniform dilatation if $\alpha > 1$. Both contraction and dilatation maps $t \mapsto f_\alpha(t)$ are constructed by assuming the end point b fixed, and the time instants in the interval $[a, b]$ can move forward and backward in time, respectively, depending on α linearly, i.e.

$$f_\alpha(t) := \begin{cases} t + (1-\alpha)(b-a) & \text{for } t < a, \\ \alpha t + (1-\alpha) b & \text{for } a \leqslant t < b, \\ t & \text{for } t \geqslant b. \end{cases} \tag{3.27}$$

If $\mathbf{E} : [a, b] \to \mathrm{Sym}$ is a deformation process, the process $\mathbf{E}_\alpha$

$$\mathbf{E}_\alpha(f_\alpha(t)) := \mathbf{E}(t), \quad t \in \mathbb{R}, \tag{3.28}$$

is also a deformation process, called the α-acceleration of $\mathbf{E}$ in $[a, b]$ if $\alpha < 1$, and the α-retardation of $\mathbf{E}$ in $[a, b]$ if $\alpha > 1$. In particular, for $t \geqslant b$

$$\mathbf{E}_\alpha(t) = \mathbf{E}(t) \quad \forall \alpha > 0, \tag{3.29}$$

and, for $t < b$

$$\mathbf{E}_\alpha(t) := \begin{cases} \mathbf{E}(t - (1-\alpha)(b-a)) & \text{for } t < f_\alpha(a), \\ \mathbf{E}(b - \alpha^{-1}(b-t)) & \text{for } t \geqslant f_\alpha(a). \end{cases} \tag{3.30}$$

From (3.26) it follows that the limiting process $\mathbf{E}_0$ for $\alpha \to 0$ is characterized by:

$$\mathbf{E}_0 := \begin{cases} \mathbf{E}(t) & \text{for } t \geqslant b, \\ \mathbf{E}(t - b + a) & \text{for } t < b \end{cases} \tag{3.31}$$

and the limiting process $\mathbf{E}_\infty$ for $\alpha \to \infty$ is characterized by

$$\mathbf{E}_\infty(t) := \begin{cases} \mathbf{E}(t) & \text{for } t \geqslant b, \\ \mathbf{E}(b^-) & \text{for } t < b. \end{cases} \tag{3.32}$$

For the time derivative of the rescaled deformation process we have

$$\dot{\mathbf{E}}_\alpha(t) = \begin{cases} \dot{\mathbf{E}}(t - (1-\alpha)(b-a)) & \text{for } t < f_\alpha(a), \\ \frac{1}{\alpha}\dot{\mathbf{E}}\left(b + \frac{t-b}{\alpha}\right) & \text{for } t \geqslant f_\alpha(a). \end{cases} \tag{3.33}$$

