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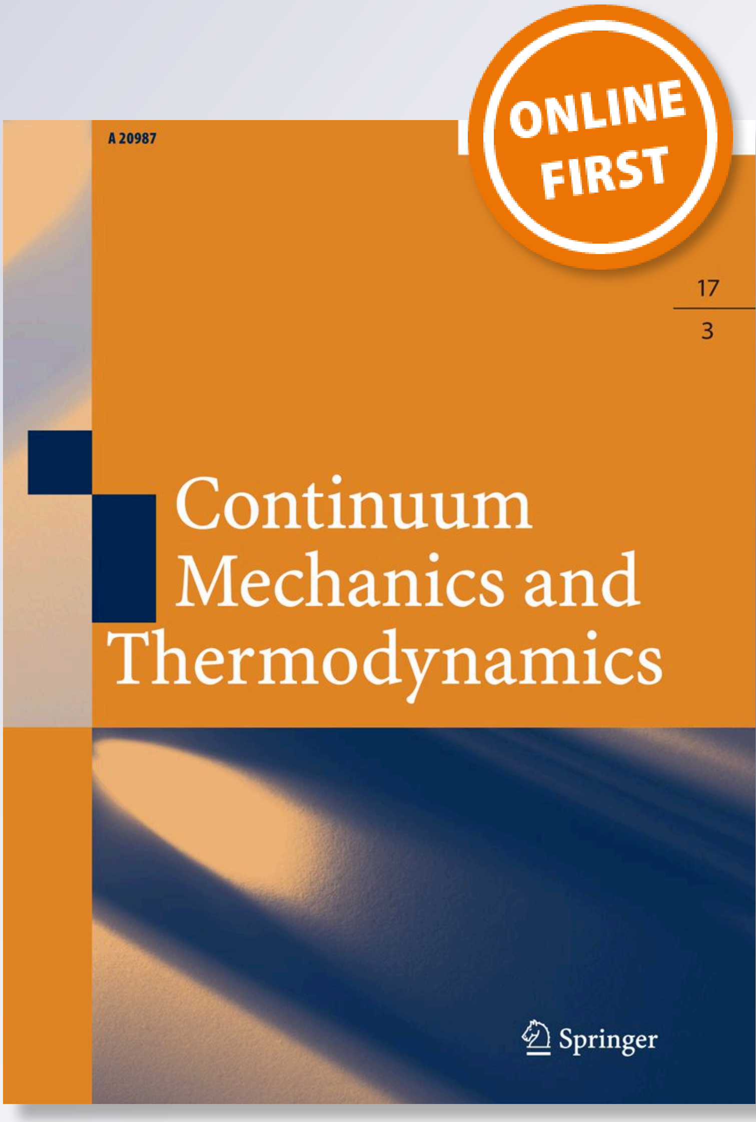

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ORIGINAL ARTICLE

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Moving interfaces that separate loose and compact phases of elastic aggregates: a mechanism for drastic reduction or increase in macroscopic deformation

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Abstract Granular materials such as sand may be viewed as continuous bodies composed of much smaller elastic bodies. The multiscale geometry of structured deformations captures the contribution at the macrolevel of the smooth deformation of each small body in the aggregate (deformation without disarrangements) as well as the contribution at the macrolevel of the non-smooth deformations such as slips and separations between the small bodies in the aggregate (deformation due to disarrangements). When the free energy response of the aggregate depends only upon the deformation without disarrangements, is isotropic, and possesses standard growth and semi-convexity properties, we establish (i) the existence of a compact phase in which every small elastic body deforms in the same way as the aggregate and, when the volume change of macroscopic deformation is sufficiently large, (ii) the existence of a loose phase in which every small elastic body expands and rotates to achieve a stress-free state with accompanying disarrangements in the aggregate. We show that a broad class of elastic aggregates can admit moving surfaces that transform material in the compact phase into the loose phase and vice versa and that such transformations entail drastic changes in the level of deformation of transforming material points.

Keywords Granular materials · Phase transitions · Elastic aggregates · Structured deformations

1 Introduction

The purpose of this article is threefold:

- to use multiscale geometry in the form of structured deformations [1] to provide a field theory for continuous bodies that submacroscopically may be thought of as aggregates of elastic bodies,
- to show for a broad class of elastic bodies and for the standard class of macroscopic deformations in those bodies the existence of a "compact phase" in which the aggregate of elastic bodies deforms as a single elastic body, and to show for each of a large subclass of macroscopic deformations the existence of a "loose phase" in which each small elastic body within the aggregate is free of stress and may deform differently from the aggregate,

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In honor of Gianpietro Del Piero, our teacher, colleague, and mentor, with admiration and enduring friendship.

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- to exhibit and describe moving interfaces that separate the loose and compact phases of the aggregate.

To illustrate in a concrete setting the main issues underlying our goals, we note that each of the pages of a paper-back book may be thought of as a thin elastic body, while the bound aggregate of pages may be considered as a continuous body with properties different from those of a single page. The compact phase of the aggregate in this case can be achieved by applying equal and opposing compressive forces to the opposite covers of the book: each page of the book undergoes a compressive stress and is flattened slightly; moreover, the compressive stress prevents small spaces from forming between the pages and provides frictional forces between the pages that prevent the pages from sliding relative to each other. In the compact phase, the aggregate of pages is rather stiff and resists bending and shearing. The loose phase of the aggregate can be attained from the compact phase by hanging the book from the bound edges of the pages and removing the compressive forces on the covers. Each page of the book no longer is compressed, and the lower edges of the hanging pages spontaneously separate slightly from each other. In this loose phase, the aggregate of pages is less stiff and less resistent to bending and shearing, because the frictional forces between adjacent pages are absent, and substantial changes in the shape of the aggregate can arise under the action of relatively small applied forces. The act of bending the loose aggregate of pages gives rise to compressive forces that can restore the frictional forces between pages and cause subcollections of pages to return to the compact phase. Reversing the bending of the aggregate can reverse the transformations. The aggregate in the compact phase viewed under normal light appears nearly purely white in color, while in the loose phase the aggregate of pages appears gray in color. The transformations between the loose and compact phases that arise upon bending the loose aggregate are quite vivid: white regions of pages in the compact phase appear and disappear within the gray aggregate of pages in the loose phase, and the gray and white regions are separated by interfaces that move as the amount of bending changes.[1]

Additional examples of elastic aggregates that appear in both loose and compact phases are powdered or granular materials such as talc, breakfast cereal, sand, packaging aggregates, and "powder snow." In these materials, the behavior in the loose phase can include fluid-like behavior in which drastic changes in shape and large displacements occur. For example, a pile of sand can separate into a compact region at rest and a loose region of flowing sand, as the steepness of the pile exceeds a critical value. Similarly, breakfast cereal in a container can flow from the opening of the container through a narrow band that is adjacent to regions in which no flow occurs. The ubiquity and usefulness of such aggregates in everyday life has generated over the decades a wide-ranging, diverse literature on experimental observations and theoretical descriptions of specific materials within this broad class (see, for example, the lecture notes [2] for a variety of approaches to mathematical modeling of granular materials as well as the references cited later in this introduction).

The theoretical tools for modeling such aggregates include continuum mechanics, statistical mechanics, discrete mechanics, and computational mechanics. Here, we focus on using structured deformations of continua [1] via the continuum field theory [3] of elastic bodies undergoing disarrangements (non-smooth submacroscopic geometrical changes such as slips and void formation) to identify a broad class of elastic bodies that exhibit both loose and compact phases. The process of identifying elastic bodies with both loose and compact phases begins in Sect. 2 where the field theory [3] is summarized briefly. In this theory, the multiscale geometrical changes of a body are described by pairs (g, G) called structured deformations in which g is the macroscopic deformation field and G is a tensor field that measures the contribution to macroscopic deformation arising from smooth, submacroscopic geometrical changes. In order for the macroscopic geometrical changes to provide enough room to accommodate the submacroscopic changes associated with G, the pair (g, G) is required to satisfy the Accommodation Inequality $0 < \det G \leq \det \nabla g$, that is, the volume changes associated with the deformation without disarrangements cannot exceed the macroscopic volume changes. As a consequence of these requirements, one can prove that the tensor field $M = \nabla g - G$ represents the contributions to the macroscopic deformation arising from disarrangements [1]. In the field theory for elastic bodies undergoing disarrangements [3], the Helmholtz free energy response function depends upon both the deformation without disarrangements G *and* the deformation due to disarrangements M, and this field theory

[1] Typically, in addition to the small spaces between pages that appear in the loose phase, large gaps between pages can form during deformation of the aggregate of pages. It is not our intention to take into account explicitly in the present analysis such large gaps or to predict their formation, but the approach that we take here can be broadened to address this phenomenon. More significantly, we shall restrict our attention in the following analysis to elastic aggregates that behave in the compact phase as an isotropic elastic body. Because the aggregate of pages of a book in the compact phase has the symmetry of a transversely isotropic body, it does not fall within the scope of our analysis. Nevertheless, the observed features of the aggregate of pages described in this paragraph arise naturally in the course of our analysis and provide concrete intuition and motivation for questions that we raise and study.

includes the standard field theory of nonlinear elasticity in which the free energy depends upon the single variable $\nabla g = G + M$.

In Sect. 3, the field theory in Sect. 2 is specialized to the case of "purely dissipative disarrangements" in which non-smooth submacroscopic geometrical changes do not affect the energy stored in the body. This specialization leads to a class of free energy response functions, each of which depends only upon the single tensor variable G, the deformation without disarrangements, and which can be ascribed growth and semi-convexity properties known to be significant from the vast literature on nonlinear elasticity. The assumed growth and smoothness properties of Ψ imply that it attains a minimum $\Psi_{\min}$ on the set of tensors G with positive determinant. We further require in Sect. 3 that the free energy response be isotropic, and this additional requirement implies that there is a purely dilatational deformation $G = \zeta_{\min} I$ at which the minimum $\Psi_{\min}$ is attained, so that $D\Psi(\zeta_{\min} I) = 0$. Consequently, every structured deformation of the form $(g, \zeta_{\min} Q)$, with Q an orthogonal-valued tensor field, has vanishing stress field: $S = D\Psi(\zeta_{\min} I) = 0$ and minimizes the free energy. The Accommodation Inequality requires that the structured deformation $(g, \zeta_{\min} Q)$ satisfy $\zeta_{\min}^3 \leq \det \nabla g$, and we conclude in Sect. 3 that: *the given body $\mathcal{B}$ attains a stress-free configuration $g(\mathcal{B})$ with minimum free energy for every macroscopic deformation field g whose gradient has determinant no less than $\zeta_{\min}^3$*. We use the term "dilatationally cohesionless" to describe such a body, because sufficiently large macroscopic volume changes allow the body to attain stress-free configurations, no matter what the shape of the body.

Section 4 begins with an application of the Approximation Theorem for structured deformations [1] to the structured deformations $(g, \zeta_{\min} Q)$ satisfying $\zeta_{\min}^3 \leq \det \nabla g$, and we conclude that a dilatationally cohesionless body that undergoes $(g, \zeta_{\min} Q)$ can be viewed (approximately but to within any desired accuracy) as an aggregate of small elastic bodies, each undergoing a dilatation that leaves it stress free, along with a rotation and translation that avoids interpenetration with other small bodies in the aggregate. We use the term "loose phase associated with g" to describe the aggregate in the configuration associated with the structured deformation $(g, \zeta_{\min} Q)$. We point out in Sect. 4 that an equilibrium configuration in which the aggregate is in the loose phase is submacroscopically stable in the sense of [4]. Thus, *the loose phase can be attained if the macroscopic deformation g allows enough room for each small body in the aggregate to relax to a stress-free configuration and, once the loose phase and equilibrium are attained, the aggregate prefers not to undergo further submacroscopic changes.*

In contrast to the loose phase associated with g when $\zeta_{\min}^3 \leq \det \nabla g$, we may consider for each macroscopic deformation g, without any restriction on its volume change $\det \nabla g$, the "classical deformation" $(g, \nabla g)$. For this special structured deformation, there holds $G = \nabla g$, so that $M = 0$ and no disarrangements occur. In particular, no voids are formed and no slips occur, and we may view the classical deformation $(g, \nabla g)$ as producing geometrical changes in which each small body in the aggregate deforms in exactly the same way as the aggregate, itself. Consequently, we use the term "compact phase associated with g" to describe the aggregate in the configuration associated with the structured deformation $(g, \nabla g)$.

The compact and loose phases just described are instances of "solid-like" and "fluid-like" states of granular matter discussed in the literature. Khakhar [5] surveys a number of continuum models of granular materials undergoing processes that entail solid-like states and fluid-like states, either in isolation or in coexistence. For example, steady flow of a granular material in a circular cylinder rotated slowly about its axis entails at each instant a solid-like region undergoing a rigid-body rotation and a fluid-like region undergoing a nearly uni-directional flow with particles moving continuously from one region to the other. Employing the widely used discrete element method of computational mechanics [6], Souroush and Ferdowsi [7] simulate a granular material undergoing cyclical, triaxial deformation processes at constant volume and at strain rates slow enough that the aggregate of particles has solid-like behavior over the initial cycles of the processes. The simulations show that "fluidization" occurs after relatively few cycles, in that the deviatoric stress vanishes throughout subsequent cycles. In statistical mechanical simulations of a dense sheared granular fluid [8], "crystallization" occurs in an aggregate of smooth inelastic hard disks through the formation of a central region of particles that are triangularly packed. This solid-like region is bordered above and below by fluid-like regions undergoing nearly uniform shearing and exchanging particles with the solid region. The coexistent solid-like and fluid-like phases arising in the above simulations of granular aggregates are reflected in recent experiments in a parallel plate shear cell designed for the study of large-scale shear flows [9]. These observed flows reveal that the process of cyclical shearing of spherical particles results in the gradual appearance of a layered structure in the granular medium that eventually crystallizes into hexagonal-close-packed layers forming three-dimensional face-centered-cubic crystals. The availability of both compact and loose phases in the dilatationally cohesionless elastic bodies introduced here provides a new continuum framework for the theoretical study of the interplay between solid-like and fluid-like phases arising in simulations and experiments in the current literature.

We view the concepts and results described in Sects. 2–4 as having accomplished the first two of the three goals set forth at the beginning of this Introduction, and we devote the remainder of the paper to providing a rather broad material context and two specific geometrical contexts (of possibly many) in which the remaining goal is accomplished. The material context is that of dilatationally cohesionless bodies described at the end of Sect. 3. In Sects. 5–10, two different geometrical contexts are described and studied, both of which exclude the possibility that a shock wave forms in either phase. The geometrical context introduced in Sect. 5 requires that each of the motions g_c and g_ℓ of the compact and loose phases represents a time-independent, homogeneous change in shape of the body followed by a translational motion with constant acceleration, so that all of the material points in the compact phase move together as a pre-deformed, rigid body, and all of the points in the loose phase move as a second rigid body. The specific forms chosen for g_c and g_ℓ in Sect. 5 imply that the deformation gradients ∇g_c and ∇g_ℓ differ by a rank-one tensor $(\xi_c - \xi_\ell) Fa \otimes n$. Here, ξ_c and ξ_ℓ are scalar measures of macroscopic deformation in the compact and loose phases, F is a given tensor with positive determinant, and a and n are given unit vectors.

Our goals in Sects. 6–9 are

- to provide a description of the geometry and motion of interfaces in the aggregate that can separate the loose and compact phases moving according to the motions g_c and g_ℓ introduced in Sect. 5, and
- to relate to one another the deformation gradients ∇g_c, ∇g_ℓ and the velocities $\dot{g}_c$, $\dot{g}_\ell$ that appear on the two sides of such interfaces.

We examine interfaces that are space-time hypersurfaces on which g_c and g_ℓ are equal, but on which ∇g_c and ∇g_ℓ may differ and on which $\dot{g}_c$ and $\dot{g}_\ell$ may differ. Although we shall analyze the motions of the loose and compact phases and interfaces separating them at a fixed temperature θ, this analysis requires knowledge of properties of the free energy response on a small interval of temperatures containing θ along with related thermodynamical considerations. In Sect. 6 we provide the forms of the First and Second Laws used in our subsequent analysis as well as the additional constitutive relations required to broaden appropriately the field theory [3] so as to cover temperatures other than the temperature θ. Within this broader context we show in Sect. 6 that the laws of thermodynamics are satisfied in the motions g_c and g_ℓ defined in Sect. 5 when the temperature field is constant. In Sect. 7 we exploit the assumed semi-convexity and growth properties of the free energy to deduce monotonicity and growth properties of the component of the stress field in the compact phase associated with $Fa \otimes n$, the diad describing (to within a scalar factor) the difference $\nabla g_c - \nabla g_\ell$.

