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Accepted 1999 November 11. Received 1999 October 6; in original form 1999 August 3

SUMMARY
We present a new numerical method to solve the heterogeneous elastic anisotropic wave equation with arbitrary high order accuracy in space and time on unstructured tetrahedral meshes. Using the most general Hooke’s tensor we derive the velocity-stress formulation leading to a linear hyperbolic system which accounts for the variation of the material properties depending on direction. This approach allows for the modeling of triclinic anisotropy, the most general crystalline symmetry class. The proposed method combines the Discontinuous Galerkin method with the ADER time integration approach using arbitrary high order derivatives of the piecewise polynomial representation of the unknown solution. In contrast to classical Finite Element methods discontinuities of this piecewise polynomial approximation are allowed at element interfaces, which allows for the application of the well-established theory of Finite Volumes and numerical fluxes across element interfaces obtained by the solution of derivative Riemann problems. Due to the ADER time integration technique the scheme provides the same approximation order in space and time automatically. Furthermore, through the projection of the tetrahedral elements of the physical space onto a canonical reference tetrahedron an efficient implementation is possible as many three-dimensional integral computations can be carried out analytically beforehand. A numerical convergence study confirms that the new scheme provides arbitrary high order accuracy even on unstructured tetrahedral meshes and shows that computational cost and storage space can be reduced by higher order schemes. We additionally include a way to couple anisotropy with viscoelastic attenuation based on the Generalized Maxwell Body rheology and the mean and deviatoric stress concepts. Besides, we present a new Godunov-type numerical flux for anisotropic material and compare its accuracy with a computationally simpler Rusanov flux. Finally, we validate the new scheme by comparing the results of our simulations to an analytic solution as well as to Spectral Element computations.

Key words: anisotropy, Discontinuous Galerkin method, high order accuracy, tetrahedral meshes

1 INTRODUCTION
Anisotropy is a common feature of materials and appears in the Earth at all scales. Exploration geophysics have for a long time paid attention to the anisotropic behavior of the seismic waves in the soil in order to resolve, for example, crack alignment (Crampin, Chesnokov & Hipkin 1984; Helbig 1994) in hydrocarbon reservoirs. For larger scales and global seismology it has been shown the existence of anisotropy in the crust and upper mantle induced by fluid-filled cracks (Crampin & Booth 1985) as well as strong stress regimes (Christensen 1984; Sharma & Garg 2006). Further observation of mantle anisotropy appears from SKS and SKKS shear wave splitting (Silver & Chan 1991). However these are not the only parts of the Earth were anisotropy has been observed and measured, as inner core travel time anomalies have also found a successful explanation in terms of material anisotropy (Morelli, Dziewonski & Woodhouse 1986; Woodhouse, Giardini & Li 1986). That observational proof has lead to thorough studies on the the influence of the anisotropic material properties on seismic wave propagation (Backus 1962; Cara 2002; Carcione 2002).

In addition, the improvements of our knowledge of the geological and geophysical properties of subsurface models in seismologically interesting regions often show highly complex geometries. This increasing complexity still presents a challenge for numerical methods based on...
regular, structured griding. On the other hand, numerical methods on geometrically more flexible unstructured tetrahedral meshes until today could not provide high order accuracy. Therefore, most approaches were forced to find a compromise between preserving the complexity of the models and having highly accurate results.

In the past, many approaches describing anisotropic wave propagation have been developed. Early attempts aimed at the simplification of anisotropic effects for some weakly anisotropic media (Thomsen 1986; Song, Every & Wright 2001). Analytical and quasi-analytical solutions to simplified cases exist and ray theory can handle the problem to some extent (Červený 1972). However, when heterogeneous materials and complex geometrical structures are involved only three-dimensional, full wave form simulations are able to provide satisfying results. The most widely used method, the Finite Difference (FD) method, has successfully been extended from isotropic (Madariaga 1976; Virieux 1984; Virieux 1986) to anisotropic problems using staggered (Mora 1989; Igel, Mora & Riollet 1995) or rotated staggered grids (Saenger & Bohlen 2004). However, both approaches are forced to interpolate stress and strain off-diagonal values as they are not defined in the same grid points. Pseudospectral (PS) methods (Carcione 1994; Tessmer & Kosloff 1994; Fornberg 1996; Igel 1999) have been extended to handle anisotropic material in (Carcione, Kosloff & Kosloff 1988; Tessmer 1995; Hung & Forsyth 1998). More recently, the Spectral Element Method (SEM) has considerably gained in popularity due to its accuracy and efficiency on deformable hexahedral elements (Komatitsch & Tromp 1999) and has also been further developed for anisotropic problems in (Seriani, Priolo & Pregarz 1995; Komatitsch, Barnes & Tromp 2000) and successfully been applied to the case of global seismic wave propagation in (Komatitsch & Tromp 2002). Recent attempts of incorporating anisotropy in fully unstructured grids in (Gao & Zhang 2006) represent an alternative approach.

We also aim at accurately coupling anisotropic with viscoelastic effects for wave propagation problems. Initial attempts have centered themselves in using complex values for each of the 21 independent coefficients of the general Hooke’s tensor (Auld 1990). Alternatively the concept of eigenstresses and eigenstiffnesses were developed in (Carcione & Cavallini 1994) which can be implemented in the time domain but can be computationally expensive. We choose to follow a third approach, similar to that presented in (Carcione 1995), based on the concepts of mean and deviatoric stresses. This formulation ensures the existence of only four different attenuating modes, one purely dilatational and three shear modes. This way a viscoelastic-anisotropic scheme can be implemented based on memory variables, making it both accurate and efficient in time-domain numerical modeling.

In this paper, we extend the Discontinuous Galerking (DG) approach with a time integration approach using Arbitrary high order DERivatives (ADER) presented in (Käser & Dumbser 2006; Dumbser & Käser 2006; Käser, Dumbser, de la Puente & Igel 2006) to the three-dimensional anisotropic case for the most general triclinic crystalline symmetry class. The ADER-DG method provides arbitrary high order accuracy in space and time on unstructured tetrahedral meshes, which makes it very attractive when complicated model geometries are involved. To our knowledge the ADER-DG approach provides the first numerical scheme achieving high order polynomial approximation for anisotropic seismic wave propagation on three-dimensional unstructured meshes.

The paper is structured as follows. In Section 2 we present the linear hyperbolic system of the anisotropic seismic wave equations in velocity-stress formulation. In Section 3 we show the extension of the ADER-DG scheme to anisotropic material with particular focus on a new Godunov-type numerical flux. The coupling of anisotropy and viscoelastic attenuation is derived in Section 4. A convergence study is presented in Section 5 in order to validate the high order accuracy of the new ADER-DG scheme for anisotropic material. Finally, in Section 6 we demonstrate different application examples to confirm the performance of the proposed method by comparisons of our results with analytical solutions and results of the Spectral Element Method. Section 7 summarizes the work presented and provides concluding remarks.

## 2 ANISOTROPIC SEISMIC WAVE EQUATIONS

The most general linear elastic stress-strain relation can be expressed through a tensorial constitutive law (Hooke’s Law), see e.g. (Stein & Wysession 2003), of the form

\[
\sigma_{ij} = c_{ijkl} \varepsilon_{kl} \tag{1}
\]

The entries of the fourth-order elasticity tensor \(c_{ijkl}\) can be reduced to 21 independent real coefficients in the most general case due to symmetry considerations. Using matrix notation, the stresses \(\sigma_{ij}\) and strains \(\varepsilon_{ij}\) are defined as vectors \(\vec{\sigma} = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yx}, \sigma_{xy}, \sigma_{zx}, \sigma_{zy}, \sigma_{yz}, \sigma_{xz})^T\) and \(\vec{\varepsilon} = (\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{yx}, \varepsilon_{xy}, \varepsilon_{zx}, \varepsilon_{zy}, \varepsilon_{yz}, \varepsilon_{xz})^T\) and we can rewrite (1) using an anisotropic elastic matrix \(M_{ij}\) as

\[
\vec{\sigma} = M_{ij} \vec{\varepsilon} \tag{2}
\]
which extended in more detail reads as

\[
\begin{pmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\sigma_{yx} \\
\sigma_{xy} \\
\sigma_{xz}
\end{pmatrix} = \begin{pmatrix}
c_{11} & c_{12} & c_{13} & 2c_{14} & 2c_{15} & 2c_{16} \\
c_{12} & c_{22} & c_{23} & 2c_{24} & 2c_{25} & 2c_{26} \\
c_{13} & c_{23} & c_{33} & 2c_{34} & 2c_{35} & 2c_{36} \\
c_{14} & c_{24} & c_{34} & c_{44} & 2c_{45} & 2c_{46} \\
c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & 2c_{56} \\
c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66}
\end{pmatrix} \begin{pmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz} \\
\varepsilon_{yx} \\
\varepsilon_{xy} \\
\varepsilon_{xz}
\end{pmatrix}.
\]

Considering all 21 independent coefficients in \( M_{ij} \) we can represent a triclinic material, which is the most general case of anisotropy and includes as special cases all other crystalline symmetry classes, i.e. monoclinic, trigonal, tetragonal, orthorhombic, hexagonal, cubic and isotropic, as shown in (Nye 1958; Okaya & McEvilly 2003). Therefore, isotropy can be understood as the particular case in which \( c_{11} = c_{22} = c_{33} = \lambda + 2\mu, c_{12} = c_{13} = c_{23} = \lambda, c_{44} = c_{55} = c_{66} = \mu \) and all other coefficients equal to zero. In non-isotropic cases the actual values of the coefficients of the matrix \( M_{ij} \) in (3) depend on the orientation of the reference system we use. Certain anisotropic symmetry classes exhibit symmetry axes. Therefore, appropriate reference systems can be chosen in a way that structured grids are aligned to these symmetry axes. However, when modeling anisotropic wave propagation on unstructured meshes the reference coordinate system for the physical coordinates of mesh nodes. In the following, however, we consider the elastic properties of the anisotropic medium referring to the underlying physical coordinate system that also defines the orientation of stresses and strains or the physical coordinates of mesh nodes.