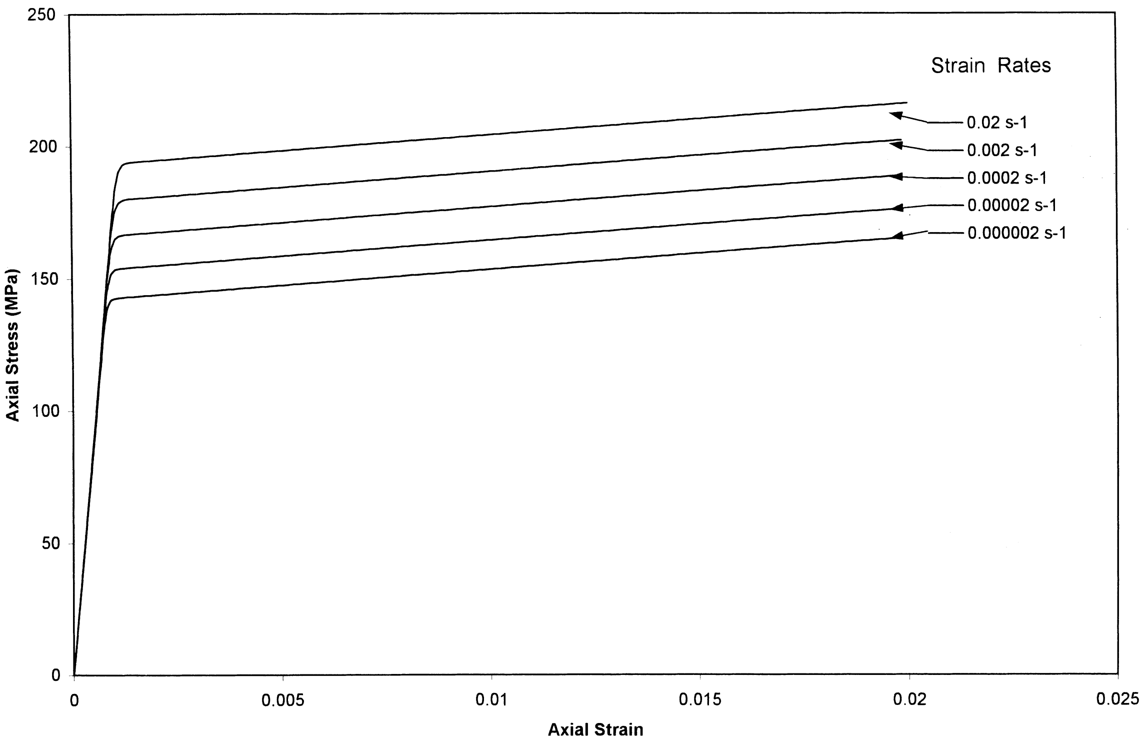

Fig. 3. Uniaxial stress–strain curves at different strain rates with g_1 overstress function.

In the following, deformation processes are meant to be bounded. The time-rescaling of the overstress function g can be written as:

$$\tilde{g}_\alpha(t) = \hat{g}_\alpha(\mathbf{\Omega}_\alpha(t)) = \tilde{g}_\alpha(\|\mathbf{T}_\alpha(t) - \mathbf{T}_\infty(t)\|). \qquad (3.34)$$

In order to explore the response of the material undergoing very slow processes a reasonable further assumption on the penalization function φ^+ appearing in (3.3) may be introduced. We have the following theorem.

Theorem. *Let* $\mathbf{E}_\alpha$ *be given as in* (3.29) *and* (3.30), *and bounded. Moreover, let* φ^+ *be piecewise twice differentiable and bounded from above. Then*

$$\lim_{\alpha\to+\infty} \tilde{g}_\alpha(t) = 0 \qquad (3.35)$$

i.e.

$$\lim_{\alpha\to+\infty} \mathbf{T}_\alpha(t) = \mathbf{T}_\infty. \qquad (3.36)$$

Proof. An integration by parts of (3.25) yields to

$$g_\alpha(t) = \mathbf{L}(t,a)\cdot\mathbf{E}_\alpha(t) - \mathbf{L}(t,b)\cdot\mathbf{E}_\alpha(a) - \int_a^b \frac{\partial \mathbf{L}}{\partial t'}(t,t')\cdot\mathbf{E}_\alpha(t')\,\mathrm{d}t' \qquad (3.37)$$

and by (3.26), we have

$$\frac{\partial \mathbf{L}(t,t')}{\partial t'} = \left[-\frac{1}{\tau}\int_{t'}^{t} K(\theta)\,\mathrm{d}\theta\right] K(t)\mathbf{L}(t,t'). \qquad (3.38)$$

To prove (3.35), the limit for $\alpha \to +\infty$ of integral appearing on the right-hand side of (3.37) must be evaluated. To do this, because the dependence on α enters inside the integral in $\mathbf{E}_\alpha$, the Lebesgue dominated convergence theorem may be needed. Recalling the definitions

$$g(\|\mathbf{\Omega}\|) := \frac{\mathrm{d}\varphi^+}{\mathrm{d}\|\mathbf{\Omega}\|}, \quad \hat{g}(\mathbf{\Omega}) := g(\|\mathbf{\Omega}\|),$$