Section 8 contains the core of our analysis of moving interfaces separating compact and loose phases of elastic aggregates. We describe the phase interface as a space-time hypersurface and provide in Sect. 8.1 the interfacial jump conditions corresponding to the balance of linear momentum, the First Law of Thermodynamics, and the Second Law of Thermodynamics by writing each relation in a space-time divergence form and citing a standard argument employing a space-time region that contains the interface in its interior. The jump condition corresponding to the Second Law is an inequality that amounts to the assertion that the jump in entropy density experienced as a material point changes from one phase to the other is non-negative, while the jump conditions corresponding to the balance of linear momentum and the First Law are relations with equality. These three jump conditions together with the requirement that g_c and g_ℓ agree on a phase interface are the relations that restrict the form of the phase interface and the values of the velocities and deformation gradients in the contiguous compact and loose phases. In Sect. 8.2 we show that, given the tensor F, the two unit vectors a and n, and the positive number θ, the assumption that the phase interface is planar along with the relations obtained in Sect. 8.1 allow one to determine (i) $\hat{t}(X)$, the time at which the phase interface passes through a given point X in the reference configuration, (ii) $[\dot{g}]\,(X, \hat{t}(X))$, the jump in velocity across the interface at that point, (iii) ξ_ℓ, the scalar deformation parameter for the loose phase, and (iv) Ξ, the speed of the phase interface, all four as functions of F, a, n, θ, and of ξ_c, the corresponding scalar deformation parameter for the compact phase. Moreover, the jump conditions imply that the internal energy density in the compact phase is no less than that in the loose phase.

Section 8.3 is devoted to the study of three inequalities that arise in the analysis up to this point: (a) the condition that the entropy cannot decrease as a given material point undergoes the phase transition, (b) the Accommodation Inequality in the loose phase, and (c) the condition that the internal energy density in the compact phase is no less than that in the contiguous loose phase, derived in Sect. 8.2. For given F, a, n, and θ, the three inequalities lead to sufficient conditions to be satisfied by the parameter ξ_c in order that a moving interface separating the two phases be present. For each of the three different types of transitions, (1) "loose-to-compact," (2) "compact-to-loose," and (3) "reversible," we determine in Sect. 8.3 sufficient conditions on

ξ_c in order that the inequalities (a)–(c) are satisfied. The results in Sect. 8.3 suggest that loose-to-compact transitions are available under a broader range of conditions than compact-to-loose transitions which, in turn, are more broadly available than reversible transitions.

For the case of reversible transitions, we construct in Sect. 8.4 an example of a motion of an elastic aggregate that admits an infinite succession of reversible transitions between the loose and compact phases of the aggregate. At regular times, planar phase interfaces travel from one end of a rectangular slab (the reference configuration) to the opposite end, with a new interface starting at the first end as the previous one reaches the second end. The deformation gradient and the velocity within each phase are constant, and a schematic figure showing trajectories of points in the opposing ends of the slab is presented in Sect. 8.4 for the case when the velocity in the compact phase is zero.

The relation (64) in Sect. 8.2 that determines the deformation parameter ξ_ℓ as a function of ξ_c, along with the monotonicity properties of the component of stress associated with $Fa \otimes n$ established in Sect. 7, underlies the analysis of compact-to-loose and loose-to-compact transitions given in Sect. 8. In particular, the behavior of the ratio of the jump in internal energy to that stress component as the deformation parameter ξ_c tends to ξ_0, the point where the stress component vanishes, determines the behavior of ξ_ℓ as ξ_c tends to ξ_0. In Sect. 9 we highlight the essential feature of that limiting behavior exploited in Sect. 8: $|\xi_\ell|$ tends to ∞ as ξ_c tends to ξ_0. This feature tells us that the transitions, whose existence are established in Sect. 8, can drastically increase the deformation for the compact-to-loose transition and drastically reduce the deformation for the loose-to-compact transition.

A second geometrical context for the study of moving phase interfaces is provided in Sect. 10, where the class of homogeneous deformations for the compact phase studied in Sects. 5–9 is broadened to admit non-homogeneous deformations in the form of plane progressive waves with small associated strains superposed on a homogeneous, background state of deformation, while the restriction to homogeneous deformations in the loose phase is maintained. Accordingly, satisfaction of the balance laws in the compact phase and satisfaction of the jump conditions on phase interfaces is imposed in an approximate sense, consistent with the context of small strains. Moreover, in order to simplify and make more explicit the analysis in Sect. 10, we assume that the phase boundary and the progressive wave in the compact phase have the same orientation. Our analysis provides sufficient conditions on the temperature and on the background strain, orientation, and direction of the progressive wave for the existence of a moving phase boundary.

In Sect. 11, we illustrate the results obtained in Sects. 8–10 when the free energy function has the special form

$$\Psi(G,\theta) = \frac{1}{2}\alpha(\theta)(\det G)^{-2} + \frac{1}{2}\beta(\theta)G\cdot G,$$

with $\alpha(\theta)$ and $\beta(\theta)$ temperature-dependent elastic modulii. This class of free energies has appeared in various studies of nonlinear elastic behavior (see, for example, [10], Section 4.10), because of its simple analytic form and its growth and semi-convexity properties. Moreover, this class of free energy functions satisfies the requirements we make for elastic aggregates and so falls within the present theory. In Sect. 11.2 we specialize the sufficient conditions for the existence of moving phase interfaces obtained in Sects. 8 and 9 (where both phases to within a translation undergo time-independent homogeneous deformations) to such free energy functions and to particular choices of the parameters F, a, and n compatible with a simple shear in the compact phase. In Sect. 11.3, we carry out an analogous specialization of the results in Sect. 10 (where the compact phase can undergo non-homogeneous, time-dependent deformations in the form of small-strain, progressive waves) to this class of free energy functions, to a particular family of background deformations in the compact phase, and to the case where the waves in the compact phase are shear waves.

In the results obtained in Sect. 11, the inequality $\beta(\theta) \le \alpha(\theta)$ relating the two elastic modulii plays a central role: this inequality is a necessary condition for the presence of a progressive shear wave in the compact phase that is adjacent to the homogeneously deformed loose phase, while the opposite inequality is a necessary condition for the presence of a static, homogeneous deformation in the compact phase that is adjacent to the homogeneously deformed loose phase. Moreover, when $\beta(\theta) \le \alpha(\theta)$, so that a progressive shear wave may be present in the compact phase, the deformation in the loose phase necessarily is a uniaxial extension, while for the opposite inequality the deformation in the loose phase is a simple shear. Thus, the relative stiffness of each piece of the aggregate in distortion versus that in dilatation determines whether the loose phase may undergo a simple shear or an extension as a phase interfaces moves through the body.

One notable feature of our analysis is the absence of a "kinetic relation," a constitutive assumption that relates the speed of a moving interface to the states of the phases separated by the interface (see [11] for background, motivation, and detailed analysis of the role played by kinetic relations in studies of wave propagation and phase changes in solids). From the point of view taken in [11], the structure of a "kinetic relation" rests on the notion of "driving force" that is identified by means of a "work–energy relation," formulated for an arbitrary subregion of a body containing a surface across which discontinuities in the basic continuum fields appear. A kinetic relation relates the driving force to the speed of a moving interface and, when the stress-deformation relation is not monotone, eventually selects from a number of alternatives a particular deformation that can be present at a given level of stress. We note that the jump condition associated with the First Law of Thermodynamics in our study is a counterpart of the "work–energy relation" that identifies the driving force in [11].

There appear to be two features in our analysis in Sects. 5–9 that obviate the need for a kinetic relation:

- the only stress component in the compact phase that enters into our analysis is associated with a spatially constant diad (measuring the direction of the jump in deformation gradient), and the assumed rank-one convexity of the free energy implies that this stress component is a monotone function of a corresponding scalar measure of deformation;
- the Accommodation Inequality in the loose phase actually restricts the level of deformation in the compact phase to an interval close to a point where the above stress component vanishes; consequently, monotonicity of the stress component as a function of the measure of deformation in the compact phase only is needed when the stress component is small.

2 Field relations for elastic bodies undergoing disarrangements

According to the field theory formulated for elastic bodies undergoing disarrangements [3] in the context of structured deformations (g, G), the macroscopic deformation g, the deformation without disarrangements G, and the deformation due to disarrangements $M = \nabla g - G$ satisfy the following relations:

$$\rho_0\, \ddot{g} = \mathrm{div}(D_G\Psi + D_M\Psi) + b, \tag{1}$$
$$D_G\Psi\ M^T + D_M\Psi\ (\nabla g)^T = 0, \tag{2}$$
$$sk(D_G\Psi\ M^T + D_M\Psi\ G^T) = 0, \tag{3}$$
$$D_G\Psi \cdot \dot{M} + D_M\Psi \cdot \dot{G} \geq 0, \tag{4}$$
$$0 < \det G \leq \det \nabla g. \tag{5}$$

All of the fields in these relations are defined on pairs (X, t) with X in a given reference configuration for the body and with t in a given open interval. The relations are formulated in the special context of isothermal deformations; accordingly, the temperature field is assumed to be constant in space and time. We assume also that ρ_0, the density in the reference configuration, also is a constant. The vector field b in (1) is the body force measured per unit volume in the reference configuration. The Helmholtz free energy function $(G, M) \mapsto \Psi(G, M)$ determines the free energy per unit volume in the reference configuration ψ through the constitutive relation

$$\psi(X, t) = \Psi(G(X, t), M(X, t)) \tag{6}$$

for all points X in the reference configuration and times t of interest. The sum of partial derivatives $D_G\Psi + D_M\Psi$ turns out to be the Piola–Kirchhoff stress field S in the field theory [3],

$$S = D_G\Psi + D_M\Psi, \tag{7}$$

and the relation (1) is the local form of the balance of linear momentum in the reference configuration. Because the Piola Kirchhoff stress S not only can be decomposed additively as $D_G\Psi + D_M\Psi$ but also multiplicatively as $D_G\Psi\ K^T$, with $K := (\nabla g)^{-1} G$ (see [3] for details), the consistency relation (2) must hold throughout the reference configuration at all times. The additive decompositions $S = D_G\Psi + D_M\Psi$ and $\nabla g = G + M$ imply that the rate of dissipation $S \cdot \nabla \dot{g} - \dot{\psi}$ per unit volume in the reference configuration is given by $D_G\Psi \cdot \dot{M} + D_M\Psi \cdot \dot{G}$, and the relations (3) and (4) express the requirement that the rate of dissipation be frame-indifferent and non-negative. The final field relation is the Accommodation Inequality [1,12] that expresses the requirement that disarrangements not result in the interpenetration of matter.

3 Dilatationally cohesionless elastic materials

Suppose that the Helmholtz free energy ψ for a structured deformation (g, G) obeys the special constitutive relation

$$\psi(X, t) = \Psi(G(X, t)). \tag{8}$$

The relation (8) is the statement that Helmholtz free energy (per unit volume in the reference configuration) associated with the structured deformation (g, G) does not depend upon the disarrangement tensor $M = \nabla g - G$, so that, while disarrangements can contribute to the dissipation occurring in the material, those disarrangements cannot contribute to the free energy in the material. Accordingly, we may call the disarrangements in such a material *purely dissipative*.

Because (8) implies that $D_M \Psi = 0$, the field relations for a general elastic body reduce in the case of purely dissipative disarrangements to the relations

$$\rho_0 \ddot{g} = \operatorname{div}(D_G \Psi) + b, \tag{9}$$

$$D_G \Psi \; M^T = 0, \tag{10}$$

$$sk(D_G \Psi \; M^T) = 0, \tag{11}$$

$$D\Psi_G \cdot \dot{M} \geq 0, \tag{12}$$

$$0 < \det G \leq \det \nabla g. \tag{13}$$

We note that in the present context of purely dissipative disarrangments (8) the consistency relation (10) implies the frame indifference relation (11), and we may omit the latter from the list of requirements of the field theory in what follows. Moreover, the stress relation (7) reduces to

$$S = D_G \Psi. \tag{14}$$

If the material is isotropic, then

$$\Psi(GQ) = \Psi(G) \quad \text{for all } Q \in \text{Orth}^+, \tag{15}$$

where Orth^+ denotes the proper orthogonal group for the underlying Euclidean space $\mathcal{E}$ and its translation space $\mathcal{V}$. The polar decomposition $G = V_G R_G$ permits one to write (8) in the form (without displaying all arguments):

$$\psi = \Psi(V_G). \tag{16}$$

We assume further that the response function Ψ is smooth and satisfies the two limit conditions

$$\lim_{\zeta \longrightarrow 0+} \Psi(\zeta I) = \lim_{\zeta \longrightarrow +\infty} \Psi(\zeta I) = +\infty \tag{17}$$

These conditions state that under extreme contraction and under extreme expansion the stored energy increases without bound. Because Ψ is smooth, the limit conditions (17) tell us that there exists $\zeta_{\min} > 0$ such that $\Psi(\zeta_{\min} I) = \min_{\zeta > 0} \Psi(\zeta I)$, and consequently

$$0 = \frac{\mathrm{d}}{\mathrm{d}\zeta} \Psi(\zeta I) \mid_{c=\zeta_{\min}} = D_G \Psi(\zeta_{\min} I) \cdot I$$

that is, $\operatorname{tr} D\Psi(\zeta_{\min} I) = 0$. Because the response of the material is assumed to be isotropic, the frame indifference of the response implies

$$D_G \Psi(Q V Q^T) = Q D_G \Psi(V) Q^T \quad \text{for all } Q \in \text{Orth and } V \in Sym^+, \tag{18}$$

where Orth denotes the orthogonal group, and Sym^+ denotes the positive definite tensors on $\mathcal{V}$. Applying (18) when $V = \zeta_{\min} I$ we conclude that

$$D_G \Psi(\zeta_{\min} I) = Q D_G \Psi(\zeta_{\min} I) Q^T \quad \text{for all } Q \in \text{Orth}$$

and, therefore, that there exists $\widetilde{p} \in \mathbb{R}$ satisfying

$$D_G \Psi(\zeta_{\min} I) = \widetilde{p} I.$$

The conclusion tr $D\Psi(\zeta_{\min} I) = 0$ obtained above yields $\widetilde{p} = 0$ and therefore, according to the stress relation (14),

$$S = D_G \Psi(\zeta_{\min} I) = 0. \tag{19}$$

We observe here that every classical deformation $(g, \nabla g)$ satisfies the consistency relation (10) (since in this case $M = 0$). Moreover, every structured deformation (g, G) for which the stress response $S = D_G \Psi(G)$ is zero also satisfies the consistency relation. Furthermore, the Accommodation Inequality (5) is satisfied with equality by every classical deformation; it is satisfied by a structured deformation of the form $(g, \zeta_{\min} I)$, or more generally of the form $(g, \zeta_{\min} Q)$ with $Q \in \mathrm{Orth}^+$, if and only if

$$\zeta_{\min}^3 \leq \det \nabla g. \tag{20}$$

We may now state the following remark:

Remark 1 Let the Helmholtz free energy Ψ and the positive number $\zeta_{\min}$ satisfy not only the conditions (8), (15), and (17) above but also the following strengthened version of $\Psi(\zeta_{\min} I) = \min_{\varsigma > 0} \Psi(\varsigma I)$:

$$\Psi(\zeta_{\min} I) = \min_{\zeta > 0} \Psi(\zeta I) = \min_{G \in \mathrm{Lin}^+} \Psi(G), \tag{21}$$

with Lin^+ the group of linear mappings on $\mathcal{V}$ with positive determinant. Then for each macroscopic deformation g satisfying (20) and each $Q \in \mathrm{Orth}^+$ the structured deformation $(g, \zeta_{\min} Q)$ provides a stress-free configuration for the body in which the free energy of the body is a minimum. In other words, if the free energy attains a minimum at the tensor $\zeta_{\min} I$, then for sufficiently large volume changes $\det \nabla g$ associated with the macroscopic deformation g, the stress-free, final configuration associated with $(g, \zeta_{\min} Q)$ (with Q any proper orthogonal tensor) is geometrically admissible in the body and minimizes the free energy. In addition, if strict inequality holds in (20), then the stress-free configuration associated with $(g, \zeta_{\min} Q)$ does not correspond to a classical deformation: $\nabla g \neq \zeta_{\min} Q$.

We note that the orthogonal tensor Q above can be replaced by any orthogonal valued tensor field on the body and the same conclusions hold. Therefore, when the body force is zero and g is independent of time, an equilibrium configuration can be achieved via a macroscopic deformation g satisfying (20) together with a time-independent "texturing" dilatational field $G = \zeta_{\min} Q$, because all of the field relations are then satisfied. As indicated in Remark 1, these equilibria need not be classical deformations. In fact, $(g, \zeta_{\min} Q)$ is a classical deformation if and only if $\nabla g = \zeta_{\min} Q$, which by Euler's Theorem amounts to the requirement that Q be a constant field equal to $\frac{1}{\zeta_{\min}} \nabla g$, so that g, itself, is a scalar multiple of a rotation.