Combining the constitutive relation in (3) with the equations of motion, see e.g. (LeVeque 2002), leads to a complete partial differential equation system of the shape

\[
\frac{\partial Q}{\partial t} + A_{pq} \frac{\partial Q}{\partial x} + B_{pq} \frac{\partial Q}{\partial y} + C_{pq} \frac{\partial Q}{\partial z} = 0,
\]

where \( Q \) is the vector of the unknown stresses and velocities, i.e. \( Q = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, u, v, w)^T \). Note, that here the ordering of stresses in the vector \( Q \) is different from the one typically used for the stress-strain relation in eq. (3). This ordering is chosen in order to be consistent with the formulation introduced in previous work (Käser & Dumbser 2006; Dumbser & Käser 2006; Käser, Dumbser, de la Puente & Igel 2006). In addition, classical tensor notation is used in (4), which implies summation over each index that appears twice. The matrices \( A_{pq} = A_{pq}(\vec{x}), B_{pq} = B_{pq}(\vec{x}), \) and \( C_{pq} = C_{pq}(\vec{x}), \) where \( \vec{x} = (x, y, z), \) are the space dependent Jacobian matrices, with \( p, q = 1, \ldots, 9, \) and are given through

\[
A_{pq} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -c_{11} & -c_{16} & -c_{15} \\
0 & 0 & 0 & 0 & 0 & 0 & -c_{12} & -c_{16} & -c_{25} \\
0 & 0 & 0 & 0 & 0 & 0 & -c_{13} & -c_{36} & -c_{35} \\
0 & 0 & 0 & 0 & 0 & 0 & -c_{14} & -c_{46} & -c_{45} \\
0 & 0 & 0 & 0 & 0 & 0 & -c_{15} & -c_{56} & -c_{55} \\
0 & 0 & 0 & 0 & -\frac{1}{p} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{p} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
B_{pq} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -c_{16} & -c_{12} & -c_{14} \\
0 & 0 & 0 & 0 & 0 & 0 & -c_{26} & -c_{22} & -c_{24} \\
0 & 0 & 0 & 0 & 0 & 0 & -c_{36} & -c_{33} & -c_{34} \\
0 & 0 & 0 & 0 & 0 & 0 & -c_{46} & -c_{46} & -c_{46} \\
0 & 0 & 0 & 0 & 0 & 0 & -c_{56} & -c_{56} & -c_{56} \\
0 & 0 & 0 & 0 & -\frac{1}{p} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{p} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
\[ C_{pq} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -c_{15} & -c_{14} & -c_{13} \\
0 & 0 & 0 & 0 & 0 & 0 & -c_{25} & -c_{24} & -c_{23} \\
0 & 0 & 0 & 0 & 0 & 0 & -c_{35} & -c_{34} & -c_{33} \\
0 & 0 & 0 & 0 & 0 & 0 & -c_{45} & -c_{44} & -c_{43} \\
0 & 0 & 0 & 0 & 0 & 0 & -c_{55} & -c_{54} & -c_{53} \\
0 & 0 & 0 & 0 & -\frac{1}{\rho} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{\rho} & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \]

(7)

where the coefficients \( c_{ij} \) are those of the anisotropic elastic matrix \( M_{ij} \) of (2) or (3) and \( \rho \) is the mass density of the material. We remark that analytically determining the eigenstructure the Jacobian matrices defined in (5), (6) and (7) is much more difficult for the anisotropic case than for the purely isotropic case as presented in the previous work (Dumbser & Käser 2006; Käser, Dumbser, de la Puente & Igel 2006). As shown in the following Section 3 this leads to modifications in the formulation of the ADER-DG scheme.
3 THE NUMERICAL SCHEME

The computational domain $\Omega \in \mathbb{R}^3$ is divided into conforming tetrahedral elements $T^{(m)}$ being addressed by a unique index $(m)$. Furthermore, we suppose the matrices $A_{pq}$, $B_{pq}$, and $C_{pq}$ to be piecewise constant inside an element $T^{(m)}$. The numerical solution $Q_h$ of equation (4) is approximated as shown in (Dumbser & Käser 2006) inside each tetrahedron $T^{(m)}$ by a linear combination of space-dependent but time-independent polynomial basis functions $\Phi_i(\xi, \eta, \zeta)$ of degree $N$ with support $T^{(m)}$ and with only time-dependent degrees of freedom $Q_{p}^{(m)}(t)$, i.e.,

$$\left( Q_{h}^{(m)} \right)_p (\xi, \eta, \zeta, t) = \tilde{Q}_{p}^{(m)}(t) \Phi_i(\xi, \eta, \zeta),$$

(8)

where $\xi$, $\eta$ and $\zeta$ are the coordinates in a canonical reference element $T_E$. For a detailed definition of these coordinates together with the basis functions $\Phi_i$ see (Dumbser & Käser 2006; Dumbser, Käser & de la Puente 2006). Multiplying (4) by the test function $\Phi_k$ and integrating over a tetrahedral element $T^{(m)}$ gives

$$\int_{T^{(m)}} \Phi_k \frac{\partial Q_p}{\partial t} dV + \int_{T^{(m)}} \Phi_k \left( A_{pq} \frac{\partial Q_p}{\partial x} + B_{pq} \frac{\partial Q_p}{\partial y} + C_{pq} \frac{\partial Q_p}{\partial z} \right) dV = 0.$$  

(9)

By applying integration by parts to the last integral of (9) we obtain

$$\int_{T^{(m)}} \Phi_k \frac{\partial Q_p}{\partial t} dV + \int_{\partial T^{(m)}} \Phi_k F_p^h dS - \int_{T^{(m)}} \left( \frac{\partial \Phi_k}{\partial x} A_{pq} Q_p + \frac{\partial \Phi_k}{\partial y} B_{pq} Q_p + \frac{\partial \Phi_k}{\partial z} C_{pq} Q_p \right) dV = 0,$$

(10)

where a numerical flux $F_p^h$ has been introduced in the surface integral since $Q_h$ may be discontinuous at an element boundary. Here, two major changes with respect to the isotropic case appear.

First, we need to introduce the matrix $\tilde{A}^{(m)}$ which is similar to the matrix $A$ of (5), however, with the entries $c_{ij}$ rotated from the global coordinate system to a local coordinate system of a tetrahedron’s face. This local coordinate system is defined by the normal vector $\vec{n} = (n_x, n_y, n_z)^T$ and the two tangential vectors $\vec{s} = (s_x, s_y, s_z)^T$ and $\vec{t} = (t_x, t_y, t_z)^T$, which lie in the plane determined by the face of the tetrahedron and are orthogonal to each other and to the normal vector $\vec{n}$. Usually we define vector $\vec{s}$ such that it points from the local face node 1 to the local face node 2. The exact definitions of the vectors $\vec{n}$, $\vec{s}$ and $\vec{t}$ as well as the local vertex numbering of a tetrahedral element can be found in (Dumbser & Käser 2006). The rotation to this local coordinate system is done by applying the so-called Bond’s matrix $\mathcal{N}$ (Bond 1976; Okaya & McEvilly 2003)

$$\mathcal{N} = \begin{pmatrix} n_x^2 & n_y^2 & n_z^2 & 2n_x n_y & 2n_x n_z & 2n_y n_x \\ s_x^2 & s_y^2 & s_z^2 & 2s_x s_y & 2s_x s_z & 2s_y s_x \\ t_x^2 & t_y^2 & t_z^2 & 2t_x t_y & 2t_x t_z & 2t_y t_x \\ s_x t_x & s_y t_y & s_z t_z & s_x t_y + s_z t_x & s_y t_x + s_z t_y & s_y t_x + s_z t_y \\ t_x n_x & t_y n_y & t_z n_z & n_x t_x + n_z t_y & n_x t_y + n_z t_x & n_x t_y + n_z t_x \\ n_x s_x & n_y s_y & n_z s_z & n_x s_y + n_z s_x & n_x s_z + n_y s_x & n_x s_z + n_y s_y \end{pmatrix}$$

(11)

To the Hooke’s matrix $\mathcal{C}$ of the global reference system

$$\mathcal{C} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{pmatrix}$$

(12)

leading to the Hooke’s matrix $\tilde{\mathcal{C}}$ in the local reference system of the tetrahedron’s boundary face

$$\tilde{\mathcal{C}} = \mathcal{N} \cdot \mathcal{C} \cdot \mathcal{N}^T.$$

(13)

We remark that in the isotropic case matrix $\mathcal{C}$ is invariant under coordinate transformation, i.e. $\tilde{\mathcal{C}}_{iso} = \mathcal{C}_{iso}$. Therefore, this rotation could be skipped for the isotropic case discussed in (Dumbser & Käser 2006; Käser, Dumbser, de la Puente & Igel 2006).

The second modification comes through the different approaches for the numerical flux computation. The general definition of our numerical flux incorporating anisotropic material can be written as

$$F^h_p = \frac{1}{2} \left( T_{pq} \tilde{Q}^{(m)}_{q} (T_{rs})^{-1} - \Theta_{ps} \right) Q_{st}^{(m)} \phi^{(m)}_{t} + \frac{1}{2} \left( T_{pq} \tilde{Q}^{(m)}_{q} (T_{rs})^{-1} - \Theta_{ps} \right) Q_{st}^{(m)} \phi^{(m)}_{t},$$

(14)

where $\tilde{Q}^{(m)}$ and $\phi^{(m)}$ are the boundary extrapolated values of the numerical solution from element $T^{(m)}$ and its $j$-th side neighbor $T^{(m)j}$, respectively. To simplify notation, in the following, we drop the index $j$ indicating the $j$-th face of the tetrahedron $T^{(m)}$. The rotation matrix $T_{pq}$ that transforms all variables of $Q_p$ from (4) into the reference system associated to the tetrahedron’s $j$-th face has...
the same expression as in the isotropic case in (Dumbser & Käser 2006) and reads as
\[
T_{pq} = \begin{pmatrix}
  n_x^2 & s_x^2 & t_x^2 & 2n_x s_x & 2s_x t_x & 2n_x t_x & 0 & 0 & 0 \\
  n_y^2 & s_y^2 & t_y^2 & 2n_y s_y & 2s_y t_y & 2n_y t_y & 0 & 0 & 0 \\
  n_z^2 & s_z^2 & t_z^2 & 2n_z s_z & 2s_z t_z & 2n_z t_z & 0 & 0 & 0 \\
  n_x n_x & s_x s_x & t_x t_x & n_x s_x + n_x s_y & s_x t_x + s_x t_y & n_x t_x + n_x t_y & 0 & 0 & 0 \\
  n_y n_y & s_y s_y & t_y t_y & n_y s_y + n_y s_z & s_y t_y + s_y t_z & n_y t_y + n_y t_z & 0 & 0 & 0 \\
  n_z n_z & s_z s_z & t_z t_z & n_z s_z + n_z s_x & s_z t_z + s_z t_x & n_z t_z + n_z t_x & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & n_x & s_x & t_x \\
  0 & 0 & 0 & 0 & 0 & 0 & n_y & s_y & t_y \\
  0 & 0 & 0 & 0 & 0 & 0 & n_z & s_z & t_z
\end{pmatrix} .
\]

(15)

The matrix \( \Theta_{ps} \) is a numerical viscosity term whose particular form determines the flux type we wish to use and depends on the orientation of the interface with the j-th side neighbor. In the following, we introduce the Godunov flux and the Rusanov flux which have a numerical viscosity matrix of the form
\[
(\Theta_{ps})_{\text{Godunov}} = T_{pq} \left| \bar{A}_{qr} \right| T_{rs} ,
\]

(16)

\[
(\Theta_{ps})_{\text{Rusanov}} = \alpha_{\text{max}} I_{ps} ,
\]

(17)

where \( I_{ps} \) is the identity matrix. The computation of the Godunov flux requires knowledge of the eigenstructure of the Jacobian matrix \( \bar{A}_{qr} \).