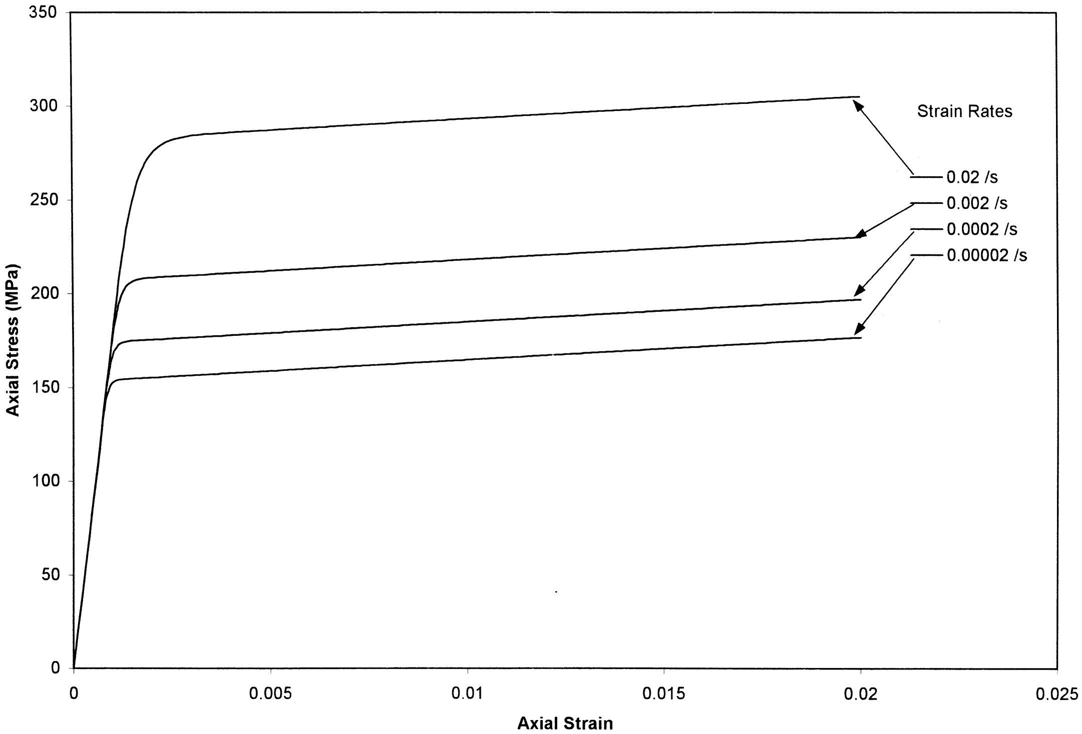

Fig. 4. Uniaxial stress–strain curves at different strain rates with g_2 overstress function.

we have

$$\frac{\partial \hat{g}}{\partial \mathbf{\Omega}} = \frac{d^2\varphi^+}{d\|\mathbf{\Omega}\|^2}(\|\mathbf{\Omega}\|)\frac{\boldsymbol{G}_0^{-1}\mathbf{\Omega}}{\|\mathbf{\Omega}\|} \tag{3.39}$$

and, by assumption, there exists $M > 0$ such that

$$\left|\frac{d^2\varphi^+}{d\|\mathbf{\Omega}\|^2}(\|\mathbf{\Omega}\|)\right| \leqslant M$$

for all $\mathbf{\Omega} \in \mathrm{Sym}$, so that the inequalities

$$\left|\frac{\partial \hat{g}}{\partial \mathbf{\Omega}}\right| \leqslant \frac{M}{\|\mathbf{\Omega}\|}\|\boldsymbol{G}_0^{-1}\mathbf{\Omega}\| \leqslant M|||\boldsymbol{G}_0^{-1}||| \tag{3.40}$$

follows.