It is important to note also that the condition (21) follows from the weaker condition $\Psi(\zeta_{\min} I) = \min_{\zeta > 0} \Psi(\zeta I)$ provided that the free energy response Ψ satisfies standard regularity and growth conditions known to be significant in the solution of variational problems associated with equilibrium configurations of isotropic elastic bodies (see [13], Theorem 1). Specifically, the condition

$$\min_{G \in \mathrm{Lin}^+} \Psi(G) = \min_{\zeta > 0} \Psi(\zeta I) \tag{22}$$

is satisfied if the function Ψ is isotropic, rank-one convex, and has the following growth properties:

$$\lim_{\det G \longrightarrow 0} \Psi(G) = \lim_{|G| \longrightarrow \infty} \Psi(G) = \infty. \tag{23}$$

These conditions on Ψ provide a broad and well-studied class of free energy response functions for which the conclusions of Remark 1 are valid.

We call an elastic material *dilatationally cohesionless* relative to the positive number $\zeta_{\min}$ if its free energy response function satisfies the conditions of purely dissipative disarrangements (8), isotropy (15), growth (17), and coincidence of minimizers (21) that form the hypotheses of the previous remark. Our use of the term "dilatationally cohesionless" underscores the fact that all macroscopic deformations g that cause sufficiently large expansion of the material are realizable in stress-free equilibria of the form $(g, \zeta_{\min} Q)$ with $\det \nabla g \geq \zeta_{\min}^3$ and with Q any proper orthogonal-valued field. The class of dilatationally cohesionless elastic materials is intended to describe materials such as, for example, dry sand, powdered snow, dry cereals and grains, and packaging materials.

4 Compact and loose phases of elastic aggregates

We suppose that a body composed of a dilatationally cohesionless elastic material relative to $\zeta_{\min}$ undergoes a macroscopic deformation g with $\det \nabla g \geq \zeta_{\min}^3$. Not only is the deformed configuration associated with the classical deformation $(g, \nabla g)$ geometrically admissible in the body, but so also is the deformed configuration associated with the structured deformation $(g, \zeta_{\min} Q)$ with Q any proper orthogonal-valued field. The appearance of g in the first entry in both pairs signifies that the macroscopic deformations associated with the pairs are one and the same. In attaining the deformed configuration associated with the classical deformation $(g, \nabla g)$, the body undergoes no disarrangements and, in particular, no voids have been formed. Moreover, the stress $S = D_G \Psi(\nabla g)$ experienced in that configuration need not be zero. In order for the body to attain the deformed configuration associated with $(g, \zeta_{\min} Q)$ when $\nabla g \neq \zeta_{\min} Q$, disarrangements must occur in the body, voids must be created when $\det \nabla g > \zeta_{\min}^3$, and the stress $D_G \Psi(g, \zeta_{\min} Q)$ in that configuration must vanish. For a given macroscopic deformation g satisfying $\det \nabla g \geq \zeta_{\min}^3$, we use the term *the compact phase for g* to describe the body in the deformed configuration associated with the classical deformation $(g, \nabla g)$ and the term *the loose phase for g* to describe the body in the deformed configuration associated with the generally non-classical structured deformation $(g, \zeta_{\min} Q)$.

Suppose alternatively that the macroscopic deformation g satisfies $\det \nabla g < \zeta_{\min}^3$. The pair $(g, \nabla g)$ is again a structured deformation in which the body undergoes no disarrangements and may experience non-zero stresses, and we use as above the term *compact phase for g* to describe the body in the deformed configuration associated with $(g, \nabla g)$. In contrast, the pairs $(g, \zeta_{\min} Q)$ in this case are not structured deformations, because the Accommodation Inequality (5) is violated, and we describe this situation by saying that, for the given macroscopic deformation g, the loose phase is not geometrically admissible in the body. In what follows, when we speak of the loose phase for a macroscopic deformation g, it is understood that g satisfies the inequality $\det \nabla g \geq \zeta_{\min}^3$.

We note that the Approximation Theorem for structured deformations [1] permits us to associate with each structured deformation (g, G) a sequence of injective, piecewise smooth deformations $n \mapsto f_n$ satisfying $\lim_{n \longrightarrow \infty} f_n = g$ and $\lim_{n \longrightarrow \infty} \nabla f_n = G$. In the case $\det \nabla g \geq \zeta_{\min}^3$, we apply this result to the loose phase for g, that is, to the structured deformation $(g, \zeta_{\min} Q)$ with Q any proper orthogonal-valued field, to obtain a sequence $n \mapsto f_n$ satisfying $\lim_{n \longrightarrow \infty} f_n = g$ and $\lim_{n \longrightarrow \infty} \nabla f_n = \zeta_{\min} Q$. Each injective, piecewise smooth deformation f_n in such a sequence provides an approximate view of the complex submacroscopic geometrical changes associated with the structured deformation $(g, \zeta_{\min} Q)$. Specifically, the geometrical changes associated with f_n in this case amount to breaking the body into many small pieces, expanding each piece an amount associated with the positive number $\zeta_{\min}$ and rotating each piece according to the value of the orthogonal tensor field Q at a point in the center of each piece; the pieces finally must be translated appropriately (without interpenetration) to meet the requirement that f_n be injective and be close to the macroscopic deformation g. Consequently, we may view the body in the deformed configuration associated with the loose phase as consisting of an aggregate of small subbodies, each undergoing its own rotation along with a common expansion. For this reason, we use the term *elastic aggregate* to describe an elastic body that is dilatationally cohesionless with respect to $\zeta_{\min}$.

When a body undergoes the classical deformation $(g, \nabla g)$, the sequence of piecewise smooth deformations in the Approximation Theorem can be taken to be the constant sequence $n \mapsto g$, so that the body as a whole undergoes the smooth deformation g at every stage in the approximating sequence. We may then describe the deformed configuration associated with the compact phase as one obtained when the elastic aggregate behaves as a single, coherent body. In other words, the individual pieces of the aggregate are not distinguished in the compact phase.

The appearance of piecewise smooth deformations of an isotropic elastic body in which each piece of the body undergoes a dilatation is described in a different setting by Mizel [13]. In that article as well as here, the distinguished role of dilatations arises through the class of stored energy functions under study. Here, the piecewise smooth deformations arise as approximations to the structured deformations that describe the loose phase, while in the article [13] these discontinuous deformations arise as a broad class of functions of bounded variation that minimize the energy under particular circumstances.

As we described above, the loose phase for g is geometrically admissible if and only if the macroscopic deformation satisfies $\det \nabla g \geq det(\zeta_{\min} Q) = \zeta_{\min}^3$. Thus, if the macroscopic deformation g satisfies the opposite inequality $\det \nabla g < \zeta_{\min}^3$, the compact phase, but not the loose phase, for that g may appear. However, for the case $\det \nabla g \geq \zeta_{\min}^3$ the loose phase for g not only may appear but also has the additional property of submacroscopic stability defined and analyzed in [4]: for a given macroscopic deformation g, among all candidates G for the deformation without disarrangements for the structured deformation (g, G), the tensor field $G = \zeta_{\min} Q$ minimizes an augmented energy functional that is stationary with respect to g when the body is in equilibrium and whose rate of change in a prescribed class of processes equals the excess of the rate at which energy is stored over the rate at which energy is dissipated during each process in the class. In summary, *when the macroscopic deformation g does not provide enough room for each of the pieces of the aggregate to appear in its stress-free state of deformation $\zeta_{\min} Q$, then only the compact phase for g can appear; when enough room is provided by g, then both phases can appear and the loose phase for g determines a submacroscopically stable equilibrium for the aggregate.*

Because both the loose phase and the compact phase are available in every dilatationally cohesionless elastic body, we may describe them as "universal phases" for the class of dilatationally cohesionless elastic bodies. However, for a given body in this class, other phases may be available for a given macroscopic deformation g, that is, there are other choices of G than ∇g and $\zeta_{\min} Q$ such that the pair (g, G) satisfies (9)–(13). Some of these additional phases may also be universal for the class of dilatationally cohesionless bodies, and the subsequent considerations that we undertake here also may be relevant to these additional phases. However, we do not attempt here to identify explicitly additional universal phases.

5 Field relations governing the loose and compact phases: the case of homogeneous deformations in each phase

In view of the special form of the structured deformations $(g, \nabla g)$ and $(g, \zeta_{\min} Q)$ that determine the compact and loose phases, the field relations (9)–(13) that govern a dilatationally cohesionless elastic material in the compact phase for g are all satisfied identically, except for the balance of linear momentum:

$$\rho_0\, \ddot{g} = \operatorname{div}(D_G \Psi(\nabla g)) + b, \tag{24}$$

while for every choice of the orthogonal-valued field $Q : \mathcal{E} \longrightarrow \text{Orth}$ the field relations in the loose phase for g all are satisfied identically except for the balance of linear momentum and the Accommodation Inequality:

$$\rho_0\, \ddot{g} = b, \tag{25}$$
$$\zeta_{\min}^3 \leq \det \nabla g. \tag{26}$$

We now restrict our attention to the case in which the body force field b is a constant and define two families of structured deformations $(g_c, \nabla g_c)$ and $(g_\ell, \zeta_{\min} Q)$ that generate special solutions of the field relations in the case of the compact and loose phases, respectively. Let $F \in \text{Lin}^+$, $X_0 \in \mathcal{E}$, $a, n, v_c, v_\ell \in \mathcal{V}$ with a, n unit vectors, and $\xi_c, \xi_\ell \in \mathbb{R}$ be given, and define $g_c, g_\ell : \mathcal{E} \times \mathbb{R} \to \mathcal{E}$ by

$$g_c(X, t) = X_0 + F(I + \xi_c\, a \otimes n)(X - X_0) + t v_c + \frac{t^2}{2\rho_0} b, \tag{27}$$
$$g_\ell(X, t) = X_0 + F(I + \xi_\ell\, a \otimes n)(X - X_0) + t v_\ell + \frac{t^2}{2\rho_0} b. \tag{28}$$

For each t, $X \mapsto g_c(X, t)$ and $X \mapsto g_\ell(X, t)$ are homogeneous deformations with

$$\nabla g_c(X, t) = F(I + \xi_c\, a \otimes n)$$
$$\nabla g_\ell(X, t) = F(I + \xi_\ell\, a \otimes n),$$

so that $\nabla g_c(X, t) - \nabla g_\ell(X, t) = (\xi_c - \xi_\ell) F a \otimes n$ is a rank-one tensor. This rank-one property of the difference in deformation gradients is a necessary condition in order that two given homogeneous deformations determine a continuous field that agrees with g_c on one side of a smooth surface and with g_ℓ on the other. Since $\nabla g_c = F(I + \xi_c\, a \otimes n)$ is independent of X and $\ddot{g}_c = b/\rho_0$, $(g_c, \nabla g_c)$ satisfies balance of linear momentum (24) in the compact phase. Because of the choice of $\zeta_{\min}$, the relation (19) and the relation $\ddot{g}_\ell = b/\rho_0$ tell us that $(g_\ell, \zeta_{\min} Q)$ satisfies the balance of linear momentum (25). The Accommodation Inequality (26) is satisfied by $(g_\ell, \zeta_{\min} Q)$ if and only if

$$\zeta_{\min}^3 \leq \det(F(I + \xi_\ell\, a \otimes n)) = (\det F)\,(1 + \xi_\ell\, a \cdot n). \tag{29}$$

Similarly, the restriction $0 < (\det F)\,(1 + \xi_c\, a \cdot n)$ on $F,\ \xi_c\ ,\ a$, and n, or, equivalently,

$$-1 < \xi_c\, a \cdot n \tag{30}$$

amounts to the condition $0 \ < \det \nabla g_\ell$ that is required of every macroscopic deformation. If $a \cdot n > 0$, (30) is equivalent to the requirement

$$\xi_c \ \in (-(a \cdot n)^{-1}, \infty);$$

if $a \cdot n < 0$, the inequality (30) is equivalent to the requirement

$$\xi_c \ \in (-\infty,\ -(a \cdot n)^{-1});$$

if $a \cdot n = 0$, then (30) places no restriction on ξ_c. In conclusion, we may express the content of (30) by writing

$$\xi_c \ \in I_{a,n} \qquad \text{where } I_{a,n} = \begin{cases} (-(a \cdot n)^{-1}, \infty) & \text{if } a \cdot n > 0 \\ (-\infty,\ -(a \cdot n)^{-1}) & \text{if } a \cdot n < 0 \\ \mathbb{R} & \text{if } a \cdot n = 0. \end{cases} \tag{31}$$

We shall use the following easily established results in the next section:

$$\begin{cases} \lim\limits_{\xi_c \longrightarrow -(a \cdot n)^{-1}} \det \nabla g_c = 0 \quad \text{and} \quad \lim\limits_{\xi_c \longrightarrow \infty} |\nabla g_c| = \infty & \text{if } a \cdot n > 0 \\ \lim\limits_{\xi_c \longrightarrow -\infty} |\nabla g_c| = \infty \quad \text{and} \quad \lim\limits_{\xi_c \longrightarrow -(a \cdot n)^{-1}} \det \nabla g_c = 0 & \text{if } a \cdot n < 0 \\ \lim\limits_{\xi_c \longrightarrow -\infty} |\nabla g_c| = \infty \quad \text{and} \quad \lim\limits_{\xi_c \longrightarrow +\infty} |\nabla g_c| = \infty & \text{if } a \cdot n = 0. \end{cases} \tag{32}$$

6 Thermodynamical considerations

We pause to consider two additional field relations, the local forms of the First and Second Laws of Thermodynamics, formulated in the reference configuration for a general elastic body, for a general structured deformation (g, G), and for a not necessarily constant temperature field θ:

$$\dot{\varepsilon} = S \cdot \nabla \dot{g} - \operatorname{div} q + r \tag{33}$$

$$\dot{\psi} \le S \cdot \nabla \dot{g} - \eta\, \dot{\theta} - \frac{q \cdot \nabla \theta}{\theta}. \tag{34}$$

In recording the local version of the First Law, we denote by r the external radiation field on the body. In these relations, the internal energy ε and the free energy ψ are assumed constitutively to be functions of $G,\ M = \nabla g - G$, and θ, while the heat flux q is assumed to be a function of $G,\ M,\ \theta$ and $\nabla \theta$. We assume further that q vanishes when $\nabla \theta$ vanishes, as in the case of Fourier's constitutive relation for heat conduction: $q = -\kappa(\theta, \nabla g)\nabla \theta$. Because $\varepsilon,\ \psi,\ \theta$, and the entropy η are related by

$$\varepsilon = \psi + \theta \eta, \tag{35}$$

the entropy in this context also is determined constitutively as a function of $G,\ M$, and θ. We assume as in Sect. 2 that the stress S is determined by the relation

$$S = D_G \Psi + D_M \Psi \tag{36}$$

where now the free energy response function Ψ may also depend upon the temperature θ, and we make the additional constitutive assumption:

$$\eta = -D_\theta \Psi. \tag{37}$$

The stress relation (36), the entropy relation (37), and the Second Law (34) yield the dissipation inequality

$$D_G \Psi \cdot \dot{M} + D_M \Psi \cdot \dot{G} - \frac{q \cdot \nabla \theta}{\theta} \ge 0, \tag{38}$$

which we regard as a restriction on the thermodynamical processes that can be realized in the body. [For the case of uniform temperature fields, this inequality reduces to (4)]. The entropy relation (37) and the formula (35) imply that the internal energy ε is given constitutively by

$$\varepsilon = \Psi - \theta D_\theta \Psi. \tag{39}$$

Let us now return to our earlier assumption that the temperature field θ is a constant in space and time, and we assume now that the radiation field r vanishes. We resume our consideration of loose and compact phases of a dilatationally cohesionless elastic body undergoing the motions (27) and (28), respectively, and recall that, in this context, the deformation due to disarrangements M drops out of all of the constitutive relations (36), (37), and (39). Moreover, the relations $\nabla \dot{g}_c = \nabla \dot{g}_\ell = 0$ and the constitutive assumption on the heat flux field q formulated above tell us that the First and Second Laws (33), (34) and the dissipation inequaltiy (38) are satisfied by $(g_c, \nabla g_c)$ and $(g_\ell, \zeta_{\min} Q)$ in the form "0 = 0". In particular, no dissipation occurs in the phases individually, and the only source of dissipation within the body will turn out to be the moving interfaces that separate the phases.