This, for the anisotropic case, is a non-trivial issue as it requires the computation of \( |\bar{A}_{qr}| \), which in turn usually requires the knowledge of the left and right eigenvectors of \( \bar{A}_{qr} \). A new method to obtain the Godunov flux in (17) for anisotropic material is presented in the Appendix A. Alternatively, the Rusanov flux (LeVeque 2002) requires only the knowledge of the maximum eigenvalue of \( \bar{A}_{qr} \). This value is \( \alpha_{\text{max}} = \max(\alpha_i) \), where \( \alpha_i \) are the roots of the following polynomial of \( \alpha \)
\[
XYZ - Xc_{15}^2 - Yc_{16}^2 - Zc_{16}^2 + 2c_{15}c_{16}c_{56} = 0 ,
\]

(18)

where the coefficients \( c_{ij} \) are the entries of the Hooke’s matrix \( \bar{C} \) of (13) rotated into the local reference system of the j-th side of the tetrahedral element. Furthermore, we used the substitutions \( X = c_{33} - \alpha^2 \rho, Y = c_{56} - \alpha^2 \rho \) and \( Z = c_{56} - \alpha^2 \rho \). As can be seen from (18) we are searching the maximum value of the possibly 6 roots of polynomial of degree 6. However, the substitutions using \( X, Y \) and \( Z \) tell us, that there are only three different values to search for, as (18) represents a cubic polynomial of \( \alpha^2 \). We can exclude the possibility of having complex eigenvalues, i.e. \( \alpha^2 < 0 \), as this would imply the loss of hyperbolicity of the PDE system in (4). The physical interpretation of the eigenvalues is that they represent the speed at which the resulting waves are propagating in normal direction through the j-th element face. This is a known result for the anisotropic phase wave speeds (Crampin 1981) which appears here naturally from the eigendecomposition of the jacobians of our scheme (5). In general the resulting waves are called quasi-waves \( qP, qS_1 \) and \( qS_2 \); ordered in decreasing magnitude of their velocities (Crampin 1981). For the isotropic case we would get the positive and negative P-wave velocity and two positive and negative S-wave velocity of the same absolute value. These values correspond to the two differently polarized S-waves, perpendicular to each other.

Once the maximum eigenvalue \( \alpha_{\text{max}} \) is determined, the Rusanov flux is given via (16). As the full derivation of the numerical scheme would go beyond the scope of this work we refer the reader to previous work (Käser & Dumbser 2006; Dumbser & Käser 2006) for the mathematical details. Instead we give the final form of the fully discrete ADER-DG scheme, which after transformation into the canonical reference element \( T_E \) and time integration over one time step \( \Delta t \) from time level \( n \) to the following time level \( n + 1 \) reads as
\[
\begin{align*}
\left[ \left( \hat{Q}_{mt}^{(n)} \right)^{n+1} - \left( \hat{Q}_{mt}^{(n)} \right)^{n} \right] \left| J \right| M_{kl} & + \\
+ \frac{1}{2} \sum_{j=1}^{4} \left( T_{pq} \bar{A}_{qr} \left( T_{rs} \right)^{-1} + \Theta_{ps} \right) \left| S_j \right| F_{kl}^{-j} \cdot I_{qlmn} \left( \Delta t \right) \left( \hat{Q}_{ml}^{(n)} \right)^{n} & + \\
+ \frac{1}{2} \sum_{j=1}^{4} \left( T_{pq} \bar{A}_{qr} \left( T_{rs} \right)^{-1} - \Theta_{ps} \right) \left| S_j \right| F_{kl}^{+j} \cdot I_{qlmn} \left( \Delta t \right) \left( \hat{Q}_{ml}^{(n)} \right)^{n} & - \\
- A_{pq} \left| J \right| K_{kl} \cdot I_{qlmn} \left( \Delta t \right) \left( \hat{Q}_{ml}^{(n)} \right)^{n} - B_{pq} \left| J \right| K_{kl}^{*} \cdot I_{qlmn} \left( \Delta t \right) \left( \hat{Q}_{ml}^{(n)} \right)^{n} - C_{pq} \left| J \right| K_{kl}^{*} \cdot I_{qlmn} \left( \Delta t \right) \left( \hat{Q}_{ml}^{(n)} \right)^{n} & = 0 ,
\end{align*}
\]

(19)

where \( I_{qlmn}(\Delta t) \) represents the high order ADER time integration operator that is applied to the degrees of freedom \( \left( \hat{Q}_{ml}^{(n)} \right)^{n} \) at time level \( n \). The matrices \( M_{kl}, F_{kl}^{-j} \) and \( K_{kl} \) are the mass matrix, flux and stiffness matrices, respectively, and include space integrations of the basis functions that can be computed beforehand as shown in more detail in (Dumbser & Käser 2006). \( A_{pq}, B_{pq} \) and \( C_{pq} \) are the Jacobian matrices transformed into the reference tetrahedron \( T_E \). Furthermore, \( |J| \) is the determinant of the Jacobian matrix of this transformation, see the Appendixin (Dumbser & Käser 2006), and \( |S_j| \) denotes the area of the j-th face of tetrahedron \( T \).
the unknown variables from a time-step $t^n$ to a following $t^{n+1}$ without storing any intermediate values as typically necessary for classical multi-stage Runge-Kutta time-stepping schemes. Furthermore, the scheme has a very local character, as the evolution of the variables in time within the element $T^{(m)}$ depends only on the variables associated to the element $T^{(m)}$ itself and its direct neighbours $T^{(m, j)}$, with $j = 1, ..., 4$.

## 4 COUPLING OF ANISOTROPY AND VISCOELASTICITY

Anisotropy and viscoelastic attenuation play an important role as secondary effects in seismic wave propagation modeling. The incorporation of anisotropy into the ADER-DG framework has been discussed in the previous Sections 2 and 3. Viscoelastic attenuation, however, was introduced in (Käser, Dumbser, de la Puente & Igel 2006). In order to accurately couple both effects we use the concepts of mean and deviatoric stresses first presented in (Carcione 1995) and adapt them to the rheological model of the Generalized Maxwell Body (GMB) as suggested in (Emmerich & Korn 1987). At first we define the underlying physical theory of viscoelastic anisotropy. Then we present in detail how viscoelasticity changes the anisotropic PDE system as given in (4). Finally, we explain how the ADER-DG scheme presented in Section 3 has to be modified in order to couple anisotropy and viscoelasticity.

### 4.1 Viscoelastic Anisotropic Wave Propagation

The mean stress $\mathbf{\sigma}$ and mean strain $\mathbf{\varepsilon}$, as well as the deviatoric stress $\mathbf{\sigma}^D$ and deviatoric strain $\mathbf{\varepsilon}^D$ are defined as

$$\mathbf{\sigma} \equiv \frac{1}{3} (\sigma_{xx} + \sigma_{yy} + \sigma_{zz}),$$

$$\mathbf{\varepsilon} \equiv \frac{1}{3} (\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}),$$

$$\mathbf{\sigma}^D \equiv \mathbf{\sigma} - \mathbf{\varepsilon},$$

$$\mathbf{\varepsilon}^D \equiv \mathbf{\varepsilon} - \mathbf{\sigma},$$

where we remark that the mean stress and strain are invariant under coordinate transformation. As shown in (Carcione 2002) we need a total of four attenuation moduli to model viscoelastic attenuation in an anisotropic medium: one purely dilatational modulus and three shear moduli. Those are associated to the four possible viscoelastic modes we can have in an anisotropic medium, and which we will refer to as modes $k = 1$ (dilatational) and $k = 2, 3, 4$ (shear). It can be shown that the mean stress $\mathbf{\sigma}$ depends only on the dilatational modulus while the deviatoric stress $\mathbf{\sigma}^D$ only depends on the shear moduli. The stress-strain relation in the general anisotropic case can be expressed in the frequency domain or in the time domain, e.g. see (Moczo, Kristek & Halada 2004) for the isotropic case, which reads in the anisotropic case (Carcione 1995) as

$$\sigma_i(\omega) = M_{ij}(\omega)\varepsilon^j(\omega),$$

$$\sigma_i(t) = \frac{\partial}{\partial t} (\Psi_{ij}(t)) * \varepsilon^j(t) = \Psi_{ij}(t) * \varepsilon^j(t),$$

where * denotes the convolution operator and the relaxation matrix $\Psi_{ij}(t)$ is given by

$$\Psi_{ij}(t) = \begin{pmatrix}
\Psi_{11}(t) & \Psi_{12}(t) & \Psi_{13}(t) & 2c_{14} & 2c_{15} & 2c_{16} \\
\Psi_{21}(t) & \Psi_{22}(t) & \Psi_{23}(t) & 2c_{24} & 2c_{25} & 2c_{26} \\
\Psi_{31}(t) & \Psi_{32}(t) & \Psi_{33}(t) & 2c_{34} & 2c_{35} & 2c_{36} \\
c_{14} & c_{24} & c_{34} & 2\Psi_{44}(t) & 2c_{45} & 2c_{46} \\
c_{15} & c_{25} & c_{35} & 2c_{45} & 2\Psi_{55}(t) & 2c_{56} \\
c_{16} & c_{26} & c_{36} & 2c_{46} & 2c_{56} & 2\Psi_{66}(t)
\end{pmatrix} \cdot H(t).$$