By (3.38), by using the result

$$\left|\frac{1}{\tau}\int_{t'}^{t} K(\theta)\,d\theta\right| \leqslant \frac{1}{\tau}\int_{t'}^{t}|K(\theta)|\,d\theta$$

and recalling the definition of $K(t) = (\partial\hat{g}/\partial\mathbf{\Omega})\cdot\mathbf{\Omega}(t)$, we have

$$\begin{aligned}
&\left\|\frac{\partial \mathbf{L}(t,t')}{\partial t'}\right\| \\
&\leqslant \frac{1}{\tau}\int_{t'}^{t}|K(\theta)|d\theta|K(t')|\,\|\mathbf{L}(t,t')\| \\
&\leqslant \frac{1}{\tau}|||\boldsymbol{G}_0^{-1}|||M\int_{t'}^{t}\|\mathbf{\Omega}(\theta)\|d\theta M \sup_{\theta\in(a,b)}\|\mathbf{\Omega}(\theta)\|\,\|\mathbf{L}(t,t')\| \\
&\leqslant |||\boldsymbol{G}_0^{-1}|||\frac{M^2(b-a)}{\tau}\left(\sup_{\theta\in(a,b)}\|\mathbf{\Omega}(\theta)\|\right)^2|||\bar{\boldsymbol{G}}_0|||\left\|\frac{\partial\hat{g}}{\partial\mathbf{\Omega}}\right\| \\
&\leqslant |||\boldsymbol{G}_0^{-1}|||^2|||\bar{\boldsymbol{G}}_0|||\frac{M^3(b-a)}{\tau}\left(\sup_{\theta\in(a,b)}\|\mathbf{\Omega}(\theta)\|\right)^2
\end{aligned} \tag{3.41}$$

after using inequality (3.40), where $\bar{\boldsymbol{G}}_0 = \boldsymbol{G}_0 - (\partial\mathbf{T}_\infty/\partial\mathbf{E})$. Inequalities (3.41) show that $\partial\mathbf{L}(t,t')/\partial t'$ is uniformly bounded on $[a,b]$.

Moreover, because deformation processes are assumed to be bounded, we note that the following inequality hold:

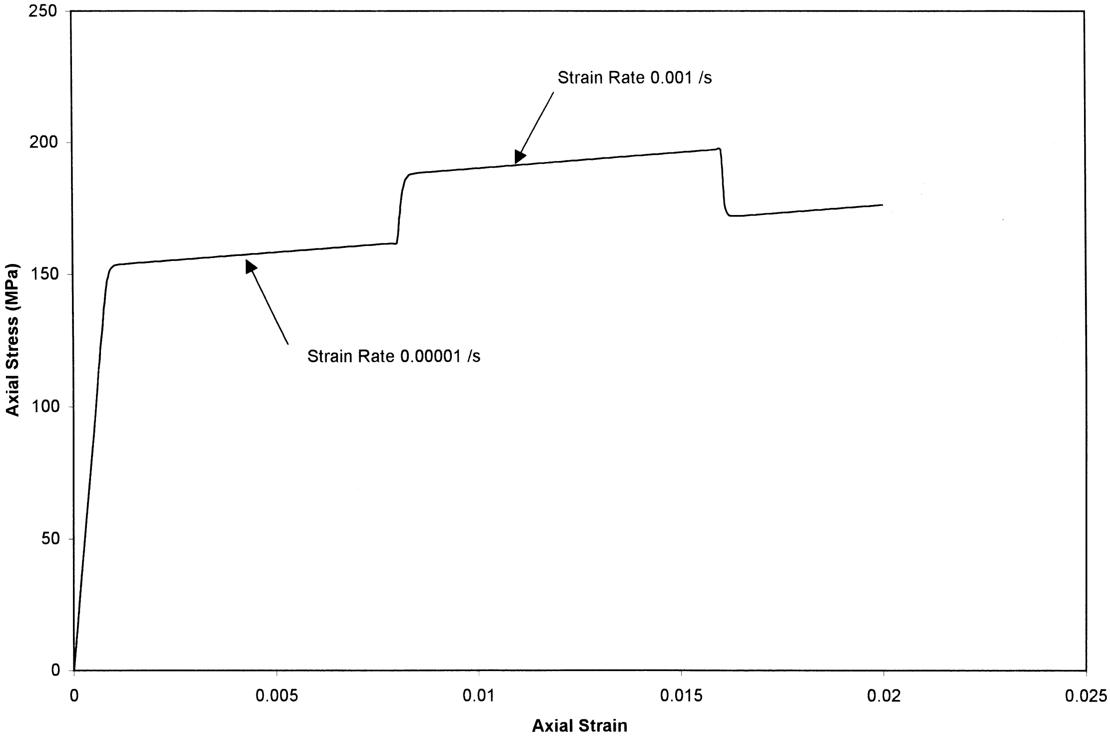

Fig. 5. Response curves under the loading condition of repeated strain rate change.