7 Consequences of rank-one convexity and growth

In the analysis of the following sections, we consider a dilatationally cohesionless elastic material relative to the positive number $\zeta_{\min}$, so that according to the definition given at the end of Sect. 3, the free energy response function Ψ satisfies the conditions of purely dissipative disarrangements (8), isotropy (15), growth under extreme dilatations (17), coincidence of minimizers (21), and, consequently, the relation (19). We suppress in the notation of this section the dependence of Ψ upon the temperature θ. In addition, we assume the stronger growth properties (23) as well as the condition of rank-one convexity for Ψ (stated here in a form that exploits the assumed smoothness of Ψ [14], p. 17): for all $A \in \mathrm{Lin}^+$ and $u, v \in \mathcal{V}$

$$D_G \Psi(A) \cdot (u \otimes v) \leq \Psi(A + u \otimes v) - \Psi(A). \tag{40}$$

As we noted following Remark 1, the stronger growth properties along with rank-one convexity imply the coincidence of minimizers (21), as well as other relations that we shall use in this section. For example, rank-one convexity of Ψ implies the following monotonicity property for particular components of its derivative: for each $F \in \mathrm{Lin}^+$, $a, n \in \mathcal{V}$, and $\xi, \eta \in \mathbb{R}$,

$$\xi \leq \eta \Longrightarrow D_G \Psi(F(I + \xi a \otimes n)) \cdot (Fa \otimes n) \leq D_G \Psi(F(I + \eta a \otimes n)) \cdot (Fa \otimes n). \tag{41}$$

In fact, if we put

$$f(\xi) := \Psi(F(I + \xi a \otimes n)) = \Psi(F + \xi Fa \otimes n) \quad \text{for each } \xi \in I_{a,n} \tag{42}$$

[with $I_{a,n}$ the unbounded interval defined in (31)], then

$$f'(\xi) = D_G \Psi(F(I + \xi a \otimes n)) \cdot (Fa \otimes n). \tag{43}$$

The rank-one convexity of Ψ implies that f is convex, and the convexity and smoothness of f imply that f' is continuous and monotone increasing, that is, (41) holds.

Applying the rank-one convexity condition (40) with $A = F + \xi Fa \otimes n$, $u = -\xi Fa$, and $v = n$ yields the inequality

$$\begin{aligned} &D_G \Psi(F + \xi Fa \otimes n)) \cdot (-\xi Fa \otimes n) \\ &\quad \leq \Psi(F + \xi Fa \otimes n + -\xi Fa \otimes n) - \Psi(F + \xi Fa \otimes n) \\ &\quad = -(\Psi(F + \xi Fa \otimes n) - \Psi(F)) \end{aligned}$$

or, equivalently,

$$D_G \Psi(F + \xi Fa \otimes n) \cdot (\xi Fa \otimes n) \geq \Psi(F + \xi Fa \otimes n) - \Psi(F). \tag{44}$$

This relation, the limit relations (32), and the strong growth conditions (23) imply that the right-hand member of (44) tends to plus infinity as ξ approaches either end of the interval $I_{a,n}$, and we conclude from these assertions and from (42) that

$$\begin{aligned}\lim_{\xi\longrightarrow\inf I_{a,n}} f(\xi) &= \lim_{\xi\longrightarrow\sup I_{a,n}} f(\xi)\\ &= \lim_{\xi\longrightarrow\inf I_{a,n}} \xi f'(\xi) = \lim_{\xi\longrightarrow\sup I_{a,n}} \xi f'(\xi) = +\infty. \end{aligned} \tag{45}$$

We conclude that f attains a minimum at at least one point in $I_{a,n}$. Because $\inf I_{a,n}$ is negative and $\sup I_{a,n}$ is positive, the relation (45) also implies that the continuous, monotone increasing function f' must have negative values near $\inf I_{a,n}$ and positive values near $\sup I_{a,n}$. Therefore, the stress component $f'(\xi) = D_G\Psi(F(I+\xi a\otimes n))\cdot(Fa\otimes n)$ is a monotone increasing function of ξ that increases from negative values to positive values as ξ increases through $I_{a,n}$, and the stress component vanishes on the (possibly singleton) interval of points where f attains its minimum on $I_{a,n}$ (and at no others). For convenience, we assume in what follows that the stress component $f'(\xi) = D_G\Psi(F(I+\xi a\otimes n))\cdot(Fa\otimes n)$ vanishes at exactly one point $\xi=\xi_0$ in $I_{a,n}$, and we note from the definition of $\zeta_{\min}$ and from the relation (21) that

$$f(\xi_0) := \min_{\xi\in I_{a,n}} f(\xi) = \min_{\xi\in I_{a,n}} \Psi(F+\xi Fa\otimes n) \geq \min_{G\in \mathrm{Lin}^+} \Psi(G) = \Psi(\zeta_{\min} I).$$

8 Moving interfaces separating loose and compact phases

Each of the macroscopic deformations (27) and (28) considered here has the geometrical effect of causing a homogeneous deformation that changes the shape of the body followed by a rigid translation at constant acceleration. Our main interest in the sequel is to delimit ways in which the body can be partitioned into adjacent regions, one of which is occupied by material in the compact phase and the other by material in the loose phase, with the possibility that material points in one phase can be transformed into material points of the other as the interface separating the phases moves through the reference configuration. As we noted earlier, the loose phase has the property of providing a submacroscopically stable equilibrium configuration for the body, and this suggests that the loose phase be preferred. However, the Accommodation Inequality (26) provides a lower bound for the amount of macroscopic volume change that must occur in order that the loose phase be present. In particular, when the macroscopic volume change is too small, the body cannot appear in the loose phase and may select as an alternative the compact phase. The details of the competition on energetic and geometric grounds between the loose and compact phases are the subject of the present section.

8.1 Jump conditions at a phase interface

We assume now that a smooth function $\hat{t}$ defined on a subset of the reference configuration $\mathcal{B}$ determines points $(X,\hat{t}(X))$ that form a space-time surface $\mathcal{I}$ for which the normal vector field $(X,\hat{t}(X)) \longmapsto (-\nabla\hat{t}(X),1)$ on $\mathcal{I}$ has the property that one of the two structured deformations $(g_c,\nabla g_c)$ and $(g_\ell,\zeta_{\min}Q)$ is specified to act on a space-time region $\mathcal{R}_+ = \{(X,\hat{t}(X))+\gamma(-\nabla\hat{t}(X),1) \mid 0<\gamma<\gamma_+(X)\}$ on the side of the surface $\mathcal{I}$ into which $(-\nabla\hat{t}(X),1)$ points, while the other of the two structured deformations is specified to act on a space-time region $\mathcal{R}_- = \{(X,\hat{t}(X))-\gamma(-\nabla\hat{t}(X),1) \mid 0<\gamma<\gamma_-(X)\}$ on the opposite side of the surface $\mathcal{I}$. Here, γ_+ and γ_- are positive-valued functions such that the points $X-\gamma\nabla\hat{t}(X)$ for $\gamma\in(0,\gamma_+(X))$ and the points $X+\gamma\nabla\hat{t}(X)$ for $\gamma\in(0,\gamma_-(X))$ all are in the reference configuration $\mathcal{B}$, and we call the surface $\mathcal{I}$ a *(space-time) phase interface*. If $(g_c,\nabla g_c)$ acts on the space-time region $\mathcal{R}_+$, we say that *the normal vector* $(-\nabla\hat{t}(X),1)$ *points into the compact phase*, and we refer to the region $\mathcal{R}_+$ as the compact phase. Similarly, if $(g_\ell,\zeta_{\min}Q)$ acts on $\mathcal{R}_+$, then we say that $(-\nabla\hat{t}(X),1)$ *points into the loose phase* and call the region $\mathcal{R}_+$ the loose phase. We note that when, for example, the normal vector $(-\nabla\hat{t}(X),1)$ points into the compact phase, the material at X transforms at time $\hat{t}(X)$ from the loose phase to the compact phase. Of course, the normal vector $(-\nabla\hat{t}(X),1)$ is not a unit vector, in general.

For each $t_0\in\mathbb{R}$ the set of points X such that $\hat{t}(X)=t_0$ is the surface $\mathcal{I}_{t_0} = \{X \mid (X,t_0)\in\mathcal{I}\}$ in the reference configuration, and we call $\mathcal{I}_{t_0}$ the *phase interface at time* t_0. For each $X\in\mathcal{I}_{t_0}$ and $\gamma\in(0,\gamma_+(X))$ we have

$$((X-\gamma\nabla\hat{t}(X),t_0)-(X,\hat{t}(X)))\cdot(-\nabla\hat{t}(X),1) = \gamma\left|\nabla\hat{t}(X)\right|^2 > 0,$$

and this tells us that $(X-\gamma\nabla\hat{t}(X), t_0) \in \mathcal{R}_+$ for every $\gamma \in (0, \gamma_+(X))$. Given a point $X \in \mathcal{I}_{t_0}$, and if, for example, $(g_c, \nabla g_c)$ acts on the space-time region $\mathcal{R}_+$, then all of the points $(X-\gamma\nabla\hat{t}(X), t_0)$ with $\gamma \in (0, \gamma_+(X))$ are in $\mathcal{R}_+$ and, hence, in the domain of g_c. Consequently, all of the points $X-\gamma\nabla\hat{t}(X)$ with $\gamma \in (0, \gamma_+(X))$ are in the domain of $g_c(\cdot, t_0)$; by the same reasoning, all of the points $X+\gamma\nabla\hat{t}(X)$ for $\gamma \in (0, \gamma_-(X))$ are in the domain of $g_\ell(\cdot, t_0)$. We then are justified in stating that the vector $-\nabla\hat{t}(X)$ normal at X to $\mathcal{I}_{t_0}$, the phase interface at time t_0, points into the same phase as does the normal $(-\nabla\hat{t}(X), 1)$ at $(X, t_0) = (X, \hat{t}(X))$ to $\mathcal{I}$, the space-time interface, and we may also assert without ambiguity that $-\nabla\hat{t}(X)$ points into a particular phase at time t_0. In the sequel we often will not use the modifiers "space-time" and "at time t_0" to distinguish the precise sense in which terms such as phase interface, loose phase, and compact phase are employed.

For each piecewise smooth field h defined for each time t on the entire body, we denote by $[h](X, \hat{t}(X))$ the jump in h at $(X, \hat{t}(X))$, that is,

$$[h](X, \hat{t}(X)) := \lim_{(y,\tau)\longrightarrow(X,\hat{t}(X))+} h(y,\tau) - \lim_{(y,\tau)\longrightarrow(X,\hat{t}(X))-} h(y,\tau) \tag{46}$$

where $(y,\tau) \longrightarrow (X, \hat{t}(X))+$ indicates that, in taking the limit, h is restricted to points (y, τ) near $(X, \hat{t}(X))$ such that the inner product $(y-X, \tau-\hat{t}(X))\cdot(-\nabla\hat{t}(X), 1)$ is positive, while $(y,\tau) \longrightarrow (X, \hat{t}(X))-$ indicates that the restriction is made with $(y-X, \tau-\hat{t}(X))\cdot(-\nabla\hat{t}(X), 1) < 0$. Alternatively, for example, if the normal $(-\nabla\hat{t}(X), 1)$ points into the compact phase, we may write $[h](X, \hat{t}(X)) = h_c(X, \hat{t}(X)) - h_\ell(X, \hat{t}(X))$, where the subscript c indicates that in taking the limit, h is evaluated at points in the compact phase, and the subscript ℓ indicates that h is evaluated at points in the loose phase.

The balance of linear momentum (1) can be written in the divergence form

$$\mathrm{div}_4(S, -\rho_0\,\dot{g}) + b = 0 \tag{47}$$

where $\mathrm{div}_4(A, v) := \mathrm{div}A + \dot{v}$ with A a tensor field and v a vector field. A standard argument involving integration of both sides of (47) over cylinders centered at $(X, \hat{t}(X))$ with axes parallel to the normal $(-\nabla\hat{t}(X), 1)$ yields the relation

$$[S](X, \hat{t}(X))(-\nabla\hat{t}(X)) - \rho_0\,[\dot{g}](X, \hat{t}(X)) = 0. \tag{48}$$

If the chosen normal points into the compact phase, this jump relation for linear momentum becomes

$$D_G\Psi(F(I+\xi_c a\otimes n, \theta)\nabla\hat{t}(X) + \rho_0(v_c - v_\ell) = 0. \tag{49}$$

(The same relation results in the case where the normal points into the loose phase.) The fact that the stress in the loose phase vanishes accounts for the fact that ξ_ℓ does not appear in the jump condition (49) corresponding to the balance of linear momentum.

The First Law of Thermodynamics (33) in local form can be written as follows:

$$\begin{aligned}\dot{\varepsilon} &= S\cdot\nabla\dot{g} - \mathrm{div}q + r = \mathrm{div}(S^T\dot{g} - q) - \mathrm{div}S\cdot\dot{g} + r\\ &= \mathrm{div}(S^T\dot{g} - q) - \rho_0\,\ddot{g}\cdot\dot{g} + r + b\cdot\dot{g}\\ &= \mathrm{div}(S^T\dot{g} - q) + (b\cdot g - \tfrac{1}{2}\rho_0\,|\dot{g}|^2)^{\cdot} + r,\end{aligned}$$

so that the First Law assumes the divergence form

$$\mathrm{div}_4(-S^T\dot{g} + q, \varepsilon + \tfrac{1}{2}\rho_0\,|\dot{g}|^2 - b\cdot g) = r.$$

This relation leads in the standard way to the jump condition

$$[(-S^T\dot{g} + q)]\cdot(-\nabla\hat{t}(X)) + [\varepsilon + \tfrac{1}{2}\rho_0\,|\dot{g}|^2 - b\cdot g] = 0. \tag{50}$$

Because we have assumed that the temperature field is a constant, it follows from our earlier assumptions on q that q is zero in both the loose and compact phases, so that its jump also vanishes. Because we have assumed that b is a constant, the jump condition (56) introduced below implies that $[g\cdot b] = 0$, and (50) has the following form when the chosen normal points into the compact phase:

$$\begin{aligned}&(D_G\Psi(F(I+\xi_c a\otimes n), \theta)^T\left(v_c + \frac{\hat{t}(X)}{\rho_0}b\right)\cdot\nabla\hat{t}(X) + \varepsilon_c - \varepsilon_\ell\\ &+\frac{1}{2}\rho_0\left(\left|v_c + \frac{\hat{t}(X)}{\rho_0}b\right|^2 - \left|v_\ell + \frac{\hat{t}(X)}{\rho_0}b\right|^2\right) = 0\end{aligned} \tag{51}$$

Here, we have used the fact that the stress in the loose phase is zero. (The same relation results when the normal points into the loose phase.)

The jump condition corresponding to the Second Law is most easily obtained by rewriting the local form of the Second Law in the divergence form

$$\dot{\eta} \geq -\mathrm{div}\left(\frac{q}{\theta}\right) + \frac{r}{\vartheta}$$

or, equivalently,

$$\mathrm{div}_4\left(\frac{q}{\theta}, \eta\right) \geq \frac{r}{\vartheta}. \tag{52}$$

This relation yields the jump condition

$$\left[\frac{q}{\theta}\right](X, \hat{t}(X)) \cdot (-\nabla\hat{t}(X)) + [\eta](X, \hat{t}(X)) \geq 0,$$

and the vanishing of the heat flux q on both sides of the interface $t = \hat{t}(X)$ along with the entropy relation (37) implies the simpler relation

$$[D_\theta \Psi] \leq 0. \tag{53}$$

Taking into account the fact that the free energy response function Ψ depends upon G and θ and recalling the values of G in the loose and compact phases, we conclude that when the chosen normal $(-\nabla\hat{t}(X), 1)$ points into the compact phase the jump condition corresponding to the Second Law becomes

$$D_\theta \Psi(F(I + \xi_c a \otimes n), \theta) \leq D_\theta \Psi(\zeta_{\min} I, \theta). \tag{54}$$

[The opposite relation results when the chosen normal $(-\nabla\hat{t}(X), 1)$ points into the loose phase.] We note here that the formula for the internal energy (39) implies that the jump condition corresponding to the Second Law (53) is equivalent to the assertion that the jump in internal energy at the interface is no less than the jump in free energy:

$$\begin{aligned}[\varepsilon] &= [\psi + \theta\eta] = [\psi] + \theta[\eta] \\ &\geq [\psi]\end{aligned} \tag{55}$$

where we have used the fact that $[\theta] = 0$. For example, if the normal at X points into the compact phase, the free energy cannot decrease [by virtue of the relation (21)]; consequently, (55) tells us that the internal energy also cannot decrease as a result of the transformation from the loose phase to the compact phase: $\varepsilon_c - \varepsilon_\ell \geq 0$.

Finally, we impose the requirement that the position of material points not experience discontinuities at a phase interface:

$$[g] = 0. \tag{56}$$

The formulas (27) and (28) yield the following form of the jump relation for position:

$$\hat{t}(X)(v_c - v_\ell) = (\xi_\ell - \xi_c)(Fa \otimes n)(X - X_0). \tag{57}$$

Taking the gradient of both sides of (57) we obtain the useful relation

$$(v_c - v_\ell) \otimes \nabla\hat{t}(X) = (\xi_\ell - \xi_c)Fa \otimes n. \tag{58}$$

[Both of these relations also are valid when the normal $(-\nabla\hat{t}(X), 1)$ to the phase interface points into the loose phase.] When $\xi_\ell - \xi_c \neq 0$, the last relation implies that the velocity difference $v_c - v_\ell$ and the vector Fa are colinear, as are the gradient $\nabla\hat{t}(X)$ and the vector n.