Here, $H(t)$ is the Heaviside step function and the components $\Psi_{ij}(t)$ can be expressed as

$$\Psi_{ij}(t) = \sum_{k=0}^{\infty} \eta_{ij}^{(k)} \chi^{(k)}(t)$$

where $\eta_{ij}^{(k)}$ are real numbers, combinations of the $c_{ij}$ entries of the elastic Hooke’s tensor and the $\chi^{(k)}$, called relaxation functions, contain the time functionality of the relaxation matrix’s entries, normalized such that $\chi^{(k)} = 1$ for $t = 0$ and by defining the mode’s complex modulus as $M^{(k)}(t) = d(\chi^{(k)}(t)H(t))/dt$, this modulus in the frequency domain behaves as $M^{(k)}(\omega) \to 1$ for $\omega \to \infty$. In (Moczo, Kristek & Halada 2004) we can find a formulation of the GMB relaxation mechanisms that, once normalized, can be used to express the $\chi^{(k)}(t)$ as

$$\chi^{(k)}(t) = 1 - \sum_{\ell=1}^{\infty} Y^{(k)}_\ell (1 - e^{-\omega t}),$$

for $k = 1, 2, 3, 4$

$$\chi^{(k)}(t) = 1,$$ for $k = 0$
where \( n \) is the number of attenuating mechanisms used. The \( Y^{(k)}_i \) are the anelastic coefficients of each mechanism for the mode \( k \) which are related to the strength of the attenuation. Finally the \( \omega_\ell \) are the relaxation frequencies associated to each mechanism, which in the following we assume to be the same for all the viscoelastic modes. These GMB relaxation functions fulfill the conditions discussed above. The \( k = 0 \) case is shown for completion but doesn’t represent a relaxation function but, more accurately, a lack of it. As we have a constant \( \chi^{(0)} \) value we obtain an instantaneous response, so that we are talking about an elastic mode. We remark that in the elastic case all \( g^{(k)}_{ij} = 0 \), if \( k \neq 0 \), thus having exclusively that instantaneous response and, as a consequence, no energy losses. In the viscoelastic isotropic case we have \( g^{(k)}_{ij} = 0 \), except for \( k = 0 \), \( k = 1 \) (dilatational mode) and \( k = 2 \) (first shear mode).

Finally we can find the coefficients \( g^{(k)}_{ij} \) that ensure the separation of the dilatational and shear modes of the attenuation (Carcione 2002) giving us the entries of (27) by

\[
\begin{align*}
\Psi_{ii}(t) &= c_{ii} - (\chi + 2\mu) + (\chi + 4\mu) \chi^{(1)}(t) + (\frac{4}{3}\mu) \chi^{(2)}(t), & \text{for } i \leq 3 \\
\Psi_{ij}(t) &= c_{ij} - \chi + (\chi + \frac{4}{3}\mu) \chi^{(1)}(t) - \frac{4}{3}\mu \chi^{(2)}(t), & \text{for } i, j \leq 3 \text{ and } i \neq j \\
\Psi_{44}(t) &= c_{44} \chi^{(2)}(t) \\
\Psi_{55}(t) &= c_{55} \chi^{(3)}(t) \\
\Psi_{66}(t) &= c_{66} \chi^{(4)}(t)
\end{align*}
\]

with the definitions

\[
\begin{align*}
\overline{\mu} &= \frac{1}{3} (c_{44} + c_{55} + c_{66}), \\
\overline{\chi} &= \frac{1}{3} (c_{11} + c_{22} + c_{33}) - 2\mu.
\end{align*}
\]

We can also define new anelastic coefficients related to those appearing in (28) as

\[
\overline{\lambda}Y^{(i)}_\ell \equiv \left( \chi + \frac{2}{3}\mu \right) Y^{(i)}_\ell - \frac{2}{3}\mu Y^{(i)}_\ell ,
\]

\[
Y^{(i)}_\ell^{(i)} \equiv Y^{(i)}_\ell ,
\]

\[
Y^{(i)}_\ell^{(x)} \equiv Y^{(i)}_\ell ,
\]

\[
Y^{(i)}_\ell^{(y)} \equiv Y^{(i)}_\ell ,
\]

Now, making use of the last identity in (25) we derive in time the components of the tensor \( \Psi_{ij}(t) \) that are given in (29) to obtain the anisotropic viscoelastic stress-strain relation

\[
\begin{pmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\sigma_{xy}
\end{pmatrix} = \begin{pmatrix}
c_{11} & c_{12} & c_{13} & 2c_{14} & 2c_{15} & 2c_{16} & \varepsilon_{xx} \\
c_{12} & c_{22} & c_{23} & 2c_{24} & 2c_{25} & 2c_{26} & \varepsilon_{yy} \\
c_{13} & c_{23} & c_{33} & 2c_{34} & 2c_{35} & 2c_{36} & \varepsilon_{zz} \\
c_{14} & c_{24} & c_{34} & 2c_{44} & 2c_{45} & 2c_{46} & \varepsilon_{xy}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{zz} \\
\varepsilon_{xy}
\end{pmatrix}
\]  \[ − \sum_{\ell=1}^{3} \begin{pmatrix}
\chi Y^{(\ell)}_\ell + \frac{2}{3}\mu Y^{(\ell)}_\ell \\
\chi Y^{(\ell)}_\ell + \frac{2}{3}\mu Y^{(\ell)}_\ell \\
\chi Y^{(\ell)}_\ell + \frac{2}{3}\mu Y^{(\ell)}_\ell \\
\chi Y^{(\ell)}_\ell + \frac{2}{3}\mu Y^{(\ell)}_\ell
\end{pmatrix}
\begin{pmatrix}
Y^{(i)}_\ell^{(i)} \\
Y^{(i)}_\ell^{(x)} \\
Y^{(i)}_\ell^{(y)}
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c_{44}Y^{(i)}_\ell^{(i)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c_{55}Y^{(i)}_\ell^{(y)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c_{66}Y^{(i)}_\ell^{(x)}
\end{pmatrix}
\]  \[
\begin{pmatrix}
d_{xx} \\
d_{yy} \\
d_{zz} \\
d_{xy}
\end{pmatrix}
\]

Here, the anelastic functions \( \tilde{\varphi} \equiv (\tilde{\varphi}^{(i)}_{xx}, \tilde{\varphi}^{(y)}_{yy}, \tilde{\varphi}^{(z)}_{xx}, \tilde{\varphi}^{(x)}_{yy}, \tilde{\varphi}^{(y)}_{xx}, \tilde{\varphi}^{(z)}_{xy})^T \) are defined by

\[
\delta_{ij}^{(m)}(t) = \omega_\ell \frac{\partial}{\partial t} \left( \int_{-\infty}^{t} \varepsilon_j(\tau) e^{-\omega_\ell(t-\tau)} d\tau \right),
\]

as shown in (Moczo, Kristek & Halada 2004). The anelastic coefficients have to be fitted to the particular \( Q \)-law over a desired frequency range by using a number of relaxation frequencies \( \omega_\ell \) as outlined in more detail in (Käser, Dumbser, de la Puente & Igel 2006).

Notice here that this formulation even admits anisotropic attenuation, meaning that we can have different \( Q \) values for each of the 3 shear attenuating modes. However, our knowledge of the quality factors \( Q \) inside the Earth is often poor and rarely would allow us to consider any dependence on direction of the values of the \( Q \)-factors. Therefore, in the following we limit ourselves to the case in which attenuation is considered as an isotropic effect, even if the medium is anisotropic. This means, that \( Q^{(x)} = Q^{(y)} = Q^{(z)} \) and therefore we can define \( Y^{(1)}_\ell = Y^{(2)}_\ell = Y^{(3)}_\ell \). Note, that the stress-strain relation in (36) provides the general case from which we can infer the anisotropic elastic case by setting \( Y^{(1)}_\ell = 0 \) and \( Y^{(2)}_\ell = 0 \), thus recovering (3). The viscoelastic isotropic case is obtained by setting \( c_{11} = c_{22} = c_{33} = \lambda + 2\mu, \lambda \).
The Coupled Equation System

Section 4.2

The use of the anelastic functions \( \phi_j^c \) requires the storage of 6 new variables per attenuation mechanism, one per each stress component as shown in (36), that have to be updated at every time step, as already shown in (Käser, Dumbser, de la Puente & Igel 2006) for the viscoelastic isotropic case. This is done by solving an additional set of 6n linear partial differential equations given by

\[
\frac{\partial}{\partial t} \phi_j^c(t) + \omega_l \phi_j^c(t) = \omega_l \frac{\partial}{\partial y} \epsilon_j(t),
\]

where \( j = 1, \ldots, 6 \) for the 6 stress components according to (36). A detailed description of the resulting coupled linear system of equations is given in Section 4.2.

4.2 The Coupled Equation System

As shown in (Käser, Dumbser, de la Puente & Igel 2006), the new enlarged system of \( n_v = 9 + 6n \) partial differential equations including 9 elastic and 6n anelastic variables can be written in the compact form

\[
\frac{\partial \mathbf{Q}_a}{\partial t} + \mathbf{A}_a \frac{\partial \mathbf{Q}_a}{\partial x} + \mathbf{B}_a \frac{\partial \mathbf{Q}_a}{\partial y} + \mathbf{C}_a \frac{\partial \mathbf{Q}_a}{\partial z} = \mathbf{E}_a \mathbf{Q}_a,
\]

where \( \mathbf{E} \) denotes the so-called reaction term and takes into account the energy losses introduced by the viscoelastic medium. Note that the dimensions of the variable vector \( \mathbf{Q} \), the Jacobian matrices \( \mathbf{A}, \mathbf{B}, \mathbf{C} \) and the source matrix \( \mathbf{E} \) now depend on the number \( n \) of attenuation mechanisms. To keep the notation as simple as possible and without loss of generality, in the following we assume that the order of the equations in (39) is such, that \( p, q \in [1, \ldots, 9] \) denote the elastic part and \( p, q \in [10, \ldots, n_v] \), denote the anelastic part of the system, represented by the variables in (37) and the corresponding equations in (38).

As the Jacobian matrices \( \mathbf{A}, \mathbf{B} \) and \( \mathbf{C} \) as well as the source matrix \( \mathbf{E} \) are sparse and show some particular symmetry pattern and as their dimensions may become impractical for notation, we will use the block-matrix syntax. Therefore, we decompose the Jacobian matrices as follows:

\[
\mathbf{A} = \begin{bmatrix} A & 0 \\ A_n & 0 \end{bmatrix} \in \mathbb{R}^{n_v \times n_v}, \quad \mathbf{B} = \begin{bmatrix} B & 0 \\ B_n & 0 \end{bmatrix} \in \mathbb{R}^{n_v \times n_v}, \quad \mathbf{C} = \begin{bmatrix} C & 0 \\ C_n & 0 \end{bmatrix} \in \mathbb{R}^{n_v \times n_v},
\]

where \( A, B, C \in \mathbb{R}^{9 \times 9} \) are the Jacobians of the purely anisotropic elastic part as given in (5)-(7). The matrices \( A_n, B_n, C_n \) include the anelastic part and exhibit themselves a block structure of the form:

\[
A_n = \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix} \in \mathbb{R}^{6n \times 9}, \quad B_n = \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} \in \mathbb{R}^{6n \times 9}, \quad C_n = \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix} \in \mathbb{R}^{6n \times 9},
\]

where each sub-matrix \( A_\ell, B_\ell, C_\ell \in \mathbb{R}^{6 \times 9} \), with \( \ell = 1, \ldots, n \), contains the relaxation frequency \( \omega_\ell \) of the \( \ell \)-th attenuation mechanism in the form:

\[
A_\ell = \omega_\ell \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\]

\[
B_\ell = \omega_\ell \cdot \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\]

with

\[
c_{12} = c_{13} = c_{23} = \lambda \quad \text{and} \quad c_{44} = c_{55} = c_{66} = \mu \quad \text{with all other coefficients} \quad c_{ij} \quad \text{equal to zero. This way, we also obtain} \quad \lambda = \ell \quad \text{and} \quad \mu = \mu \quad \text{from (30) and (31) as a consequence.}
\]
\( C_\ell = \omega_\ell \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \).  