$$\left|\frac{\partial \mathbf{L}(t,t')}{\partial t'}\cdot \mathbf{E}_\alpha(t')\right| \leqslant \left\|\frac{\partial \mathbf{L}(t,t')}{\partial t'}\right\| \sup_{\theta\in(a,b)} \|\mathbf{E}(\theta)\|, \tag{3.42}$$

after using definition (3.28), and the obtained estimate is uniform on $[a,b]$ as well. Thus the Lebesgue dominated convergence theorem allows to take the limit for $\alpha \to +\infty$ inside the integral on the right-hand side of (3.37). Recalling (3.32) for $t < b$, we can write

$$\begin{aligned} g_\infty(t) &= \mathbf{L}(t,t)\cdot \mathbf{E}_\infty(t) - \mathbf{L}(t,a)\cdot \mathbf{E}_\infty(a) \\ &\quad - \int_a^t \frac{\partial \mathbf{L}(t,t')}{\partial t'}\cdot \mathbf{E}_\infty(t')\,\mathrm{d}t' \\ &= \left(\mathbf{L}(t,t) - \int_a^t \frac{\partial \mathbf{L}(t,t')}{\partial t'}\,\mathrm{d}t'\right)\cdot \mathbf{E}_\infty(t) \\ &\quad - \mathbf{L}(t,a)\cdot \mathbf{E}_\infty(a) \\ &= \left[\mathbf{L}(t,t) - \left(\mathbf{L}(t,t')\big|_a^t\right)\right]\cdot \mathbf{E}_\infty(t) \\ &\quad - \mathbf{L}(t,a)\cdot \mathbf{E}_\infty(a) = 0, \end{aligned} \tag{3.43}$$

which completes the proof of the theorem. □

It is worth to emphasize that the constitutive equation for the model discussed in the present paper exhibits rate-independent behavior under extremely retarded deformation processes.

4. Numerical simulation

In order to test the ability of the model to reproduce the experimental observations described in Section 2, a numerical simulation of the response of an axially loaded specimen is investigated.

In the calculations made by Haupt and Lion (1993), and Krempl (1987) the material constants of AISI304 stainless steel at room temperature were used. This is a material which is known to exhibit rate sensitivity even at room temperature.

The equations that characterize the class of constitutive models in the one-dimensional case