8.2 Implications of the jump conditions

The jump condition associated with the First Law (51) contains the term

$$(D_G\Psi(F(I+\xi_c a\otimes n),\theta)^T\, v_c\cdot\nabla\hat{t}(X) = v_c\cdot D_G\Psi(F(I+\xi_c a\otimes n),\theta)\,\nabla\hat{t}(X),$$

which leads us to take the inner product of both sides of the jump condition for balance of linear momentum (49) with the vector v_c. When the two resulting expressions for $v_c\cdot D_G\Psi(F(I+\xi_c a\otimes n),\theta)\,\nabla\hat{t}(X)$ are equated, we find that

$$\begin{aligned}\varepsilon_c-\varepsilon_\ell+\frac{1}{2}\rho_0\left(\left|v_c+\frac{\hat{t}(X)}{\rho_0}b\right|^2-\left|v_\ell+\frac{\hat{t}(X)}{\rho_0}b\right|^2\right)&\\ =\rho_0(v_c-v_\ell)\cdot v_c-\rho_0^{-1}\hat{t}(X)\,D_G\Psi(F(I+\xi_c a\otimes n),\theta)\,\nabla\hat{t}(X)\cdot b&\end{aligned}$$

which is equivalent to the relation

$$2\hat{t}(X)\;\{(v_\ell-v_c)-\rho_0^{-1}\,D_G\Psi(F(I+\xi_c a\otimes n),\theta)\,\nabla\hat{t}(X)\}\cdot b = 2(\varepsilon_c-\varepsilon_\ell)-\rho_0\,|v_c-v_\ell|^2\,.$$

We assume for the sake of simplicity that the function $\hat{t}$ is affine, so that $\nabla\hat{t}(X)$ is independent of X. Because the previous relation must hold for every point X in an open region in the reference configuration, and because only $\hat{t}$ depends upon X, there can be a non-trivial phase interface only if

$$\{(v_\ell-v_c)-\rho_0^{-1}\,D_G\Psi(F(I+\xi_c a\otimes n),\theta)\,\nabla\hat{t}\}\cdot b=0, \tag{59}$$

and the jump relation for the First Law reduces to

$$\rho_0\,|v_c-v_\ell|^2=2(\varepsilon_c-\varepsilon_\ell). \tag{60}$$

This relation strengthens the conclusion drawn above from the relation (55) about $\varepsilon_c-\varepsilon_\ell$. That conclusion amounted to the assertion that when the normal at $(X,\hat{t}(X))$ points into the compact phase, then $\varepsilon_c-\varepsilon_\ell\geq 0$. However, the relation (60) does not depend upon whether or not the normal to the phase interface points into the compact phase and also leads to the conclusion $\varepsilon_c-\varepsilon_\ell\geq 0$. Therefore, at every point $(X,\hat{t}(X))$ on the phase interface there holds $\varepsilon_c-\varepsilon_\ell\geq 0$, independent of the orientation of the normal $(-\nabla\hat{t}(X),1)$ relative to the loose and compact phases.

The relations (39) and (60) yield the formula

$$\rho_0\,|v_c-v_\ell|^2=2(\Psi_c-\theta(D_\theta\Psi)_c)-2(\Psi_\ell-\theta(D_\theta\Psi)_\ell) \tag{61}$$

where, in detail, the right-hand side of (61) is the expression

$$\begin{aligned}&2\,(\Psi(F(I+\xi_c\,a\otimes n),\theta)-\theta D_\theta\Psi(F(I+\xi_c\,a\otimes n),\theta))\\ &\quad-2\,(\Psi(\zeta_{\min}I,\theta)-\theta D_\theta\Psi(\zeta_{\min}I,\theta)).\end{aligned} \tag{62}$$

Consequently, the formula (61) expresses the relative speed of the compact phase and of the loose phase as a function of the measure of deformation ξ_c in the compact phase.

In order to relate the measure of deformation ξ_ℓ to the corresponding measure ξ_c in the compact phase, we use (58) and (49) in the following calculation:

$$\begin{aligned}&(\xi_\ell-\xi_c)D_G\Psi(F(I+\xi_c\,a\otimes n),\theta)\cdot(Fa\otimes n)\\ &\quad=D_G\Psi(F(I+\xi_c\,a\otimes n),\theta)\cdot\big((v_c-v_\ell)\otimes\nabla\hat{t}(X)\big)\\ &\quad=\operatorname{tr}\big(D_G\Psi(F(I+\xi_c\,a\otimes n),\theta)\,\big(\nabla\hat{t}(X)\otimes(v_c-v_\ell)\big)\big)\\ &\quad=\operatorname{tr}\big(D_G\Psi(F(I+\xi_c\,a\otimes n),\theta)\nabla\hat{t}(X)\otimes(v_c-v_\ell)\big)\\ &\quad=D_G\Psi(F(I+\xi_c\,a\otimes n),\theta)\nabla\hat{t}(X)\cdot(v_c-v_\ell)\\ &\quad=-\rho_0\,|v_c-v_\ell|^2\,.\end{aligned} \tag{63}$$

From the discussion at the end of the previous section, the stress component $D_G\Psi(F(I+\xi_c a\otimes n),\theta)\cdot(Fa\otimes n)$ vanishes at exactly one point $\xi_c=\xi_0$ in $I_{a,n}$, so that (63) and (61) yield

$$\xi_\ell=\xi_c-\frac{2(\Psi_c-\theta(D_\theta\Psi)_c)-2(\Psi_\ell-\theta(D_\theta\Psi)_\ell)}{D_G\Psi(F(I+\xi_c\,a\otimes n),\theta)\cdot(Fa\otimes n)}\quad\text{for all }\xi_c\in I_{a,n}\backslash\{\xi_0\}.\tag{64}$$

It was pointed out above that the velocity difference v_c-v_ℓ and the vector Fa are colinear when $\xi_\ell\neq\xi_c$; this and (61) tell us that

$$\begin{aligned}v_c-v_\ell&=\pm\frac{|v_c-v_\ell|}{|Fa|}Fa\\&=\pm\frac{\{(2(\Psi_c-\theta(D_\theta\Psi)_c)-2(\Psi_\ell-\theta(D_\theta\Psi)_\ell))/\rho_0\}^{1/2}}{|Fa|}Fa\end{aligned}\tag{65}$$

for all $\xi_c\in I_{a,n}\backslash\{\xi_0\}$. The formula (57) now yields a formula for $\hat{t}(X)$ through the relations (64) and (65):

$$\begin{aligned}\hat{t}(X)(v_c-v_\ell)&=(n\cdot(X-X_0)(\xi_\ell-\xi_c)Fa\qquad\Longleftrightarrow\\&\pm\hat{t}(X)\frac{\{2\left((\Psi_c-\theta(D_\theta\Psi)_c)-(\Psi_\ell-\theta(D_\theta\Psi)_\ell)\right)/\rho_0\}^{1/2}}{|Fa|}Fa\\&=-(n\cdot(X-X_0)\frac{2\left((\Psi_c-\theta(D_\theta\Psi)_c)-(\Psi_\ell-\theta(D_\theta\Psi)_\ell)\right)}{D_G\Psi(F(I+\xi_c\,a\otimes n),\theta)\cdot(Fa\otimes n)}Fa.\end{aligned}$$

This relation reduces to the formula

$$\hat{t}(X)=\mp\frac{|Fa|\{2\rho_0\left((\Psi_c-\theta(D_\theta\Psi)_c)-(\Psi_\ell-\theta(D_\theta\Psi)_\ell)\right)\}^{1/2}}{D_G\Psi(F(I+\xi_c\,a\otimes n),\theta)\cdot(Fa\otimes n)}n\cdot(X-X_0),\tag{66}$$

which implies

$$\nabla\hat{t}(X)=\mp\frac{|Fa|\{2\rho_0\left((\Psi_c-\theta(D_\theta\Psi)_c)-(\Psi_\ell-\theta(D_\theta\Psi)_\ell)\right)\}^{1/2}}{D_G\Psi(F(I+\xi_c\,a\otimes n),\theta)\cdot(Fa\otimes n)}n.$$

The reciprocal of the magnitude of $\nabla\hat{t}(X)$ is the speed of the phase interface Ξ:

$$\Xi=\frac{|D_G\Psi(F(I+\xi_c\,a\otimes n),\theta)\cdot(Fa\otimes n)|}{|Fa|\{2\rho_0\left((\Psi_c-\theta(D_\theta\Psi)_c)-(\Psi_\ell-\theta(D_\theta\Psi)_\ell)\right)\}^{1/2}},\tag{67}$$

and we note that the ratio $\Xi/|v_c-v_\ell|$ of the speed of the phase interface Ξ to the relative speed $|v_c-v_\ell|$ of the compact and loose phases then is given by

$$\frac{\Xi}{|v_c-v_\ell|}=\frac{|D_G\Psi(F(I+\xi_c\,a\otimes n),\theta)\cdot(Fa\otimes n)|}{2\left((\Psi_c-\theta(D_\theta\Psi)_c)-(\Psi_\ell-\theta(D_\theta\Psi)_\ell)\right)}.\tag{68}$$

We recall that the relations (66), (67), and (68) were derived under the restriction $\xi_c\neq\xi_0$; moreover, the choice of signs in (65) and (66) are opposite one another.

8.3 Inequalities that restrict $\xi_c,\ a,\ n$, and F

In the previous section we obtained formulas that express quantities significant for the description of coexistent loose and compact phases in terms of a single scalar ξ_c, the two unit vectors $a,\ n$, and the tensor F that control the macroscopic deformation gradient $\nabla g_c=F(I+\xi_c a\otimes n)$ in the compact phase. Specifically, the Accommodation Inequality in the loose phase (29) in conjunction with the formula (64), the jump condition associated with the Second Law (53), and the relation (61) provide inequalities that restrict $\xi_c,\ a,\ n$, and F, and we study in this section sufficient conditions for the satisfaction of these inequalities. We recall from Sect. 8.2 that the jump conditions and the assumption that the phase interface is given by an affine mapping imply the restriction (59) on the orientation of the body force and the phase interface. In view of the relations (65) and (66), this restriction becomes

$$\left\{\frac{Fa}{|Fa|}-\frac{|Fa|}{D_G\Psi(F(I+\xi_c\,a\otimes n),\theta)\cdot(Fa\otimes n)}D_G\Psi(F(I+\xi_c a\otimes n),\theta)n\right\}\cdot b=0.\tag{69}$$

We make assumptions in what follows in order that $F,\ a,\ n,\ \xi_c$, and b satisfy this condition.

8.3.1 Sufficient conditions for a loose-to-compact transition

It is useful to treat two cases separately, and we first assume that the material at X transforms from the loose to the compact phase at time $\hat{t}(X)$, that is, the normal $(-\nabla\hat{t}(X), 1)$ points into the compact phase. The jump condition associated with the Second Law (53) then becomes (54), which we write in the abbreviated form

$$(D_\theta\Psi)_c \le (D_\theta\Psi)_\ell. \tag{70}$$

However, the inequality (21) amounts to the assertion $\Psi_c \ge \Psi_\ell$, so that if ξ_c satisfies (70), then it also satisfies

$$\begin{aligned}\varepsilon_c - \varepsilon_\ell &= (\Psi_c - \theta(D_\theta\Psi)_c) - (\Psi_\ell - \theta(D_\theta\Psi)_\ell)\\ &= (\Psi_c - \Psi_\ell) - \theta((D_\theta\Psi)_c - (D_\theta\Psi)_\ell) \ge 0.\end{aligned}$$

This inequality is precisely the one provided by the relation (61), and we conclude that in the loose to compact transition the inequality

$$\varepsilon_c = (\Psi_c - \theta(D_\theta\Psi)_c) \ge (\Psi_\ell - \theta(D_\theta\Psi)_\ell) = \varepsilon_\ell \tag{71}$$

need not be considered separately.

The Accommodation Inequality (29) and the formula (64) yield

$$\frac{\zeta_{\min}^3}{\det F} \le 1 + \xi_c a\cdot n - a\cdot n\frac{2((\Psi_c - \theta(D_\theta\Psi)_c) - (\Psi_\ell - \theta(D_\theta\Psi)_\ell))}{D_G\Psi(F(I+\xi_c\, a\otimes n),\theta)\cdot(Fa\otimes n)} \tag{72}$$

in which not only the denominator of the fraction on the right but also its numerator depends upon $\xi_c \ne \xi_0,\ a,\ n$, and F [see (62) for the explicit dependence]. For given $F,\ a$, and n we now provide sufficient conditions that the inequalities (70) and (72) be satisfied in an interval included in $I_{a,n}\backslash\{\xi_0\}$.

Remark 2 Let $F \in \mathrm{Lin}^+$ and $a,\ n$ unit vectors be given, let ξ_0 be the unique point ξ_c in $I_{a,n}$ at which the stress component $D_G\Psi(F(I+\xi_c\, a\otimes n),\theta)\cdot(Fa\otimes n)$ vanishes, and suppose that

$$D_\theta\Psi(F(I+\xi_0\, a\otimes n),\theta) < D_\theta\Psi(\zeta_{\min} I,\theta). \tag{73}$$

If $a\cdot n > 0$ then there exists $\delta > 0$ such that the inequalities (70) and (72) are satisfied for all $\xi_c \in (\xi_0-\delta,\xi_0)$. Moreover, there is a choice of sign in the formula (66) that assures that the normal $(-\nabla\hat{t}(X), 1)$ points into the compact phase. If $a\cdot n < 0$ then there exists $\delta > 0$ such that the inequalities (70) and (72) are satisfied for all $\xi_c \in (\xi_0, \xi_0+\delta)$, and the opposite choice of sign in the formula (66) assures that the normal $(-\nabla\hat{t}(X), 1)$ points into the compact phase. If $a\cdot n = 0$ and if $\zeta_{\min}^3 \le \det F$ then (72) is satisfied for all $\xi_c \in \mathbb{R}$, and there exists $\delta > 0$ such that the inequality (70) is satisfied for all $\xi_c \in (\xi_0-\delta,\xi_0)\cup(\xi_0,\xi_0+\delta)$. For each of the intervals $(\xi_0-\delta,\xi_0)$ and $(\xi_0,\xi_0+\delta)$, there is a choice of sign in the formula (66) that assures that the normal $(-\nabla\hat{t}(X), 1)$ points into the compact phase.

Proof The inequality (73) implies that there exists $\delta_1 > 0$ such that

$$D_\theta\Psi(F(I+\xi_c\, a\otimes n),\theta) < D_\theta\Psi(\zeta_{\min} I,\theta)$$

holds for all $\xi_c \in [\xi_0-\delta_1, \xi_0+\delta_1]$. It follows from this and the inequality (21) that

$$\begin{aligned}\xi \mapsto\ & 2\,(\Psi(F(I+\xi\, a\otimes n),\theta) - \theta D_\theta\Psi(F(I+\xi\, a\otimes n),\theta))\\ & -2\,(\Psi(\zeta_{\min}I,\theta) - \theta D_\theta\Psi(\zeta_{\min}I,\theta))\end{aligned}$$

has a strictly positive minimum on $[\xi_0-\delta_1, \xi_0+\delta_1]$. The properties of $\xi \mapsto D_G\Psi(F(I+\xi\, a\otimes n),\theta)\cdot(Fa\otimes n) = f'(\xi)$ established in Sect. 7 tell us that, on the interval $(\inf I_{a,n}, \xi_0)$, the function f' is negative-valued and has limit zero at ξ_0, while on the interval $(\xi_0, \sup I_{a,n})$ the function f' is positive-valued and has limit zero at ξ_0. These statements and (72) imply the desired conclusions when $a\cdot n \ne 0$. When $a\cdot n = 0$ the inequality (72) reduces to $\zeta_{\min}^3 \le \det F$, and the inequality (70) follows from the first statement in this proof. □

The inequality (73) amounts to the assertion that the entropy in the loose phase is less than the entropy in the compact phase associated with the particular value $\xi_c = \xi_0$ of the scalar deformation parameter in the compact phase. In this context it also is worth observing that, in the loose phase, all of the components of the stress vanish, while in the compact phase with $\nabla g_c = F(I + \xi_0 a \otimes n)$ only the stress component $D_G\Psi(F(I + \xi_0\, a \otimes n), \theta) \cdot (Fa \otimes n)$ need vanish. This observation suggests that the entropy in the loose phase may be lower than that in compact phase with $\nabla g_c = F(I + \xi_0 a \otimes n)$, in agreement with (73).