The matrix \( E \) in (4) representing the reactive source term that couples the anelastic functions to the original elastic system can be decomposed as

\[ E = \begin{bmatrix} 0 & E \\ 0 & E' \end{bmatrix} \in \mathbb{R}^{n_a \times n_a}, \]

with \( E \) exhibiting the block structure

\[ E = [E_1, \ldots, E_n] \in \mathbb{R}^{9 \times 6n}. \]

Here, each matrix \( E_\ell \in \mathbb{R}^{6 \times 6} \), with \( \ell = 1, \ldots, n \), contains the anelastic coefficients \( Y^\lambda_\ell \) and \( Y^\mu_\ell \) of the \( \ell \)-th mechanism in the form:

\[ E_\ell = \begin{pmatrix} \sum_{j=1}^{n} F_{kj}^{-1} \sum_{i=1}^{n} J_{qk} \sum_{h=1}^{n} \gamma_{pq} \sum_{m=1}^{n} \lambda_{km} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]

where we should remark again the different ordering of the entries with respect to what we introduced in (36) as a consequence of the different order of the anelastic variables inside the variable vector \( Q \). The matrix \( E' \) in (45) is a diagonal matrix and has the structure

\[ E' = \begin{bmatrix} E'_1 & 0 \\ \vdots & \ddots \\ 0 & E'_n \end{bmatrix} \in \mathbb{R}^{6n \times 6n}, \]

where each matrix \( E'_\ell \in \mathbb{R}^{6 \times 6} \), with \( \ell = 1, \ldots, n \), is itself a diagonal matrix containing only the relaxation frequency \( \omega_\ell \) of the \( \ell \)-th mechanism on its diagonal, i.e. \( E'_\ell = -\omega_\ell \cdot I \) with \( I \in \mathbb{R}^{6 \times 6} \) denoting the identity matrix.

As shown in the following Section 4.3, we can formulate the fully discrete ADER-DG scheme with conceptually only minor changes in order to obtain a high order numerical scheme for solving this new enlarged system of equations, that includes viscoelastic attenuation as well as the most general triclinic anisotropy.

### 4.3 The Coupled Numerical Scheme

As shown in more detail in (Käser, Dumbser, de la Puente & Igel 2006) the numerical scheme including viscoelastic attenuation changes due to the enlargement of the PDE system and the addition of the reaction term \( E \). Therefore, the discrete formulation of the ADER-DG scheme for anisotropic elastic media as given in (19) is now written as

\[ \left[ \left( \hat{Q}_{pq}^{(m)} \right)^{n+1} - \left( \hat{Q}_{pq}^{(m)} \right)^n \right] |J| M_{kl} + \]

\[ + \frac{1}{\tau} \sum_{j=1}^{4} \sum_{j=1}^{4} A_{pq} \left( \hat{T}_{qj}^{(m)} \right)^{\tau_{pq}^{(m)}} \left( \hat{T}_{qj}^{(m)} \right)^{-1} + \Theta_{pq}^{(m)} \right) |S_{kl}| \left( \hat{K}_{kl}^{(m)} \cdot I_{kl} \right) \left( \hat{Q}_{pq}^{(m)} \right)^n + \]

\[ + \frac{1}{\tau} \sum_{j=1}^{4} \sum_{j=1}^{4} A_{pq} \left( \hat{T}_{qj}^{(m)} \right)^{\tau_{pq}^{(m)}} \left( \hat{T}_{qj}^{(m)} \right)^{-1} + \Theta_{pq}^{(m)} \right) |S_{kl}| \left( \hat{K}_{kl}^{(m)} \cdot I_{kl} \right) \left( \hat{Q}_{pq}^{(m)} \right)^n - \]

\[ \tilde{A}_{pq} |J| K_{kl}^{(m)} \cdot I_{kl} \left( \hat{Q}_{pq}^{(m)} \right)^n - B_{pq} |J| K_{kl}^{(m)} \cdot I_{kl} \left( \hat{Q}_{pq}^{(m)} \right)^n - \]

\[ - \tilde{C}_{pq} |J| K_{kl}^{(m)} \cdot I_{kl} \left( \hat{Q}_{pq}^{(m)} \right)^n = |J| \tilde{E}_{pq} \cdot I_{qkl} \left( \hat{Q}_{pq}^{(m)} \right)^n M_{kl}, \]

where \( \Theta_{pq} \) is specified by the particular numerical flux in (16) or (17). The matrix \( \tilde{A}_{pq} \) now represents the enlarged matrix given in (40) with the entries of (5) which are rotated through the Bond’s transformation (13) as discussed in Section 3. We remark that \( \alpha_{max} \) remains the
same in the viscoelastic case, as the enlargement of the Jacobian matrices introduces only new eigenvalues equal to zero. Further details on the calculation of the Godunov flux in (17) for the anelastic part of the coupled system can be found in the Appendix A.

Besides, the rotation matrix $T_{pq}$ becomes larger and for the case of anelasticity in (49) has the form

$$
\tilde{T} = \begin{bmatrix}
T^d & 0 & 0 \\
0 & T^v & 0 \\
0 & 0 & T_n
\end{bmatrix} \in \mathbb{R}^{n_s \times n_s},
$$

where $T^d \in \mathbb{R}^{6 \times 6}$ is the rotation matrix responsible for the stress tensor rotation as in the purely elastic part and is given as

$$
T^d = \begin{pmatrix}
\begin{array}{ccc}
n^2_x & s^2_x & t^2_x \\
2n_x s_x & 2s_x t_x & 2n_x t_x \\
n^2_z & s^2_z & t^2_z \\
2n_z s_z & 2s_z t_z & 2n_z t_z
\end{array}
\end{pmatrix},
$$

(51)

The matrix $T^v \in \mathbb{R}^{3 \times 3}$ is the rotation matrix responsible for the velocity vector rotation as in the purely elastic part and is given as

$$
T^v = \begin{pmatrix}
\begin{array}{ccc}
n_x & s_x & t_x \\
n_y & s_y & t_y \\
n_z & s_z & t_z
\end{array}
\end{pmatrix},
$$

(52)

The matrix $T_n$ in (50) is a block diagonal matrix and has the structure

$$
T_n = \begin{bmatrix}
T^d & 0 & \cdots \\
0 & \ddots & 0 \\
0 & 0 & T^d
\end{bmatrix} \in \mathbb{R}^{6n \times 6n},
$$

(53)

where each of the $n$ sub-matrices $T^d$ is the tensor rotation matrix given in (51). A more detailed description of an efficient implementation of the ADER-DG method for the anelastic case the reader is referred to (Käsar, Dumbser, de la Puente & Igel 2006).

## 5 CONVERGENCE STUDY

In this section we present a numerical convergence study of the proposed ADER-DG approach on tetrahedral meshes, in order to demonstrate its arbitrarily high order of convergence in the presence of anisotropic material. We show results from second to seventh order ADER-DG schemes denoted by ADER-DG $Q_2$ to ADER-DG $Q_7$ respectively. We remark that the same order for space and time accuracy is obtained automatically.

Similar to previous work (Käsar & Dumbser 2006; Dumbser & Käsar 2006; Käsar, Dumbser, de la Puente & Igel 2006) we determine the convergence orders by solving the three-dimensional, anisotropic, seismic wave equations on the unit-cube as sketched in Fig. 1, i.e. on a computational domain $\Omega = [-1, 1] \times [-1, 1] \times [-1, 1] \in \mathbb{R}^3$ with periodic boundary conditions.

The homogeneous anisotropic material parameters are given in Table 1 and represent an anisotropic (orthorhombic) material, similar in its anisotropic properties to olivine as given in (Browaeys & Chevrot 2004). The analytic solution to this problem can be formulated as

$$
Q_0(x, y, z, t) = Q_0^p \cdot e^{i(\omega - k_x x - k_y y - k_z z)}, \quad p = 1, \ldots, 9
$$

(54)

where $Q_0^p$ is the initial amplitude vector of the 9 components, $\omega$ are the wave frequencies to determine and $k_x$, $k_y$ and $k_z$ are the wave numbers in $x$, $y$ and $z$-direction, respectively. To confirm that anisotropy is treated correctly, we superimpose three plane waves $Q_0^{(l)}$, $l = 1, \ldots, 3$, of the form given in (54) traveling perpendicular to each other along the coordinate axes, i.e. we have the three wave number vectors

$$
K^{(1)} = (k_x^{(1)}, k_y^{(1)}, k_z^{(1)})^T = (0, 0, 0)^T, 
$$

(55)

$$
K^{(2)} = (k_x^{(2)}, k_y^{(2)}, k_z^{(2)})^T = (0, \pi, 0)^T, 
$$

(56)

$$
K^{(3)} = (k_x^{(3)}, k_y^{(3)}, k_z^{(3)})^T = (0, 0, \pi)^T.
$$

(57)
leading to a periodic, sinusoidal waves in the unit-cube.

In the following, we briefly line out, how we determine the wave frequencies \( \omega \). With the assumption, that equation (54) is the analytic solution of the governing equation (4), we calculate the first time and space derivatives of equation (54) analytically and plug them into equation (4). From there, we can derive the eigenproblem

\[
\begin{align*}
(\tilde{\mathbf{A}}_{pq}k_x + \tilde{\mathbf{B}}_{pq}k_y + \tilde{\mathbf{C}}_{pq}k_z) \cdot \mathbf{Q}_q^0 &= \omega \cdot \mathbf{Q}_q^0, \quad p, q = 1, \ldots, 9.
\end{align*}
\]

Solving the three eigenproblem (58) for each wave \( l \) gives us the matrix \( \mathbf{R}^{(l)}_{pq} \) of right eigenvectors \( \mathbf{R}^{(l)}_{pq} \), \( \mathbf{R}^{(l)}_{pq} \), and the eigenvalues \( \omega_p^{(l)} \) for each wave.