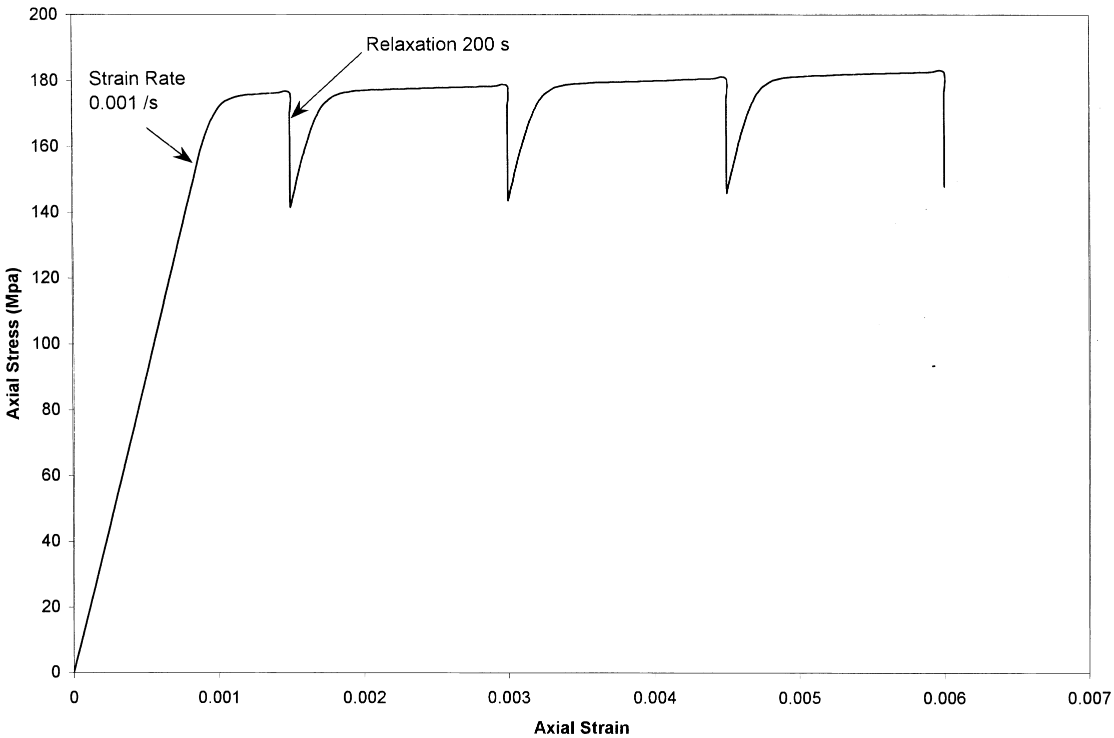

Fig. 6. Monotonic strain rate curve with relaxation.

can be obtained by particularization of (3.14), to get

$$\dot{\sigma} = \dot{\sigma}_\infty + \dot{\omega},$$
$$\dot{\omega} = G_0\dot{\varepsilon} - \dot{\sigma}_\infty - \frac{1}{\tau}g(\|\boldsymbol{\omega}\|)\boldsymbol{\omega}, \tag{4.1}$$

where σ, σ_∞, ω, ε are the scalar counterparts of $\mathbf{T}$, $\mathbf{T}_\infty$, $\boldsymbol{\Omega}$, $\mathbf{E}$ appearing in (3.14).

The form of the rate-independent functional σ_∞ used in the calculations is the elastoplastic constitutive model based on the Von Mises yield function with linear isotropic hardening; the growth equation for σ_∞ can be obtained by particularization of (3.9) and (3.10) to the one-dimensional case. The choice of the overstress function g may be done between functions used in the literature, that reproduce the experimental observations for stainless steels. The first function, proposed by Haupt and Lion (1993), is

$$g_1(\omega) = \exp\left(\frac{\omega}{\omega_0}\right), \tag{4.2}$$

whereas the second one, due to Krempl and Yao (1987), is

$$g_2(\omega) = \exp\left[R_1 \exp\left(\frac{\omega}{R_2}\right)\right], \tag{4.3}$$

where ω_0, R_1 and R_2 are material constants.

From the algorithmic point of view, backward Euler implicit integration scheme is used in order to achieve the response of the model to different loading conditions.

The obtained uniaxial stress–strain curves at different strain rates are summarized in Figs. 3 and 4 for different choices of the overstress function.

Fig. 5 shows the numerical simulation of a tensile test at strain rates ranging from 10^{-6} to 10^{-3} s^{-1}. In the test, the strain rate was instantaneously changed by three orders of magnitude at each instant in which multiples of 0.8% axial strain were attained.

A numerical experiment of repeated relaxation is shown in Fig. 6, where a monotonic loading with constant strain rate was interrupted several times