We conclude that, *under the hypotheses of Remark 2 and under the assumption $b = 0$ (so that (69) is satisfied identically in ξ_c), there is a non-trivial interval of numbers ξ_c and a choice of sign in the formula (66) such that the body admits a moving interface that transforms material from the loose phase (with $\nabla g_\ell = F(I + \xi_\ell a \otimes n)$ and with ξ_ℓ given by the formula (64)) into the compact phase (with $\nabla g_c = F(I + \xi_c a \otimes n)$). When $a \cdot n = 0$, the point $\xi_c = \xi_0$ is excluded from both intervals $(\xi_0 - \delta, \xi_0)$ and $(\xi_0, \xi_0 + \delta)$, because $\xi_c = \xi_0$ in (63) implies $v_c = v_\ell$ which, in turn, implies $\varepsilon_c = \varepsilon_\ell$, contradicting (73). In all cases, the interface separating the loose and compact phases is planar and moves with speed given by (67). Each phase moves as a (pre-strained) rigid body, and the relative velocity of motion of the two phases is given by the formula (65) with choice of sign the opposite of the choice of sign in the formula (66).*

8.3.2 Sufficient conditions for a compact-to-loose transition

We note that the hypothesis (73) and the jump condition associated with the Second Law (53) imply that the body does *not* admit a moving interface that transforms the material from the compact phase to the loose phase: when the normal to the phase boundary points from the compact phase into the loose phase, then the two inequalities (73) and (53) are opposite and, hence, contradictory. Of course, the assumption that the opposite of the inequality (73) holds, that is,

$$(D_\theta\Psi)_c = D_\theta\Psi(F(I + \xi_0\, a \otimes n), \theta) > D_\theta\Psi(\zeta_{\min} I, \theta) = (D_\theta\Psi)_\ell, \tag{74}$$

permits one to repeat the arguments that established Remark 2 to conclude that the inequalities (29) and (53) can be satisfied when the normal $(-\nabla\hat{t}(X), 1)$ points into the loose phase. However, for this orientation of the normal, the inequality

$$\varepsilon_c = (\Psi_c - \theta(D_\theta\Psi)_c) \geq (\Psi_\ell - \theta(D_\theta\Psi)_\ell) = \varepsilon_\ell \tag{75}$$

provided by (61) must be checked separately, since it generally is not a consequence of (29) and (53). To this end, we may use (74) to rewrite the inequality (75) as

$$\Psi_c - \Psi_\ell \geq \theta(D_\theta\Psi)_c - \theta(D_\theta\Psi)_\ell \geq 0.$$

This inequality, as well as both of (29) and (53), can be satisfied for ξ_c close to ξ_0 if we strengthen (74) as follows:

$$\begin{aligned}&\Psi(F(I + \xi_0\, a \otimes n), \theta) - \Psi(\zeta_{\min} I, \theta)\\ &\quad > \theta D_\theta\Psi(F(I + \xi_0\, a \otimes n), \theta) - \theta D_\theta\Psi(\zeta_{\min} I, \theta) > 0.\end{aligned} \tag{76}$$

8.3.3 Sufficient conditions for reversible transitions

The considerations in the previous subsubsections indicate in the present framework that phase transitions from loose to compact phases via moving planar interfaces are more widely available than transitions from compact to loose phases. Moreover, the sufficient conditions that we have provided for transitions in one sense and the sufficient conditions that we have provided for transitions in the opposite sense are mutually exclusive. Nevertheless, there are special circumstances under which transitions in both senses can occur.

Remark 3 Suppose that $b = 0$ and that, for a given deformation measure $\hat{\xi}_c \in I_{a,n} \setminus \{\xi_0\}$ for the compact phase, the Helmholtz free energy response function Ψ at the given, constant temperature field θ satisfies

$$D_\theta\Psi(F(I + \hat{\xi}_c\, a \otimes n), \theta) = D_\theta\Psi(\zeta_{\min} I, \theta) \tag{77}$$

and that the inequality (72), with $\xi_c = \hat{\xi}_c$, also is satisfied. It follows that both the compact-to-loose and the loose-to-compact transitions corresponding to the structured deformations $(g_c, \nabla g_c)$ and $(g_\ell, \zeta_{\min} I)$ are available to the body for $\xi_c = \hat{\xi}_c$ and for ξ_ℓ given by (64) with $\xi_c = \hat{\xi}_c$.

8.4 An infinite train of phase interfaces in the case of reversible transitions

We assume in this section that the body force field vanishes, so that (69) is satisfied identically in ξ_c, and we suppose that the sufficient conditions (77) and (72) for reversible transitions set forth in Remark 3 are satisfied. Consequently, we may choose a solution $\xi_c = \hat{\xi}_c$ of (77) that satisfies (72), and we denote by $\hat{\xi}_\ell$ the corresponding strain in the loose phase given by the formula (64) with $\xi_c = \hat{\xi}_c$. The formula (66) for $\hat{t}$ and the formula (65) for the velocity difference $v_c - v_\ell$ reduce by means of (77) to

$$\hat{t}(X) = \mp \frac{|Fa|\left\{2\rho_0(\Psi(F(I+\hat{\xi}_c\, a\otimes n))-\Psi(\zeta_{\min} I))\right\}^{1/2}}{D_G\Psi(F(I+\hat{\xi}_c\, a\otimes n),\theta)\cdot(Fa\otimes n)} n\cdot(X-X_0) \tag{78}$$

$$v_c - v_\ell = \pm \frac{\left\{2(\Psi(F(I+\hat{\xi}_c\, a\otimes n))-\Psi(\zeta_{\min} I))/\rho_0\right\}^{1/2}}{|Fa|} Fa, \tag{79}$$

respectively, and we note that, among the formulae and relations that govern jumps at the phase interface, only the formulas (78) and (79) involve the particular choice of sign $\pm$. If we put

$$\lambda := \lambda(F,\hat{\xi}_c,a,n,\theta) := \frac{|Fa|\left\{2\rho_0(\Psi(F(I+\hat{\xi}_c\, a\otimes n))-\Psi(\zeta_{\min} I))\right\}^{1/2}}{D_G\Psi(F(I+\hat{\xi}_c\, a\otimes n),\theta)\cdot(Fa\otimes n)} \tag{80}$$

and

$$\mu := \mu(F,\hat{\xi}_c,a,n,\theta) := \frac{\left\{2(\Psi(F(I+\hat{\xi}_c\, a\otimes n))-\Psi(\zeta_{\min} I))/\rho_0\right\}^{1/2}}{|Fa|}, \tag{81}$$

then (78) and (79) become

$$\hat{t}(X) = \mp\lambda n\cdot(X-X_0), \qquad v_c - v_\ell = \pm\mu Fa. \tag{82}$$

We use the notation $\hat{t}_+$ and $\hat{t}_-$ as well as $(v_c - v_\ell)_+$ and $(v_c - v_\ell)_-$ to indicate the choice of sign made in the formulas (82). We note also from (67) that the speed Ξ of a phase interface in the present context is given by

$$\Xi = \frac{\left|D_G\Psi(F(I+\hat{\xi}_c\, a\otimes n),\theta)\cdot(Fa\otimes n)\right|}{|Fa|\left\{2\rho_0\left((\Psi_c-\Psi_\ell)\right\}^{1/2}\right.} = |\lambda|^{-1}. \tag{83}$$

For definiteness, we assume that the stress component

$$D_G\Psi(F(I+\hat{\xi}_c\, a\otimes n),\theta)\cdot(Fa\otimes n)$$

is negative, so that the coefficient $\lambda = \lambda(F,\hat{\xi}_c,a,n,\theta)$ of $n\cdot(X-X_0)$ in (82) is negative. In addition, we may conclude from (64) that $\hat{\xi}_\ell > \hat{\xi}_c$. We now construct a continuous, piecewise smooth, motion of the semi-infinite reference slab of height H:

$$\mathcal{S} = \{X\in\mathcal{E} \mid 0 < n\cdot(X-X_0) < H\} \tag{84}$$

utilizing the motions g_c and g_ℓ defined in the relations (27) and (28) with $b=0$. It is convenient to indicate explicitly the dependence of these functions on v_c and v_ℓ, respectively:

$$g_c = g_c^{v_c} \quad\text{and}\quad g_\ell = g_\ell^{v_\ell}.$$

For a particular choice of velocity V in the compact phase, we require that the motion g under construction agree with g_c^V on the space-time region

$$\mathcal{R}_0 = \left\{(X,t) \mid X\in\mathcal{S} \text{ and } \lambda H \le t < \hat{t}_+(X) = \lambda n\cdot(X-X_0)\right\}, \tag{85}$$

so that we define

$$g = g_c^V \quad\text{and}\quad G = \nabla g_c^V \text{ on } \mathcal{R}_0. \tag{86}$$

We specify that a compact-to-loose transition occur on the space-time face $\mathcal{F}_{c,\ell}$ of $\mathcal{R}_0$ given by

$$\mathcal{F}_{c,\ell} := \big\{(X\,,t) \mid X \in \mathcal{S} \text{ and } t = \hat{t}_+(X) = \lambda n \cdot (X - X_0)\big\}. \tag{87}$$

Because λ is negative, this specification implies that the points in the slab nearer the plane $n \cdot (X - X_0) = H$ undergo the compact-to-loose transition before points farther from that plane. The choice of the plus sign "+" in the specification $t = \hat{t}_+(X)$ in (87) requires that we choose the velocity v_ℓ in the loose phase so that by (82)

$$V - v_\ell = v_c - v_\ell = (v_c - v_\ell)_- = -\mu F a.$$

Therefore, we have $v_\ell = V + \mu F a$, and we require that the motion g agree with $g_\ell^{V+\mu F a}$ on the space-time region

$$\mathcal{R}_1 = \left\{(X\,,t) \mid X \in \mathcal{S} \text{ and } \hat{t}_+(X) < t < \hat{t}_+(X) + \frac{H}{\Xi}\right\},$$

bounded by the parallel space-time planes "$t = \hat{t}_+(X)$" and "$t = \hat{t}_+(X) + \frac{H}{\Xi}$", that is, we define

$$g = g_\ell^{V+\mu F a} \quad \text{and} \quad G = \zeta_{\min} I \text{ on } \mathcal{R}_1. \tag{88}$$

Here, the quotient $H/\Xi = -\lambda H = |\lambda|\, H$ represents the time required for the phase interface to traverse the slab $\mathcal{S}$.

At this point we have defined g on the space-time region $\mathcal{R}_0 \cup \mathcal{R}_1$ in such a way that each point X in $\mathcal{S}$ from the time λH until the time $\hat{t}_+(X)$ is in the compact phase moving with velocity V. From the time $\hat{t}_+(X)$ until $\hat{t}_+(X) + |\lambda|\, H$ the point X is in the loose phase moving with velocity $V + \mu F a$. We note that the space-time plane "$t = \hat{t}_+(X) + |\lambda|\, H$" differs by a translation from "$t = \hat{t}_+(X)$" by amount $|\lambda|\, H$ in the t-direction. Although the untranslated plane "$t = \hat{t}_+(X)$" already has been confirmed as an interface that admits reversible transitions between the loose and compact phases, the translated plane has not been so identified. However, we here verify that the translated plane also admits such reversible transitions. In fact, if we seek a space-time surface "$t = \tilde{t}(X)$" such that a time-translated version $\tilde{g}_\ell$ of the loose phase deformation $g_\ell^{V+\mu F a}$ agrees with a corresponding time-translated version $\tilde{g}_c$ of $g_c^{v_c}$ on the space-time plane "$t = \tilde{t}(X)$", then we require

$$\begin{aligned}
&X_0 + F(I + \hat{\xi}_\ell a \otimes n)(X - X_0) + \tilde{t}(X)(V + \mu F a)\\
&= g_\ell^{v_\ell}(X, \tilde{t}(X))\\
&= \tilde{g}_c(X, \tilde{t}(X))\\
&= g_c^{v_c}(X, \tilde{t}(X) - t_1)\\
&= X_0 + F(I + \hat{\xi}_c a \otimes n)(X - X_0) + (\tilde{t}(X) - t_1)v_c.
\end{aligned}$$

We conclude from this relation that

$$\tilde{t}(X)(v_c - (V + \mu F a)) = (\hat{\xi}_\ell - \hat{\xi}_c)(n \cdot (X - X_0))F a + t_1 v_c$$

and, therefore, that

$$(v_c - (V + \mu F a)) \otimes \nabla \tilde{t}(X) = (\hat{\xi}_\ell - \hat{\xi}_c)F a \otimes n.$$

This relation agrees with (58), obtained in Sect. 8.1. The subsequent analysis in that subsection remains valid with $\nabla \hat{t}(X)$ replaced by $\nabla \tilde{t}(X)$, and we conclude that we may replace $\hat{t}(X)$ by $\hat{t}(X) + t_1$ and $g_c^{v_c}(X, t)$ by $g_c^{v_c}(X, t - t_1)$ throughout all the consequences of the jump conditions obtained above without affecting their validity.

Therefore, we may put $t_1 := H/\Xi = |\lambda|\, H = -\lambda H$ and specify that a loose-to-compact transition occur on the face

$$\mathcal{F}_{\ell,c} := \big\{(X\,,t) \mid X \in \mathcal{S} \text{ and } t = \hat{t}_+(X) + |\lambda|\, H\big\}$$

of $\mathcal{R}_1$ and, accordingly, we again must choose the minus sign "−" in relating the velocities $V + \mu F a$ in the existing loose phase and $v_c^{(1)}$ in the newly formed compact phase:

$$v_c^{(1)} - (V + \mu F a) = (v_c - v_\ell)_- = -\mu F a.$$

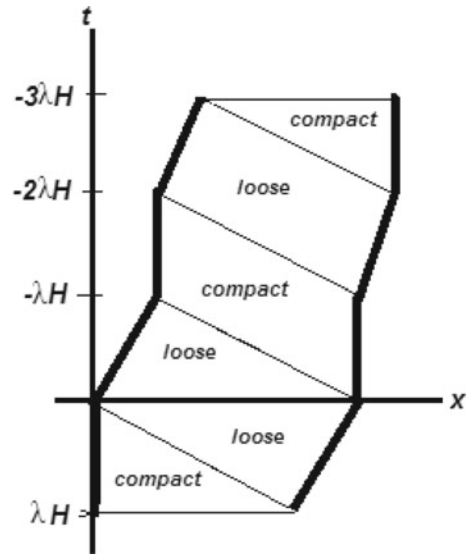

Fig. 1 Infinite train

We have $v_c^{(1)} = V$, that is, the velocity $v_c^{(1)}$ in the newly formed compact phase equals the velocity V in the original compact phase. We require that the motion g under construction be the corresponding time-translation of g_c^V on the space-time region

$$\mathcal{R}_2 = \left\{(X\,,t) \mid X \in \mathcal{S} \text{ and } \hat{t}_+(X) + |\lambda|\,H < t < \hat{t}_+(X) + 2\,|\lambda|\,H\right\},$$

so that we define

$$g(X,t) = g_c^V(X, t - |\lambda|\,H) \quad \text{and} \quad G(X,t) = \nabla g_c^V(X, t - |\lambda|\,H) \text{ on } \mathcal{R}_2. \tag{89}$$

We note that the above specification of (g, G) on the region $\mathcal{R}_0 \cup \mathcal{R}_1 \cup \mathcal{R}_2$ implies that at time $t = |\lambda|\,H$ the entire slab $\mathcal{S}$ again is in the compact phase moving with velocity V. This observation provides the basis for a recursive specification of (g, G) on $\mathcal{S} \times (0, \infty)$ in terms of an infinite train of moving phase interfaces. Although we do not provide the details of the recursive definition, we describe some features briefly. The pair (g, G) so obtained has the property that, for each non-negative integer m, at time $2m\,|\lambda|\,H$ the entire body is in the loose phase with $\xi_\ell = \hat{\xi}_\ell$, while at time $(2m\ -1)\,|\lambda|\,H$ the entire body is in the compact phase with $\xi_c = \hat{\xi}_c$. No matter what the time t, those points in the compact phase at time t move with velocity V, while those in the loose phase at time t move with velocity $V + \mu F a$. Each planar phase interface in the reference configuration forms at the end of the slab containing the point $X_0 + Hn$ and reaches the end of the slab containing the point X_0 at the same time as the next phase interface forms at the end containing $X_0 + Hn$.

For the case when $V = 0$, the velocity in the compact phase vanishes, and Fig. 1 indicates schematically the trajectories of the points X_0 (left-most broken line) and $X_0 + Hn$ (right-most broken line) for the time interval $(-|\lambda|\,H, 3\,|\lambda|\,H)$ during the motion just described.