Recalling, e.g. from (Toro 1999), that each solution of the linear hyperbolic system (4) is given by a linear combination of the right eigenvalues \( \mathbf{R}^{(l)}_{pq} \) and \( \mathbf{R}^{(l)}_{pq} \) and the eigenvalues \( \omega_p^{(l)} \) for each wave.

In the convergence test, we use the superposition of three plane \( PQ \)-waves traveling perpendicular to each other. However, the symmetry axes of the anisotropic, orthorhombic material is tilted with respect to the coordinate system, i.e. the symmetry axes point into the directions \((1, 1, 1), (1, -1, 0), \) and \((-1, -1, 2)\), respectively. The initial condition at \( t = 0 \) is given by (59) using the combination of three right eigenvectors \( \mathbf{R}^{(1)}_{pq} \), \( \mathbf{R}^{(2)}_{pq} \), and \( \mathbf{R}^{(3)}_{pq} \) with the coefficients \( \nu_1^{(l)} = \nu_2^{(l)} = \nu_3^{(l)} = 100 \) and zero otherwise.

The total simulation time \( T \) is set to \( T = 0.02s \). The CFL number is set in all computations to 50\% of the stability limit \( \frac{h}{\Delta t} \) of Runge-Kutta DG schemes. For a thorough investigation of the linear stability properties of the ADER-DG schemes via a von Neumann analysis see (Dumbser 2005).

The numerical analysis to determine the convergence orders is performed on a sequence of tetrahedral meshes as shown in Fig. 1. The mesh sequence is obtained by dividing the computational domain \( \Omega \) into a number of sub cubes, which are then subdivided into five tetrahedrons as shown in Fig. 1. This way, the refinement is controlled by changing the number of sub cubes in each dimension.

We can arbitrarily pick one of the variables of the system of the seismic wave equations (4) to numerically determine the convergence order of the used ADER-DG schemes. In Tables 2 and 3 we show the errors for the vertical velocity component \( u_z \) as given in detail in Appendix A. Fig. 2 visualizes the convergence results of Tables 2 and 3 to demonstrate the error with respect to (a) mesh width \( h \), (b) number of degrees of freedom \( N_h \) and (c) CPU time. With mesh refinement, for both choices of the numerical flux the higher order schemes converge faster as shown in Fig. 2(a). Furthermore, Fig. 2(b) demonstrates that higher order schemes reach a desired accuracy requiring a lower number of total degrees of freedom. The total number of degrees of freedom is the product of the number of mesh elements and the degrees of freedom per element. Therefore, obviously the increasing number of degrees of freedom of higher order schemes is over-compensated by the dramatic decrease of the number of required mesh elements to reach a certain error level. Also the CPU time comparisons in Fig. 2(c) show that the higher order methods reach a desired error level in less computational time. We remark that in all three plots of Fig. 2 we clearly show, that for very high accuracy, the higher order schemes with both, the Rusanov or Godunov fluxes, pay off due to their superior convergence properties.

Furthermore, we see in all plots that the Godunov flux is slightly more accurate than the Rusanov flux, which is due to well-known dissipative property of the Rusanov flux. Additionally, we want to remark, that with increasing order of the scheme the choice of the numerical flux seems to become less important. However, the Godunov flux always provides slightly more accurate results at the same CPU time as illustrated in Fig. 2.
High Order DG Method for Seismic Waves in Anisotropic Media

Figure 1. Sequence of discretizations of the computational domain $\Omega$ via regularly refined tetrahedral meshes, which are used for the numerical convergence analysis.

Table 2. Convergence rates of the vertical velocity component $w$ of the ADER-DG $O^2$ up to ADER-DG $O^7$ schemes on tetrahedral meshes with anisotropic material and Rusanov flux.

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<th>$O_{L^2}$</th>
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Table 3. Convergence rates of the vertical velocity component \( \omega \) of the ADER-DG O2 up to ADER-DG O7 schemes on tetrahedral meshes with anisotropic material and Godunov flux.

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Figure 2. Visualization of the convergence results of the vertical velocity component \( \omega \) for the Rusanov flux (dashed) of Table 2 and the Godunov flux (solid) of Table 3. The \( L^\infty \) error is plotted versus (a) the mesh spacing \( h \), (b) the number of degrees of freedom \( N_d \) and (c) the CPU time.
Table 4. Coefficients for the heterogeneous anisotropic model given in $[10^3 N \cdot m^{-2}]$ for the anisotropic and isotropic materials. All other coefficients are zero. The material density $\rho$ is given in $[kg \cdot m^{-3}]$.

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6 APPLICATION EXAMPLES

6.1 Heterogeneous Anisotropic Material

To validate the proposed ADER-DG scheme for anisotropic material in two space dimensions we show results of a heterogeneous anisotropic test case proposed by Carcione (1988) and Komatitsch et al. (2000). The computational domain $\Omega = [-32.5; 32.5]cm \times [-32.5; 32.5]cm$ is discretized by 37944 triangles with an average edge length of 0.5cm, equal to the edge length of the square shaped elements used by Komatitsch et al. (2000). Along the boundary of $\Omega$ we use absorbing boundary conditions. The domain $\Omega$ contains two materials separated by a straight line at $x = 0$. On the one side ($x < 0$) we have an anisotropic (transversely isotropic) zinc crystal with the symmetry axis in $y$-direction, whereas on the other side ($x > 0$) we use an isotropic material. The corresponding material properties are given in Table 4. The source represents a point force at location $s = (-2, 0) cm$, i.e. $2 cm$ from the material interface inside the anisotropic material and is acting in $y$-direction. The source time function is given by a Ricker wavelet with dominant frequency $f_0 = 170 kHz$ and delay $t_0 = 6 \mu s$ and acts on the vertical velocity component $v$ with a maximum amplitude of $1 \cdot 10^3 m \cdot s^{-1}$. Seismograms are calculated at four different locations $r_i = (x_i, y_i), i = 1, \ldots, 4$ with $x_1 = -10.5 cm, x_2 = -3.5 cm, x_3 = -1.0 cm, x_4 = 10.5 cm$ and $y_i = -8 cm$ for all $i = 1, \ldots, 4$ in order to compare our results with those of Komatitsch et al. (2000). The simulation is carried out using a ADER-DG O6 scheme, i.e. with polynomial basis functions of degree $N = 5$, and the Rusnovik flux presented in Section 5. The time step size was $20.58 \mu s$ such that the final simulation time $T = 100 \mu s$ was reached after 4860 iterations.

We illustrate two snapshots of the evolving wave field for a qualitative comparison. In Fig. 3(a) we show the vertical velocity component $v$ after $30 \mu s$ in a zoomed region together with the simulation mesh. Note, that the triangular elements are aligned with the material interface at $x = 0$. The locations of the source and the four receivers are also indicated by a full and empty circles, respectively. Fig. 3(b) illustrates the wave field of the velocity $v$ after $60 \mu s$ in the entire computational domain $\Omega$ together with the source and receiver locations. This visual comparison to the results of Komatitsch et al. (2000) shows, that the ADER-DG O6 scheme resolves the same wave phases. The typical cuspidal triangular wave structures and the refracted waves at the interface are clearly visible.

The seismograms calculated with the ADER-DG O6 scheme at the four receiver locations $r_i, i = 1, \ldots, 4$, are plotted in Fig. 4 (solid line). The results obtained by Komatitsch et al. (2000) with the SEM/2D PACK software and are superimposed (dashed line). The agreement is excellent for all phases. We have plotted the residuals between both computations (dotted line), amplified by a factor of 10, to show to which extent both results produce equivalent results. However we remark that for the ADER-DG computation we use a completely irregular triangular grid.

6.2 Transversely Isotropic Material with Tilted Symmetry Axis

To verify the accuracy of the proposed scheme for a fully three-dimensional problem we perform a computation of the test case proposed in (Komatitsch, Barnes & Tromp 2000) for a 3D transversely isotropic medium with tilted symmetry axis. Here, the tilt angle of $30^\circ$ with respect to the Cartesian coordinate axes creates additional complexity, as the rotation introduces a major number of non-zero entries in the Hooke’s tensor. We remark, that in our case numerical fluxes are computed with respect to a local coordinate system each aligned with a face of a tetrahedron as shown in Section 3 and therefore tilted material properties do not add additional complexity.

The computational domain $\Omega = [0; 2500]m \times [0; 2500]m \times [0; 2500]m$ is discretized with $48 \times 48 \times 48$ cubes, each subdivided in 5 tetrahedral elements, leading to a total of 532960 elements. The source is a point force placed at $(x, y, z) = (1250, 1562.5, 937.5) m$ and acting in the direction of the material’s symmetry axis. The source time function is a Ricker wavelet with dominant frequency $f_0 = 16 kHz$ and delay $t_0 = 0.07 s$. A receiver is located at $(x, y, z) = (1250, 1198.05, 1568.75) m$ to register the propagating waves. The material is homogeneous and the material parameters given in the coordinate system aligned with the anisotropic symmetry axis can be found in Table 5. Notice, that for a transversely isotropic material $c_{22} = c_{11}, c_{23} = c_{13}$ and $c_{55} = c_{44}$.

We use an ADER-DG O6 scheme, meaning that the variables are resolved with polynomials of degree $N = 5$ in space and time inside each...
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Figure 3. (a) Vertical velocity $v$ and computational mesh in the zoomed region $[-0.18; 0.1625] \times [-0.1625; 0.1625]$ at 30 $\mu$s. The source location is indicated by a full (black) circle, the four receiver locations are indicated by empty (white) circles. (b) Vertical velocity $v$ at 60 $\mu$s with the whole computational domain. A variety of different phases can be identified. The source location is indicated by a full (black) circle, the four receiver locations are indicated by empty (white) circles.

Table 5. Coefficients for the transversely isotropic material (Mesaverde clay shale) given in $[10^9 N \cdot m^{-2}]$. All other coefficients are zero. The material density $\rho$ is given in $[kg \cdot m^{-3}]$.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$c_{11}$</th>
<th>$c_{12}$</th>
<th>$c_{13}$</th>
<th>$c_{22}$</th>
<th>$c_{23}$</th>
<th>$c_{33}$</th>
<th>$c_{44}$</th>
<th>$c_{55}$</th>
<th>$c_{66}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2590</td>
<td>66.6</td>
<td>19.7</td>
<td>39.4</td>
<td>66.6</td>
<td>39.4</td>
<td>39.9</td>
<td>10.9</td>
<td>10.9</td>
<td>23.45</td>
</tr>
</tbody>
</table>

element. Furthermore, we use the Godunov flux as described in Appendix A. The time step size was 197.29 $\mu$s such that the final simulation time $T = 0.7 s$ was reached after 3548 iterations.