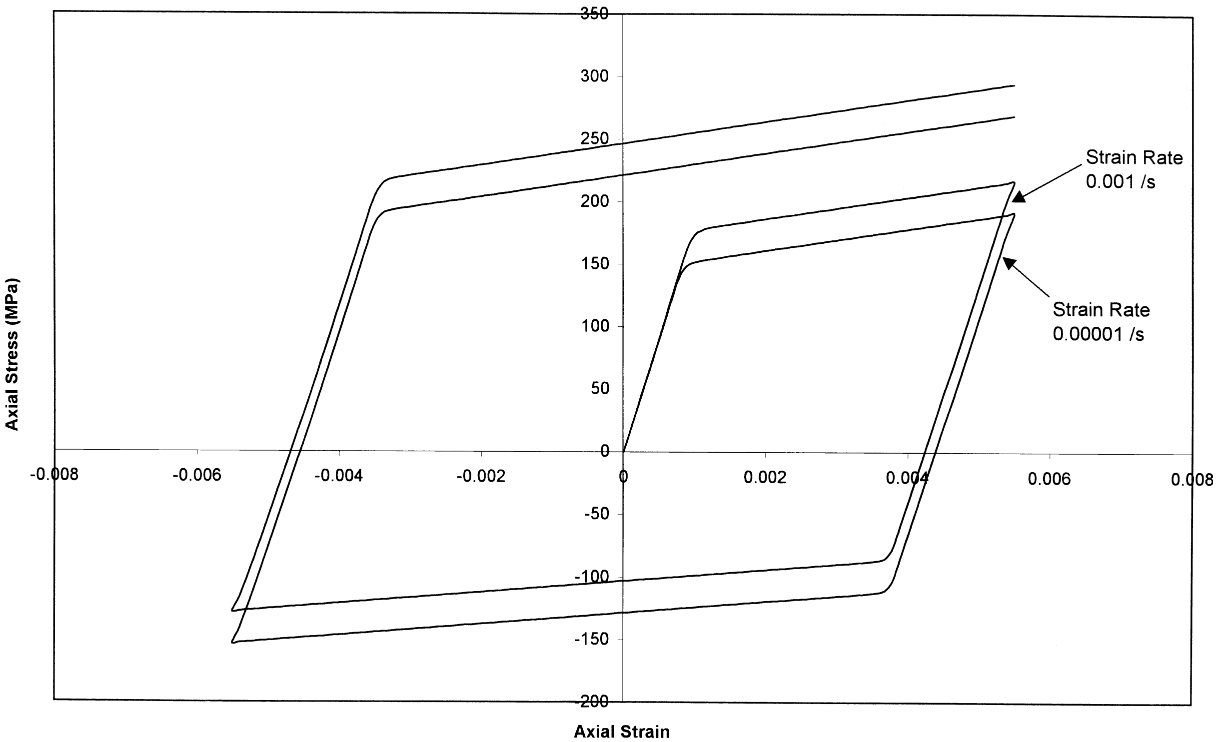

Fig. 7. Uniaxial response curve under cyclic loading at different strain rates.

by holding the reached value of the deformation for 200 s.

The response of a specimen to cyclic strain processes at different strain rates according to Eq. (4.1) is shown in Fig. 7; the test simulates a completely reversed cycling at ±0.5% axial strain.

5. Conclusions

In this paper, a class of viscoelastoplastic constitutive models, deduced from a thermodynamically consistent formulation is presented. In particular, the exploitation of a penalty version of the maximum dissipation principle leads to a class of non-linear viscoelastoplastic equations which contains the ones developed by Krempl and Yao (1987) on the one hand, and Haupt and Korzen (1987), Haupt and Lion (1993) among others. Unlike the model discussed in Haupt and Lion (1993), for the class of models derived in this paper the concept of intrinsic time developed by Valanis (1971) is not used.

History and rate dependencies are incorporated through the constitutive model by the concepts of equilibrium stress and overstress, respectively. In the previous sections, it is shown that either in the limiting cases of high viscosity or for extremely slow motions the constitutive model reduces to the one of the equilibrium stress as expected.

Further, a numerical analysis of the differential equations (3.14) describing the viscoelastoplastic behavior in the uniaxial case is investigated. The theoretical predictions obtained in this case turn out to well describe the most important effects of the variation of strain rate for stainless steels, such as abrupt changes during monotonic loading programs, monotonic repeated relaxations, and cyclic loading programs at different strain rates.

Acknowledgements

The authors wish to thank the Italian Ministero per l'Università e la Ricerca Scientifica – Progetto Cofin 98 – "Modelli Matematici per la Scienza dei Materiali" for the financial support.

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