9 Drastic reduction or increase in deformation levels via phase transitions

When $a \cdot n > 0$ the relation $\xi_c \in (\xi_0 - \delta, \xi_0)$ identified in Remark 2 for loose-to-compact transitions implies that that the stress component

$$D_G \Psi(F(I + \xi_c\, a \otimes n), \theta) \cdot (Fa \otimes n)$$

is negative, and the formula (64) tells us that $\xi_\ell > \xi_c$ and that ξ_ℓ tends to $+\infty$ as ξ_c tends to ξ_0 from below. Consequently, as the deformation parameter ξ_c varies in the bounded interval $(\xi_0 - \delta, \xi_0)$, the deformation parameter ξ_ℓ varies through an unbounded interval, and the unbounded family of macroscopic deformations $F(I + \xi_\ell a \otimes n)$ in the loose phase are transformed into the bounded family of macroscopic deformations

$F(I+\xi_c a\otimes n)$ in the compact phase. Analogously, when $a\cdot n<0$ the relation $\xi_c\in(\xi_0,\ \xi_0+\delta)$ identified in Remark 2 implies that the stress component $D_G\Psi(F(I+\xi_c\, a\otimes n),\theta)\cdot(Fa\otimes n)$ is positive, and the formula (64) yields $\xi_\ell<\xi_c$, with ξ_ℓ tending to $-\infty$ as ξ_c tends to ξ_0 from above. Similar conclusions can be drawn when $a\cdot n=0$.

Remark 4 For the family of loose-to-compact transitions in Remark 2, with the deformation parameter ξ_c near ξ_0, the deformation parameter $|\xi_\ell|$ tends to $+\infty$ as ξ_c tends to ξ_0, that is, the macroscopic deformation $F(I+\xi_\ell a\otimes n)$ in the loose phase before the transition is significantly larger than the deformation $F(I+\xi_c a\otimes n)$ present in the compact phase after the transition. Consequently, the loose-to-compact transition provides a mechanism for reducing significantly the macroscopic changes in shape associated with macroscopic deformations of the form $F(I+\xi a\otimes n)$. Similarly, for the family of compact-to-loose transitions described in Sect. 8.3.2, with the deformation parameter ξ_c near ξ_0, the deformation parameter $|\xi_\ell|$ tends to $+\infty$ as ξ_c tends to ξ_0, that is, the macroscopic deformation $F(I+\xi_c a\otimes n)$ in the compact phase before the transition is significantly smaller than the deformation $F(I+\xi_\ell a\otimes n)$ present in the loose phase after the transition. Consequently, the compact-to-loose transition provides a mechanism for increasing significantly the macroscopic changes in shape associated with macroscopic deformations of the form $F(I+\xi a\otimes n)$.

The higher level of deformation in the loose phase identified above suggests that, when overall changes in shape of the body at a given time t are limited by the environment, the dimensions of the region in the reference configuration occupied by the loose phase should be significantly smaller in the direction n normal to the phase interface $\hat{t}(X)=t$ than the dimensions of the adjacent region occupied by the compact phase. When multiple phase interfaces appear, we expect that the loose phase over time would be confined to narrow bands within the body. The formation of narrow bands of highly deformed material in a continuum usually is designated by the term "strain localization," frequently is observed and is studied in aggregates such as sand, and often is connected to a notion of material instability (see [15], for example).

10 Non-homogeneous deformations in the compact phase arising from plane progressive waves with small associated strains

Our analysis in Sects. 5–9 addresses the case in which each phase experiences a homogeneous deformation (27), (28). Here we broaden the analysis by allowing the compact phase to experience non-homogeneous deformations in the form of plane progressive waves with small associated strains. Consequently, we expect to encounter not only moving interfaces that separate loose and compact phases but also moving disturbances within the compact phase associated with the propagation of waves within a body with isotropic, linearly elastic response.

10.1 Motions in the loose and compact phases: balance laws

We replace the special class of motions (27), (28) by motions of the form

$$g_c(X,t)=X_0+F_c(X-X_0)+\varphi((X-X_0)\cdot n+st)\,e+\frac{t^2}{2\rho_0}b \tag{90}$$

$$g_\ell(X,t)=X_0+F_\ell(X-X_0)+tv_\ell+\frac{t^2}{2\rho_0}b, \tag{91}$$

where F_c and F_ℓ are tensors with positive determinant, φ is a mapping from a subset of $\mathbb{R}$ into $\mathbb{R}$, s is a real number, and v_ℓ, e, and n are vectors with $|e|=|n|=1$. Here, $|s|$ represents the speed of the progressive wave in the compact phase, n its orientation, and e its direction. Denoting the derivative of φ by φ', we have

$$\nabla g_c(X,t)=F_c+\varphi'((X-X_0)\cdot n+st)\,e\otimes n$$
$$\dot{g}_c(X,t)=s\varphi'((X-X_0)\cdot n+st)\,e+\frac{t}{\rho_0}b,$$
$$\ddot{g}_c(X,t)=s^2\varphi''((X-X_0)\cdot n+st)\,e+\frac{1}{\rho_0}b,$$

and it follows that

$$\begin{aligned} S(X,t) &= D_G\Psi(\nabla g_c(X,t),\theta) \\ &= D_G\Psi(F_c,\theta) + \varphi'((X-X_0)\cdot n + st)\, D_G^2\Psi(F_c,\theta)[e\otimes n] + o(\varphi'), \\ S(X,t)\cdot\nabla\dot{g}_c(X,t) &= s\varphi''((X-X_0)\cdot n + st)\, D_G\Psi(F_c,\theta)\cdot(e\otimes n) + O(\varphi'\varphi''), \end{aligned}$$

and

$$\mathrm{div} S(X,t) = \varphi''((X-X_0)\cdot n + st)\,(D_G^2\Psi(F_c,\theta)[e\otimes n])n + \mathrm{div}\, o(\varphi').$$

Consequently, the balance of linear momentum in the compact phase implies that

$$\varphi''((X-X_0)\cdot n + st)\,\left((D_G^2\Psi(F_c,\theta)[e\otimes n])n - \rho_0 s^2 e\right) = \mathrm{div}\, o(\varphi'),$$

and this relation will be satisfied to within terms of order $\mathrm{div}\, o(\varphi')$ if the linear mapping $v \longmapsto (D_G^2\Psi(F_c,\theta)[v\otimes n])n$ has e as an eigenvector with corresponding eigenvalue $\rho_0 s^2$. In particular, the linear mapping must have a positive eigenvalue in order that balance of linear momentum be satisfied to the given approximation for the special motion (90) with $s \neq 0$ and $\varphi''((X-X_0)\cdot n + st) \neq 0$:

$$(D_G^2\Psi(F_c,\theta)[e\otimes n])n = \rho_0 s^2 e. \tag{92}$$

This relation permits us to express the speed of the progressive wave in terms of the elasticity tensor $D_G^2\Psi(F_c,\theta)$ and the product $e\otimes n$:

$$|s| = (\rho_0^{-1} D_G^2\Psi(F_c,\theta)[e\otimes n]\cdot(e\otimes n))^{\frac{1}{2}}. \tag{93}$$

The First Law (33) in the case of a constant temperature field θ and zero external radiation r reduces to

$$(\psi + \theta\eta)^{\cdot} = S\cdot\nabla\dot{g}$$

which implies

$$D_G\Psi\cdot\dot{G} + \theta D_G(-D_\theta\Psi)\cdot\dot{G} = D_G\Psi\cdot\dot{F},$$

or, equivalently, for the compact phase (since $G = F$):

$$\begin{aligned} 0 &= D_\theta D_G\Psi(\nabla g_c,\theta)\cdot\nabla\dot{g}_c \\ &= \varphi''((X-X_0)\cdot n + st)\,\{sD_\theta D_G\Psi(F_c,\theta)\cdot(e\otimes n) + o(1)\}. \end{aligned}$$

Consequently, for the case $s \neq 0$ and for $\varphi''((X-X_0)\cdot n + st) \neq 0$, the First Law is satisfied in the compact phase to within terms of order $\varphi'\varphi''$ if and only if

$$D_\theta D_G\Psi(F_c,\theta)n\cdot e = D_\theta D_G\Psi(F_c,\theta)\cdot(e\otimes n) = 0. \tag{94}$$

This relation also guarantees the satisfaction in the compact phase of the Second Law to within terms of the same order. For the same reasons as given in Sect. 6, the field relations are satisfied exactly in the loose phase.

The Accommodation Inequality (5) is satisfied in the compact phase (with equality) and is satisfied in the loose phase if and only if

$$\zeta_{\min}^3 = \det G_\ell \leq \det\nabla g_\ell = \det F_\ell, \tag{95}$$

a condition that will be expressed below in terms of quantities defined in the compact phase by utilizing jump conditions at a phase interface.

10.2 Jump conditions at a phase interface

We assume that a phase interface as described in Sect. 8.1 separates loose and compact phases in the reference configuration, and we restrict our attention to the following special form of the mapping $\hat{t}$:

$$\hat{t}(X) = \tilde{t}((X - X_0) \cdot n) \tag{96}$$

with n a unit normal to the progressive wave (90) passing through the compact phase. In other words, the phase boundary and the progressive wave in the compact phase have the same orientation in the reference configuration. The condition that $g_\ell(X, \hat{t}(X)) = g_c(X, \hat{t}(X))$ to be satisfied for X in an open subset of the reference configuration now reads

$$(F_c - F_\ell)(X - X_0) + \varphi(\pi(X) + s\tilde{t}(\pi(X)))\, e - \tilde{t}(\pi(X))\, v_\ell = 0, \tag{97}$$

with $\pi(X) := (X - X_0) \cdot n$, and represents a restriction on the mapping $\tilde{t}$. The jump condition for balance of linear momentum (48) here takes the form

$$\begin{aligned}\tilde{t}\,'(\pi(X)) \left\{D_G\Psi(F_c, \theta)n + \varphi'(\pi(X) + s\tilde{t}(\pi(X))\, D_G^2\Psi(F_c, \theta)[e \otimes n]n + o(\varphi')\right\} \\ +\rho_0(s\varphi'(\pi(X) + s\tilde{t}(\pi(X)))\, e - v_\ell) = 0,\end{aligned}$$

and the relation (92) yields

$$\left\{\tilde{t}\,'(\pi(X)) D_G\Psi(F_c, \theta)n - \rho_0 v_\ell\right\} + \rho_0 s\varphi'\left(\pi(X)\left\{\tilde{t}\,'(\pi(X))s + 1\right\}\right) e + o(\varphi') = 0 \tag{98}$$

Sufficient conditions in order that this relation be satisfied to within terms $o(\varphi')$ are

$$v_\ell = \frac{\tilde{t}\,'(\pi(X))}{\rho_0} D_G\Psi(F_c, \theta)n \tag{99}$$

and $\tilde{t}\,'(\pi(X)) = -s^{-1}$, or

$$\hat{t}\,(X) = \tilde{t}(\pi(X))) = -\tfrac{1}{s}(X - X_0) \cdot n, \tag{100}$$

where, without loss of generality, we have imposed the requirement $\hat{t}\,(X_0) = 0$. In particular, (100) tells us that the speed of the phase interface equals the speed of the progressive wave in the compact phase. The relations (98), (99), and (100) then yield $\varphi(0) = 0$,

$$v_\ell = -\tfrac{1}{s\rho_0} D_G\Psi(F_c, \theta)n, \tag{101}$$

and

$$(F_c - F_\ell)(X - X_0) - \tfrac{1}{\rho_0 s^2}((X - X_0) \cdot n\,) D_G\Psi(F_c, \theta)n = 0.$$

The last relation implies the formula

$$\begin{aligned}F_\ell &= F_c - \tfrac{1}{\rho_0 s^2} D_G\Psi(F_c, \theta)n \otimes n \\ &= F_c - (\{(D_G^2\Psi(F_c, \theta)[e \otimes n])n\} \cdot e)^{-1} D_G\Psi(F_c, \theta)n \otimes n\end{aligned} \tag{102}$$

for the deformation F_ℓ in the loose phase in terms of quantities associated with the compact phase, and it also implies the following version of (91):

$$g_\ell(X, t) = X_0 + F_c(X - X_0) - \tfrac{1}{\rho_0 s^2}(n \cdot (X - X_0) + st) D_G\Psi(F_c, \theta)n + \frac{t^2}{2\rho_0} b. \tag{103}$$

From (102) we also have

$$\begin{aligned}\det F_\ell &= \det\left(F_c\left(I - \tfrac{1}{\rho_0 s^2} F_c^{-1} D_G\Psi(F_c, \theta)n \otimes n\right)\right) \\ &= \det F_c\left(1 - \tfrac{1}{\rho_0 s^2} D_G\Psi(F_c, \theta)n \cdot F_c^{-T} n\right),\end{aligned}$$

and the Accommodation Inequality in the loose phase (95) becomes

$$\zeta_{\min}^3 \leq \det F_c \left(1 - \tfrac{1}{\rho_0 s^2} D_G \Psi(F_c, \theta) n \cdot F_c^{-T} n\right). \tag{104}$$

The jump condition associated with the First Law

$$[(-S^T \dot{g} + q)] \cdot (-\nabla \hat{t}(X)) + \left[\varepsilon + \frac{1}{2}\rho_0 \left|\dot{g}\right|^2\right] = 0, \tag{105}$$

written when the normal to the phase interface points into the compact phase, leads to the individual terms

$$\begin{aligned}
&[(-S^T \dot{g} + q)] \cdot (-\nabla \hat{t}(X)) + o(\varphi') \\
&\quad = (s\varphi'(0)e - \tfrac{\pi(X)}{s\rho_0} b) \cdot (-\tfrac{1}{s}(D_G \Psi(F_c, \theta)n + \varphi'(0) D_G^2 \Psi(F_c, \theta)[e \otimes n]n) \\
&\quad = \tfrac{\pi(X)}{s^2 \rho_0} D_G \Psi(F_c, \theta) n \cdot b + \varphi'(0)(\pi(X) e \cdot b - D_G \Psi(F_c, \theta) n \cdot e),
\end{aligned} \tag{106}$$

$$\begin{aligned}
[\varepsilon] + o(\varphi') &= [\Psi - \theta D_\theta \Psi] \\
&= \Psi(F_c, \theta) - \theta D_\theta \Psi(F_c, \theta) - (\Psi(\zeta_{\min} I, \theta) - \theta D_\theta \Psi(\zeta_{\min} I, \theta)) \\
&\quad + \varphi'(0)\, (D_G \Psi(F_c, \theta) - \theta D_\theta D_G \Psi(F_c, \theta))\, n \cdot e,
\end{aligned} \tag{107}$$

$$\begin{aligned}
[|\dot{g}|^2] + o(\varphi') &= |\dot{g}_c|^2 - |\dot{g}_\ell|^2 \\
&= \left|s\varphi'(0)e + \tfrac{\hat{t}(X)}{\rho_0} b\right|^2 - \left|v_\ell + \tfrac{\hat{t}(X)}{\rho_0} b\right|^2 \\
&= \left|s\varphi'(0)e + \frac{-\frac{1}{s}\pi(X)}{\rho_0} b\right|^2 - \left|-\tfrac{1}{s\rho_0} D_G \Psi(F_c, \theta)n + \frac{-\frac{1}{s}\pi(X)}{\rho_0} b\right|^2 \\
&= -\tfrac{1}{s^2 \rho_0^2} \left|D_G \Psi(F_c, \theta) n\right|^2 - \tfrac{2\pi(X)}{s^2 \rho_0^2} D_G \Psi(F_c, \theta) n \cdot b \\
&\quad - 2\varphi'(0) \tfrac{\pi(X)}{\rho_0} e \cdot b.
\end{aligned} \tag{108}$$

The relations (106)–(108) then permit us to write (105) in the simple form

$$\begin{aligned}
&\Psi(F_c, \theta) - \theta D_\theta \Psi(F_c, \theta) - (\Psi(\zeta_{\min} I, \theta) - \theta D_\theta \Psi(\zeta_{\min} I, \theta)) \\
&\quad - \tfrac{1}{2\rho_0 s^2} \left|D_G \Psi(F_c, \theta) n\right|^2 - \varphi'(0) \theta D_\theta D_G \Psi(F_c, \theta) n \cdot e + o(\varphi') = 0.
\end{aligned} \tag{109}$$

In order that this relation be satisfied as $\left|\varphi'\right|$ tends to zero, it is sufficient that

$$\begin{aligned}
\Psi(F_c, \theta) - \theta D_\theta \Psi(F_c, \theta) &= (\Psi(\zeta_{\min} I, \theta) - \theta D_\theta \Psi(\zeta_{\min} I, \theta)) \\
&\quad + \tfrac{1}{2\rho_0 s^2} \left|D_G \Psi(F_c, \theta) n\right|^2
\end{aligned} \tag{110}$$

and

$$D_\theta D_G \Psi(F_c, \theta) n \cdot e = 0. \tag{111}$$

We note that the last relation is identical to the relation (94) that guarantees that the First Law is satisfied in the compact phase, so that (111) places no additional restriction on F_c, θ, n, and e.