In Fig. 5(a) we visualize the wave field of the normal stress $\sigma_{xx}$ at $t = 0.25 s$ in the $yz$-plane at $x = 1250 m$. A visual comparison with the result of Komatitsch et al. (2000) shows the characteristic wave pattern for the case of a tilted anisotropic material. A vector plot illustrating the total particle velocity at $t = 0.25 s$ in a zoomed region of one of the cuspidal triangles is shown in Fig. 5(b) to visualize the complexity of the seismic wave field in more detail. The corresponding seismogram calculated at the receiver is plotted in Fig. 6 and compared to the analytical solution derived in (Carcione, Kosloff, Behle & Seriani 1992). We can see the excellent agreement between analytical and numerical solutions, where we can observe both the early $qP$ wave followed by the stronger $qSV$ wave. Similar to Komatitsch et al. (2000) we observe a slight discrepancy in the amplitudes. Note, that we use absorbing boundaries of the domain $\Omega$ that do not create spurious reflected wave.

Additionally, Fig. 6 shows the seismogram calculated for the same anisotropic test case but coupled with viscoelastic attenuation as introduced in Section 4. We used the quality factors $Q^P = 80$ and $Q^S = 40$ to see a strong effect due to anelasticity. Attenuation is implemented with 3 relaxation mechanisms as described in detail in (Käser, Dumbser, de la Puente & Igel 2006). The frequency bandwidth of $100 Hz$ is centered at the dominant frequency $16 Hz$ of the source. The associated anelastic functions are all computed at the same order as the elastic variables. With respect to the purely elastic case the damping and dispersion caused by the viscoelastic material is clearly visible. For the computation of the numerical solution of the anisotropic-elastic case the CPU time was 11 hours on 64 Intel Itanium2 64-bit 1.6-GHz processors with shared-memory. For the anisotropic-viscoelastic case approximately 25 hours were needed on the same computer.

7 CONCLUSION

We have presented a high-order scheme for solving problems of anisotropic seismic wave propagation on unstructured tetrahedral meshes. The proposed ADER-DG method has proved to be well-suited for achieving highly accurate results for anisotropic material properties. Two numerical fluxes have been introduced and compared with respect to their accuracy and computational cost. A thorough convergence
The study confirms the high-order accuracy of the scheme independent of the choice of the numerical flux. However, the new Godunov type flux turns out to be the better choice. Additionally, we developed a formulation to couple anisotropic and viscoelastic effects for seismic wave propagation and present the necessary changes in the explicit expression of the numerical scheme. Results of different application examples in 2D and 3D involving $qP$, $qS_1$ and $qS_2$ wave propagation in both homogeneous and heterogeneous media have shown a very good agreement with analytical solutions or results obtained by the Spectral Element Method. We conclude, that the ADER-DG method represents a new numerical approach to solve seismic wave propagation problems, where geometrical flexibility and numerical accuracy are fundamental. The new scheme provides important advantages for future applications in realistic wave propagation scenarios, in particular, in areas where a clear distinction between geometry- or anisotropy-caused phase splitting is crucial, e.g. in cracked sedimentary layers or in the upper mantle or oceanic crust.

8 ACKNOWLEDGMENT

The authors thank the Marie Curie Research and Training Network SPICE (Seismic Wave Propagation in Complex Media: a European Network), part of the European Commission’s Human Resources Mobility Programme, as well as the DFG (Deutsche Forschungsgemeinschaft), as the work was supported through the Emmy Noether-program (KA 2281/1-1) and the DFG-CNRS research group FOR 508, Noise Generation in Turbulent Flows. We thank Dimitri Komatitsch for providing us with the analytical solution in Section 6.2 and Jean-Paul Ampuero for computing the SEM reference seismograms in Section 6.1. Furthermore, the support of the LRZ München is acknowledged.
Figure 5. (a) Snapshot of the normal stress $\sigma_{xx}$ at $t = 0.25s$ in the $yz$-plane at $x = 1250m$ (top). The source and receiver positions are indicated by the empty and full circles, respectively. The zoom region for Fig. 5(b) is indicated by the box. (b) Vector field of the particle velocity at $t = 0.25s$ in the zoom region.

to compute the numerical results presented in this work. The codes we compare our method with are freely available at the SPICE Digital Library (www.spice-rtn.org).
0.1 0.2 0.3 0.4 0.5 0.6 0.7

-4 -3 -2 -1 0 1 2 3

Time [s]

displacement

Figure 6. Numerical (solid) and analytical (dashed) displacements along the symmetry axis recorded at 728.9m from the source. The numerical solution is computed with an ADER-DG O6 scheme and shows excellent agreement with the analytical solution. The third seismogram shows the effects of viscoelasticity for the same receiver with clear physical dissipation and attenuation effects.

REFERENCES


APPENDIX A: GODUNOV FLUX FOR ANISOTROPIC MATERIAL

The flux formulation in eq. (14) requires a numerical viscosity to stabilize the scheme. This term can have a different form depending on the flux type. Hereby, the Godunov flux (also referred to as Roe flux) ensures the theoretical minimum viscosity, and therefore the highest resolution, using the matrix $|A|$ as the stabilizing term. The matrix $|A|$ decomposes the characteristic waves at an interface between two elements into outgoing and incoming waves and is given through

$$|A| = R |A| R^{-1},$$

where the matrix $|A|$ is a diagonal matrix containing the absolute values of the eigenvalues of the Jacobian matrix $A$, which has to be oriented in the interface normal direction. The columns of matrix $R$ in eq. (A1) contain the right eigenvectors of $A$. Note, that both matrices, $R$ and $\Lambda$, have to have the same ordering, i.e. the first eigenvector in the first column of $R$ has to correspond to the first eigenvalue of $A$ appearing in the diagonal of $\Lambda$. We remind, that the non-zero eigenvalues of $A$ in the elastic and viscoelastic cases can be found by solving the cubic equation (18) in Section 3. In the following, we assume a strictly descending ordering of the eigenvalues in $\Lambda$.

**A1 Computation of the elastic part of $|A|$**

We consider the 9 right eigenvectors $\vec{R}_i = (r_i^1, r_i^2, r_i^3, r_i^4, r_i^5, r_i^6, r_i^7, r_i^8, r_i^9)^T$, with $i = 1, \ldots, 9$, and the corresponding eigenvalues $\alpha_i$, that form the eigenproblem $A \vec{R}_i = \alpha_i \vec{R}_i$. The eigendecomposition is then obtained by explicitly solving the 9 equations

$$
\begin{align*}
\alpha_i r_i^1 + c_{15} r_i^6 + c_{15} r_i^9 &= \alpha_i r_i^1,
\alpha_i r_i^2 + c_{26} r_i^8 + c_{26} r_i^9 &= \alpha_i r_i^2,
\alpha_i r_i^3 + c_{26} r_i^8 + c_{26} r_i^9 &= \alpha_i r_i^3,
\alpha_i r_i^4 &= \alpha_i r_i^4,
\alpha_i r_i^5 &= \alpha_i r_i^5,
\alpha_i r_i^6 + c_{26} r_i^8 + c_{26} r_i^9 &= \alpha_i r_i^6,
\alpha_i r_i^7 &= \alpha_i r_i^7,
\alpha_i r_i^8 &= \alpha_i r_i^8,
\alpha_i r_i^9 &= \alpha_i r_i^9.
\end{align*}
$$

Due to the dependency of some equations, the solution of the equations in (A2) can be obtained by solving the more compact homogeneous linear system

$$
\begin{pmatrix}
X & c_{16} & c_{15} \\
c_{16} & Y & c_{56} \\
c_{15} & c_{56} & Z
\end{pmatrix}
\begin{pmatrix}
r_i^7 \\
r_i^8 \\
r_i^9
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
$$

with $X = c_{11} - \alpha_i^2 \rho$, $Y = c_{56} - \alpha_i^2 \rho$ and $Z = c_{55} - \alpha_i^2 \rho$. Note that this represents the Kelvin-Christoffel equation for anisotropic media, which is also obtained from plane-wave analysis by Carcione (2002). However, we arrive at this equation naturally through the eigendecomposition of the Jacobian $A$ in our hyperbolic system (4). We remark, that the solution of the linear system (A2) for the values $r_i^j$, $i, j = 1, \ldots, 9$, completely defines the 9 right eigenvectors $\vec{R}_i$. The fact, that the determinant of the matrix of the system in (A3) always zero is ensured by eq. (18). Therefore, there will always be a non-trivial solution of (A3). Having determined the values of $r_i^7$, $r_i^8$ and $r_i^9$ in (A3), we can use eqs (A2) to obtain all other elements of the eigenvector $\vec{R}_i$. Finally, the explicit form of matrix of right eigenvectors is given as

$$
R = \begin{pmatrix}
r_i^1 & r_i^2 & r_i^3 & 0 & 0 & 0 & -r_i^2 & -r_i^3 & -r_i^1 \\
r_i^2 & r_i^2 & r_i^3 & 1 & 0 & 0 & -r_i^2 & -r_i^3 & -r_i^1 \\
r_i^3 & r_i^2 & r_i^3 & 0 & 1 & 0 & -r_i^2 & -r_i^3 & -r_i^1 \\
r_i^4 & r_i^4 & r_i^5 & 0 & 0 & 0 & -r_i^2 & -r_i^3 & -r_i^1 \\
r_i^5 & r_i^2 & r_i^3 & 0 & 0 & 0 & -r_i^2 & -r_i^3 & -r_i^1 \\
r_i^6 & r_i^7 & r_i^5 & 0 & 0 & 0 & -r_i^2 & -r_i^3 & -r_i^1 \\
r_i^7 & r_i^7 & r_i^5 & 0 & 0 & 0 & -r_i^2 & -r_i^3 & -r_i^1 \\
r_i^8 & r_i^5 & r_i^9 & 0 & 0 & 0 & -r_i^2 & -r_i^3 & -r_i^1 \\
r_i^9 & r_i^9 & r_i^5 & 0 & 0 & 0 & -r_i^2 & -r_i^3 & -r_i^1
\end{pmatrix}.
$$