The jump condition associated with the Second Law $0 \leq [\eta] = [-D_\theta \Psi]$, for the case when the normal to the phase interface points into the compact phase, takes the form

$$\begin{aligned}
0 \leq &-D_\theta \Psi(F_c, \theta) - \varphi'(0) D_G D_\theta \Psi(F_c, \theta) n \cdot e + o(\varphi') \\
&- (-D_\theta \Psi(\zeta_{\min} I, \theta))
\end{aligned}$$

and, in view of (94), reduces to

$$D_\theta \Psi(F_c, \theta) + o(\varphi') \leq D_\theta \Psi(\zeta_{\min} I, \theta). \tag{112}$$

This jump condition is satisfied if

$$D_\theta \Psi(F_c, \theta) \neq D_\theta \Psi(\zeta_{\min} I, \theta), \tag{113}$$

and the inequality determined by this relation together with (112) tell us whether the transition is "compact-to-loose" or "loose-to-compact." The jump condition (112) also is satisfied to within terms $o(\varphi')$ when

$$D_\theta \Psi(F_c, \theta) = D_\theta \Psi(\zeta_{\min} I, \theta), \tag{114}$$

and this relation implies that the transition is reversible, that is, that both the "compact-to-loose" transition and the "loose-to-compact" transition can occur.

Remark 5 Sufficient conditions on $F_c,\ n,\ e$, and θ for the existence of a moving interface (96) that separates the compact phase undergoing a progressive wave (90) and the loose phase undergoing the homogeneous motion (91) are the relations (92), (94), (104) and (110). If these relations are satisfied, then (a) the phase interface is given by (100) with speed $|s|$ equal to the speed of the progressive wave (93) and with orientation n and direction e those of the progressive wave, (b) the velocity in the loose phase v_ℓ is given by (99), and (c) the deformation in the loose phase F_ℓ is given by (102). Relations (113) and (114) characterize the cases of irreversible phase transitions and of reversible phase transitions, respectively.

11 Illustrative example

11.1 Special choice of Ψ

We consider in this section a specific choice of the Helmholtz free energy response function Ψ:

$$\Psi(G, \theta) := \frac{1}{2}\alpha(\theta)(\det G)^{-2} + \frac{1}{2}\beta(\theta) G \cdot G \tag{115}$$

where the elastic modulii α and β are smooth, positive-valued functions of temperature. This example appears in various contexts in the literature ([10], Section 4.10) often without explicit dependence of the coefficients on temperature. For each positive number θ the mapping $G \longmapsto \Psi(G, \theta)$ is easily seen to satisfy the conditions of purely dissipative disarrangements (8), isotropy (15), growth under extreme dilatations (17), and coincidence of minimizers (21) with

$$\zeta_{\min} := r(\theta)^{\frac{1}{8}} := \left(\frac{\alpha(\theta)}{\beta(\theta)}\right)^{\frac{1}{8}}. \tag{116}$$

In fact, the formula (115) implies that for each positive number ζ the free energy satisfies

$$\Psi(\zeta I, \theta) = \frac{\alpha(\theta)}{2\zeta^6} + \frac{3\beta(\theta)\zeta^2}{2},$$

and this relation implies the condition of growth under extreme dilatations (17). The minimum value of $\Psi(\zeta I, \theta)$ for $\zeta > 0$ equals $2\beta(\theta) r(\theta)^{\frac{1}{4}} = 2\beta^{\frac{3}{4}}(\theta)\alpha(\theta)^{\frac{1}{4}}$ and is attained when $\zeta = \zeta_{\min} = r(\theta)^{\frac{1}{8}}$, and the formula

$$D_G \Psi(G, \theta) = -\alpha(\theta)(\det G)^{-2} G^{-T} + \beta(\theta) G \tag{117}$$

implies that $D_G \Psi(G, \theta) = 0$ if and only if $G = r(\theta)^{\frac{1}{8}} Q$, so that, in particular, the stress vanishes at $\zeta_{\min} I$, that is, (19) holds. Moreover, because $G \longmapsto \Psi(G, \theta)$ satisfies the growth conditions (23), it follows that the minimum value of $\Psi(G, \theta)$ for $G \in \mathrm{Lin}^+$ is attained precisely at the tensors $G = r(\theta)^{\frac{1}{8}} Q$, with $Q \in \mathrm{Orth}^+$; in addition, the minimum value of $\Psi(G, \theta)$ for $G \in \mathrm{Lin}^+$ equals $2\beta^{\frac{3}{4}}(\theta)\alpha(\theta)^{\frac{1}{4}}$, the minimum value of $\Psi(\zeta I, \theta)$ for $\zeta > 0$. Consequently, the coincidence of minimizers (21) is satisfied. Finally, the response (115) is easily seen to be isotropic, and the particular form of its dependence on $\det G$ and $G \cdot G$ shows that Ψ is polyconvex and, hence, is rank-one convex [10,14].

11.2 Specialization of the free energy in the case of simple shears in each phase

We now specialize the main results in Sect. 8 to the free energy (115), and, in doing so, we also impose the following requirements on F, a, and n in (27) and (28):

$$F = I \quad \text{and} \quad a \cdot n = 0 \quad \text{and} \quad b \cdot n = 0. \tag{118}$$

The first two requirements imply that $F_c = \nabla g_c$ and $F_\ell = \nabla g_\ell$ both are of the form $I + \xi a \otimes n$, the deformation gradient for a simple shear. The third requirement implies that the relation (69) is satisfied identically in ξ_c. In fact, a routine calculation shows that

$$D_G\Psi(I + \xi_c\, a \otimes n, \theta) = (\beta(\theta) - \alpha(\theta))I + \xi_c\,(\alpha(\theta)n \otimes a + \beta(\theta)a \otimes n), \tag{119}$$

$$D_G\Psi(I + \xi_c\, a \otimes n, \theta)n = (\beta(\theta) - \alpha(\theta))n + \xi_c\,\beta(\theta)a \tag{120}$$

$$D_G\Psi(I + \xi_c\, a \otimes n, \theta) \cdot (a \otimes n) = \beta(\theta)\xi_c. \tag{121}$$

Consequently, the relation (69) specializes to

$$\begin{aligned} 0 &= \left\{ \frac{Fa}{|Fa|} - \frac{|Fa|}{D_G\Psi(F(I+\xi_c\, a\otimes n),\theta)\cdot(Fa\otimes n)} D_G\Psi(F(I + \xi_c a \otimes n, \theta)n \right\} \cdot b \\ &= \left\{ a - \frac{1}{\beta(\theta)\xi_c}((\beta(\theta) - \alpha(\theta))n + \xi_c\,\beta(\theta)a) \right\} \cdot b \\ &= -(\beta(\theta)\xi_c)^{-1}\,(\beta(\theta) - \alpha(\theta))n \cdot b, \end{aligned}$$

which is satisfied by virtue of (118). Moreover, ξ_0, the unique point ξ_c in $\mathbb{R}$ at which this stress component $D_G\Psi(I + \xi_c\, a \otimes n, \theta) \cdot (a \otimes n)$ vanishes, is given here by

$$\xi_0 = 0. \tag{122}$$

The Accommodation Inequality (72) now reduces to

$$\alpha(\theta) \le \beta(\theta) \tag{123}$$

or, equivalently, $r(\theta) \le 1$. Because of the formulas (115) and (116), the inequality (73) in Remark 2 becomes

$$(\alpha(\theta) - \beta(\theta))(\alpha'(\theta) + 3\beta'(\theta)) < 0. \tag{124}$$

In view of (123) and (124), the conclusions in Sect. 8.3.1 specialize here to the statement: *if* $\alpha(\theta) < \beta(\theta)$ *and* $\alpha'(\theta) + 3\beta'(\theta) > 0$, *then there exists* $\delta > 0$ *such that for every* $\xi_c \in (-\delta, 0) \cup (0, \delta)$ *the body admits a moving interface that transforms the loose phase into the compact phase with* $\nabla g_c = I + \xi_c a \otimes n$. Similarly, the conclusions in Sect. 8.3.2 become: *if* $\alpha(\theta) < \beta(\theta)$ *and* $\alpha'(\theta) + 3\beta'(\theta) < 0$, *then there exists* $\delta > 0$ *such that for every* $\xi_c \in (-\delta, 0) \cup (0, \delta)$ *the body admits a moving interface that transforms the compact phase into the loose phase with* $\nabla g_c = I + \xi_c a \otimes n$.

Finally, the results on reversible transitions in Sect. 8.3.3 in the present context read: *if* $\alpha(\theta) \le \beta(\theta)$ *and if any one of the following three conditions holds*

(1) $\alpha(\theta) = \beta(\theta)$ *and* $\beta'(\theta) = 0$
(2) $\alpha(\theta) < \beta(\theta)$ *and* $\alpha'(\theta) = \beta'(\theta) = 0$
(3) $\alpha(\theta) < \beta(\theta)$ *and* $\beta'(\theta) \neq 0$ *and* $(\alpha'(\theta) + 3\beta'(\theta))\beta'(\theta) > 0$,

then there exists $\hat{\xi}_c \in \mathbb{R}\backslash\{0\}$ *such that the body admits both a moving interface that transforms the loose phase to the compact phase and a moving interface that transforms the compact phase to the loose phase, with* $\nabla g_c = I + \hat{\xi}_c a \otimes n$. *In cases (1) and (2), the shearing parameter* $\hat{\xi}_c$ *can be any non-zero real number, while in case (3) we have*

$$\hat{\xi}_c = \pm \left(\frac{1 - r(\theta)^{3/4}}{r(\theta)^{3/4}} \left(\frac{\alpha'(\theta)}{\beta'(\theta)} + 3 \right) \right)^{1/2}. \tag{125}$$

Suppose now that the shearing parameter ξ_c for the compact phase lies in one of the intervals described in the results above or corresponds to one of the numbers $\hat{\xi}_c$ in the case of reversible transitions. The formulas for the speed of the phase interface Ξ, for the relative velocity $v_c - v_\ell$ of the two phases, for the corresponding shear ξ_ℓ in the loose phase, and for the phase interface $\hat{t}$ obtained in Sect. 8.2 then specialize to the following:

$$\Xi = \frac{\beta(\theta)|\xi_c|}{\sqrt{\rho_0}\left[(\alpha(\theta)-\theta\alpha'(\theta))\left\{1-r(\theta)^{-\frac{3}{4}}\right\}+(\beta(\theta)-\theta\beta'(\theta))\left\{3+\xi_c{}^2-3r(\theta)^{\frac{1}{4}}\right\}\right]^{\frac{1}{2}}}, \tag{126}$$

$$\begin{aligned}\sqrt{\rho_0}(v_c - v_\ell) &= \pm[(\alpha(\theta) - \theta\alpha'(\theta))(1 - r(\theta)^{-\frac{3}{4}}) \\ &\quad +(\beta(\theta) - \theta\beta'(\theta))(3 + \xi_c{}^2 - 3r(\theta)^{\frac{1}{4}})]^{1/2} a, \end{aligned} \tag{127}$$

$$\begin{aligned}\xi_\ell &= \xi_c - \tfrac{1}{\beta(\theta)\xi_c}\left[(\alpha(\theta) - \theta\alpha'(\theta))\left\{1 - r(\theta)^{-\frac{3}{4}}\right\}\right. \\ &\quad \left. +(\beta(\theta) - \theta\beta'(\theta))\left\{3 + \xi_c{}^2 - 3r(\theta)^{\frac{1}{4}}\right\}\right], \end{aligned} \tag{128}$$

$$\begin{aligned}\hat{t}(X) &= \mp\tfrac{\sqrt{\rho_0}}{\beta(\theta)\xi_c}\left[(\alpha(\theta) - \theta\alpha'(\theta))\left\{1 - r(\theta)^{-\frac{3}{4}}\right\}\right. \\ &\quad \left. +(\beta(\theta) - \theta\beta'(\theta))\left\{(3 + \xi_c{}^2 - 3r(\theta)^{\frac{1}{4}}\right\}\right]^{1/2} n\cdot(X - X_0), \end{aligned} \tag{129}$$

with, as above, $r(\theta) = \alpha(\theta)/\beta(\theta)$. (We omit the lengthy but straightforward verification of these formulas.)

11.3 Specialization of the free energy in the case of shearing waves in the compact phase

Finally, we specialize the main results in Sect. 10 to the free energy (115). In doing so, we also impose the following requirements on F_c, e, and n in (90):

$$F_c = I \quad \text{and} \quad e\cdot n = 0, \tag{130}$$

so that the deformation g_c in the compact phase assumes the form

$$g_c(X,t) = X + \varphi((X - X_0)\cdot n + st)e + \frac{t^2}{2\rho_0}b \tag{131}$$

with gradient

$$\nabla g_c(X,t) = I + \varphi'((X - X_0)\cdot n + st)e\otimes n,$$

while the deformation g_ℓ in the loose phase remains that given in the formula (91).

These specializations lead directly to the formulas

$$\begin{aligned} D_G\Psi(F_c,\theta) &= (-\alpha(\theta) + \beta(\theta))I, \\ D_G^2\Psi(F_c,\theta)[e\otimes n] &= \alpha(\theta)n\otimes e + \beta(\theta)e\otimes n, \end{aligned}$$

and to the following simplified version of (92):

$$\begin{aligned}(D_G^2\Psi(F_c,\theta)[e\otimes n])n &= \rho_0 s^2 e \iff \\ \beta(\theta)e &= \rho_0 s^2 e \iff \\ s^2 &= \frac{\beta(\theta)}{\rho_0}, \end{aligned} \tag{132}$$

so that the progressive wave in the compact phase is a shear wave with speed $\sqrt{\beta(\theta)/\rho_0}$.

The requirement (94) arising from the imposition of the First Law in the compact phase here becomes

$$\begin{aligned} 0 &= D_\theta D_G\Psi(F_c,\theta)n\cdot e = (-\alpha'(\theta) + \beta'(\theta))In\cdot e \\ &= (\beta'(\theta) - \alpha'(\theta))n\cdot e, \end{aligned}$$

and this relation is verified in view of the assumption (130). The remaining requirement (110) obtained in Sect. 10.2 specializes to the relation

$$(r(\theta)^{3/4} - 1)(\alpha(\theta) + 3\beta(\theta) - \theta(\alpha'(\theta) + 3\beta'(\theta))) = r(\theta)^{3/4}\frac{(\alpha(\theta) - \beta(\theta))^2}{\beta(\theta)}, \tag{133}$$

with $r(\theta) := \alpha(\theta)/\beta(\theta)$, and amounts to a restriction on the temperature θ.

The Accommodation Inequality (95) in the loose phase now reads

$$\zeta_{\min}^3 \le \det F_c(1 - \tfrac{1}{\rho_0 s^2} D_G \Psi(F_c, \theta) n \cdot F_c^{-T} n) \Longleftrightarrow$$
$$\left(\frac{\alpha(\theta)}{\beta(\theta)}\right)^{3/8} = \zeta_{\min}^3 \le 1 - \frac{1}{\beta(\theta)}(-\alpha(\theta) + \beta(\theta)) = \frac{\alpha(\theta)}{\beta(\theta)} \Longleftrightarrow$$
$$\beta(\theta) \le \alpha(\theta). \tag{134}$$

We note that this version of the Accommodation Inequality, when the loose phase is adjacent to a shear wave in the compact phase, is opposite to the version of the Accommodation Inequality (123) when the loose phase is adjacent to a (static) simple shear in the compact phase.

In summary, *for an elastic aggregate with the free energy* (115)*, the relations* (130), (133)*, and* (134) *are sufficient in order that a moving interface separate the loose wave undergoing the motion* (91) *and the compact wave undergoing the motion* (131)*. In particular, if the temperature* θ *is such that* $\alpha(\theta) = \beta(\theta)$*, then these sufficient conditions all are satisfied.*

The formula (102) for the deformation gradient F_ℓ in the loose phase, when adjacent to a shear wave in the compact phase, specializes here to

$$\begin{aligned} F_\ell &= F_c - \tfrac{1}{\rho_0 s^2} D_G \Psi(F_c, \theta) n \otimes n \\ &= I - \frac{1}{\beta(\theta)}(-\alpha(\theta) + \beta(\theta)) I n \otimes n \\ &= I + \frac{\alpha(\theta) - \beta(\theta)}{\beta(\theta)} n \otimes n, \end{aligned} \tag{135}$$

which represents an extension in the direction n. In contrast, we note from Sect. 11.2 that the deformation gradient $F_\ell = I + \xi_\ell a \otimes n$ in the loose phase, when adjacent to a (static) simple shear in the compact phase, is a simple shear. The formula (101) for the velocity v_ℓ in the loose phase (when the body force vanishes) specializes here to the relation

$$\frac{v_\ell}{s} = \frac{\alpha(\theta) - \beta(\theta)}{\beta(\theta)} n. \tag{136}$$

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