For the left eigenvectors $\vec{L}_i = (l_i^1, l_i^2, l_i^3, l_i^4, l_i^5, l_i^6, l_i^7, l_i^8, l_i^9)$, the eigenproblem read as $\vec{L}_i A = \alpha_i \vec{L}_i$, and the eigendecomposition leads to the 9 equations

$$
\begin{align*}
c_{11} l_i^1 + c_{16} l_i^4 + c_{15} l_i^6 &= \alpha_i l_i^7,
 c_{16} l_i^1 + c_{26} l_i^8 + c_{26} l_i^9 &= \alpha_i l_i^8,
 c_{26} l_i^1 + c_{26} l_i^8 + c_{26} l_i^9 &= \alpha_i l_i^9,
 l_i^2 &= 0,
 l_i^3 &= 0,
 l_i^4 &= 0,
 l_i^5 &= 0,
 l_i^6 &= 0,
 l_i^7 &= 0,
 l_i^8 &= 0,
 l_i^9 &= 0.
\end{align*}
$$

which similarly to the case of the right eigenvectors lead to the more compact homogeneous system

$$
\begin{pmatrix}
X & c_{16} & c_{15} \\
c_{16} & Y & c_{56} \\
c_{15} & c_{56} & Z
\end{pmatrix}
\begin{pmatrix}
l_i^7 \\
l_i^8 \\
l_i^9
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
$$

We can observe a symmetry between the left and right eigenvectors, which are $r_i^1 = l_i^7$, $r_i^2 = l_i^8$, $r_i^3 = l_i^9$, $r_i^4 = l_i^1$, $r_i^5 = l_i^2$, $r_i^6 = l_i^3$ and $r_i^7 = l_i^4$. This allows us to find the left eigenvectors of $A$. Furthermore, to avoid scaling problems, we want the left eigenvectors to fulfill the condition
\( L = R^{-1}, \) which is obtained by using the normalization
\[
\left( \frac{L_i}{2\alpha_i S_i} \right) \bar{R}_i = 1. \tag{A7}
\]

From eq.(A7) we can conclude, that \( S_i = \rho \left[ (r_i^7)^2 + (r_i^8)^2 + (r_i^9)^2 \right]. \)

Then we can finally write down the matrix \( \bar{L} \) of left eigenvectors depending exclusively of the components of the right eigenvectors in the form
\[
L = R^{-1} = \begin{pmatrix}
\frac{r_i^7}{2\alpha_i S_i} & 0 & 0 & \frac{r_i^8}{2\alpha_i S_i} & 0 & \frac{r_i^9}{2\alpha_i S_i} & 0 & 0 & 0 \\
\frac{r_i^7}{2\alpha_i S_i} & 0 & 0 & \frac{r_i^8}{2\alpha_i S_i} & 0 & \frac{r_i^9}{2\alpha_i S_i} & 0 & 0 & 0 \\
\frac{r_i^7}{2\alpha_i S_i} & 0 & 0 & \frac{r_i^8}{2\alpha_i S_i} & 0 & \frac{r_i^9}{2\alpha_i S_i} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{r_i^7}{2\alpha_i S_i} & 0 & -\frac{r_i^8}{2\alpha_i S_i} & 0 & -\frac{r_i^9}{2\alpha_i S_i} & 0 & 0 \\
0 & 0 & -\frac{r_i^7}{2\alpha_i S_i} & 0 & -\frac{r_i^8}{2\alpha_i S_i} & 0 & -\frac{r_i^9}{2\alpha_i S_i} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} . \tag{A8}
\]

Using eq. (A1) and substituting \( r_j^i = \frac{\bar{r}_j^i}{\sqrt{S_i}} \) we get the final expression of \( |A| \) as
\[
|A| = \sum_{i=1}^{3} \begin{pmatrix}
\bar{r}_i^1 \bar{r}_i^7 & 0 & 0 & \bar{r}_i^2 \bar{r}_i^7 & 0 & \bar{r}_i^3 \bar{r}_i^7 & 0 & 0 & 0 \\
\bar{r}_i^1 \bar{r}_i^7 & 0 & 0 & \bar{r}_i^2 \bar{r}_i^7 & 0 & \bar{r}_i^3 \bar{r}_i^7 & 0 & 0 & 0 \\
\bar{r}_i^1 \bar{r}_i^7 & 0 & 0 & \bar{r}_i^2 \bar{r}_i^7 & 0 & \bar{r}_i^3 \bar{r}_i^7 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\bar{r}_i^1 \bar{r}_i^7 & 0 & -\bar{r}_i^2 \bar{r}_i^7 & 0 & -\bar{r}_i^3 \bar{r}_i^7 & 0 & 0 \\
0 & 0 & -\bar{r}_i^1 \bar{r}_i^7 & 0 & -\bar{r}_i^2 \bar{r}_i^7 & 0 & -\bar{r}_i^3 \bar{r}_i^7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} . \tag{A9}
\]

We point out, that we only need to know the 3 positive eigenvalues of the Jacobian matrix \( A \) in order to compute all entries of the matrix in (A9) by solving (A3) and using (A2) and (A7). Furthermore, note that the computation of \( |A| \) only depends on the material properties and therefore has to be computed only once for each tetrahedral element as long as the material does not change with time.

### A2 Computation of the anelastic part of \( |A| \)

The anelastic part of \( |A| \) can be found by a similar procedure as described in A1. Let’s consider the more general case of viscoelastic material, in which we use \( n \) attenuating mechanisms to describe the viscoelastic properties of a material. For each attenuating mechanism we introduce 6 new eigenvectors and eigenvalues (Käser, Dumbser, de la Puente & Igel 2006). However, these new eigenvalues have value zero. Following our convention of decreasing ordering the eigenvalues are now given through \( \alpha_i = -\alpha_9 + 6\alpha_6, \alpha_2 = -\alpha_8 + 6\alpha_6, \alpha_3 = -\alpha_7 + 6\alpha_6 \) and \( \alpha_i = 0 \) for \( i = 4, \ldots, 6 + 6n \).

The right and left eigenvectors now have the shape
\[
\begin{align*}
\bar{R}_i &= \left( \bar{r}_i^1, \bar{r}_i^2, \bar{r}_i^3, \bar{r}_i^4, \bar{r}_i^5, \bar{r}_i^6, \bar{r}_i^7, \bar{r}_i^8, \bar{r}_i^9, \frac{\omega_i^{17}}{\alpha_i}, 0, 0, \frac{\omega_i^{18}}{2\alpha_i}, 0, \cdots, \frac{\omega_i^{19}}{\alpha_i}, 0, \frac{\omega_i^{20}}{2\alpha_i}, \frac{\omega_i^{21}}{2\alpha_i}, \frac{\omega_i^{22}}{2\alpha_i}, \frac{\omega_i^{23}}{2\alpha_i}, \frac{\omega_i^{24}}{2\alpha_i}, \frac{\omega_i^{25}}{2\alpha_i}, \frac{\omega_i^{26}}{2\alpha_i}, \frac{\omega_i^{27}}{2\alpha_i}, \frac{\omega_i^{28}}{2\alpha_i}, \frac{\omega_i^{29}}{2\alpha_i} \right)^T, \\
L_i &= \left( \bar{r}_i^1, 0, 0, \bar{r}_i^2, 0, \bar{r}_i^3, 0, \bar{r}_i^4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right),
\end{align*}
\]

which gives us the possibility of constructing the blocks of the matrix \( |A| \) for the anelastic case. The block structure is equivalent to the one given for matrix \( A \) in eqs (40), (41) and (42) in Section 4.2 and is given as
\[
|\bar{A}| = \begin{bmatrix}
|A| & 0 \\
0 & |\bar{A}|^\dagger
\end{bmatrix} \in \mathbb{R}^{n \times n}, \quad |A|^\dagger = \begin{bmatrix}
A_1^\dagger \\
\vdots \\
A_n^\dagger
\end{bmatrix} \in \mathbb{R}^{6n \times 9}, \tag{A11}
\]

where \( |A| \in \mathbb{R}^{9 \times 9} \) is the matrix of the purely anisotropic elastic part as given in (A9) and the matrix \( |\bar{A}| \in \mathbb{R}^{n \times n} \) includes the block structured anelastic part where each sub-matrix \( A_\ell^\dagger \in \mathbb{R}^{6 \times 9} \), with \( \ell = 1, \ldots, n \), contains the relaxation frequency \( \omega_\ell \) of the \( \ell \)-th attenuation mechanism.
in the form
\[
A_i^{||} = \omega_i \sum_{i=1}^{3} \begin{pmatrix}
\frac{r_7 r_1^i}{\alpha_1} & 0 & 0 & \frac{r_7 r_4^i}{\alpha_1} & 0 & \frac{r_7 r_2^i}{\alpha_1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\tag{A12}
\]

Note that the isotropic case can be recovered by setting \((r_7^1, r_4^1, r_2^1) = (1, 0, 0), (r_7^2, r_4^2, r_2^2) = (0, 1, 0)\) and \((r_7^3, r_4^3, r_2^3) = (0, 0, 1)\).
1 Sequence of discretizations of the computational domain $\Omega$ via regularly refined tetrahedral meshes, which are used for the numerical convergence analysis.

2 Visualization of the convergence results of the vertical velocity component $w$ for the Rusanov flux (dashed) of Table 2 and the Godunov flux (solid) of Table 3. The $L^\infty$ error is plotted versus (a) the mesh spacing $h$, (b) the number of degrees of freedom $N_d$ and (c) the CPU time.

3 (a) Vertical velocity $v$ and computational mesh in the zoomed region $[-0.18; 0.1625] \times [-0.1625; 0.1625]$ at $30\mu s$. The source location is indicated by a full (black) circle, the four receiver locations are indicated by empty (white) circles. (b) Vertical velocity $v$ at $60\mu s$ with the whole computational domain. A variety of different phases can be identified. The source location is indicated by a full (black) circle, the four receiver locations are indicated by empty (white) circles.

4 Seismograms showing vertical displacements for the ADER-DG (solid) and SEM (dashed) computations. The good agreement of both solutions is shown by the amplified residuals (dotted).

5 (a) Snapshot of the normal stress $\sigma_{xx}$ at $t = 0.25s$ in the $yz$-plane at $x = 1250m$ (top). The source and receiver positions are indicated by the empty and full circles, respectively. The zoom region for Fig. 5(b) is indicated by the box. (b) Vector field of the particle velocity at $t = 0.25s$ in the zoom region.

6 Numerical (solid) and analytical (dashed) displacements along the symmetry axis recorded at 728.9m from the source. The numerical solution is computed with an ADER-DG $O(6)$ scheme and shows excellent agreement with the analytical solution. The third seismogram shows the effects of viscoelasticity for the same receiver with clear physical dissipation and attenuation effects.