NEAR-TIP FIELDS OF MODE III STEADY-STATE CRACK PROPAGATION IN ELASTIC-PLASTIC STRAIN GRADIENT SOLIDS

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Abstract The mode III asymptotic fields near a steadily propagating crack tip are investigated by employing the flow-theory version of the couple-stress strain gradient plasticity. Compared to classical plasticity results, an increase of the stress singularity is observed, also for small hardening. The performed asymptotic analysis can provide useful predictions about the increase of the traction level ahead of the crack tip due to the strain gradient effects, which have been found relevant and non negligible at the micron scale.

1. INTRODUCTION

Strain gradient plasticity theories have been proposed in order to capture the size effect exhibited by ductile materials when subject to non-uniform plastic deformation at the micron scale [1]. A key point is the incorporation in the model of characteristic lenghts which specify the range when strain gradients are dominant. These theories may also improve considerably the estimate of the stress traction level ahead of the crack tip required for the occurring of cleavage or atomic decohesion during crack growth processes, experimentally observed in ductile metals in [2]. Indeed, conventional plasticity theories are unable to predict such stress intensity which is of the order of ten times the tensile yield stress [3, 4].

In this note, the effects of strain rotation gradient on mode III crack propagation are investigated by performing an asymptotic analysis of the crack-tip fields, within the framework of the flow theory of couple-stress strain gradient plasticity [5]. By contrast with the Mode I and Mode II problems [6], a qualitative analysis shows that the leading order term of the velocity field turns out to be rotational. Moreover, it is found that the skew-symmetric stress field dominates the asymptotic crack-tip field. In particular, if the elastic strain gradients are kept sufficiently small, the singularity of the stress fields significantly increases with respect to the classical J_2-flow theory due only to the skew-symmetric stress term, with no need to invoke additional stretch gradient contributions.

2. CRACK GROWTH PROBLEM

The problem of a plane crack propagating at constant velocity V along a rectilinear path in an infinite medium under mode III condition in a couple-stress elastic-plastic medium is addressed. A cylindrical co-ordinate system (r, θ, x_3) moving with the crack tip towards the $\theta = 0$ direction is considered, where the x_3-axis coincides with the straight crack front. The condition of steady-state crack propagation yields the following time derivative rule, which holds for any scalar function φ

$$\dot{\varphi} = \frac{V}{r}(\varphi_{,\theta} \sin\theta - r\varphi_{,r} \cos\theta). \tag{2.1}$$

The considered constitutive model refers to the flow theory version of the strain gradient plasticity presented by Fleck and Hutchinson [5]. This model fits within the general framework of couple-stress theory and involves a single material length scale ℓ, which specifies the order of non-uniform deformation at which the effects of strain gradients become significant, and thus being generally small (about 4 μm for copper and 6 μm for nickel).

According to the couple-stress model [7, 8], a surface element of a body with unit area can transmit a force traction vector $\mathbf{p}$ and a couple-stress traction vector $\mathbf{q}$. These surface forces $\mathbf{p}$ and $\mathbf{q}$ can be expressed in terms of the non-symmetrical Cauchy stress tensor $\mathbf{t}$ and of the couple-stress tensor $\boldsymbol{\mu}$ as

$$\mathbf{p} = \mathbf{t}^T\mathbf{n} + \nabla\mu_{nn} \times \mathbf{n}, \quad \mathbf{q} = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\boldsymbol{\mu}^T\mathbf{n}. \tag{2.2}$$

In the following the Cauchy stress $\mathbf{t}$ will be decomposed into the symmetric part $\boldsymbol{\sigma}$ and the skew-symmetric part $\boldsymbol{\tau}$.

For an antiplane problem, the non-vanishing stress and couple-stress components in polar coordinates are $\sigma_{r3},\ \sigma_{\theta 3},\ \tau_{r3},\ \tau_{\theta 3},\ \mu_{r\theta},\ \mu_{\theta r}$ and $\mu_{\theta\theta} = -\mu_{rr}$ (since $\boldsymbol{\mu}$ is a purely deviatoric tensor), and the equilibrium equations are

$$(rt_{r3})_{,r} + t_{\theta 3,\theta} = 0, \tag{2.3}$$

$$(r\mu_{rr})_{,r} + \mu_{\theta r,\theta} - \mu_{\theta\theta} + 2r\tau_{\theta 3} = 0, \quad (r\mu_{r\theta})_{,r} + \mu_{\theta\theta,\theta} + \mu_{\theta r} - 2r\tau_{r3} = 0. \quad (2.4)$$

Moreover, the only non-vanishing component of the velocity vector $\mathbf{v}$ is v_3. Therefore, the kinematic compatibility conditions for the strain and deformation curvature rate tensors, respectively $\dot{\boldsymbol{\varepsilon}}$ and $\dot{\boldsymbol{\chi}}$, namely $\dot{\boldsymbol{\varepsilon}} = (\nabla\mathbf{v} + \nabla\mathbf{v}^T)/2$ and $\dot{\boldsymbol{\chi}} = \operatorname{curl}\dot{\boldsymbol{\varepsilon}}$, specialise in

$$\dot{\varepsilon}_{r3} = v_{3,r}/2, \quad r\dot{\varepsilon}_{\theta 3} = v_{3,\theta}/2, \tag{2.5}$$

$$\dot{\chi}_{rr} = \dot{\varepsilon}_{3\theta,r}, \quad \dot{\chi}_{r\theta} = \dot{\varepsilon}_{3\theta,\theta} + \dot{\varepsilon}_{3r}/r, \quad \dot{\chi}_{\theta r} = -\dot{\varepsilon}_{3r,r}, \tag{2.6}$$

where $\dot{\chi}_{\theta\theta} = -\dot{\chi}_{rr}$.

Within the context of small deformations incremental theory, the total strain rate $\dot{\boldsymbol{\varepsilon}}$ is the sum of elastic $\dot{\boldsymbol{\varepsilon}}^e$ and plastic $\dot{\boldsymbol{\varepsilon}}^p$ parts. Similarly, the total deformation curvature rate $\dot{\boldsymbol{\chi}}$ is the sum of elastic $\dot{\boldsymbol{\chi}}^e$ and plastic $\dot{\boldsymbol{\chi}}^p$ contributions. Both elastic parts are related to stress and couple-stress rates through the incremental relations

$$\dot{\boldsymbol{\varepsilon}}^e = \frac{1}{E}[(1+\nu)\dot{\boldsymbol{\sigma}} - \nu(\operatorname{tr}\dot{\boldsymbol{\sigma}})\,\mathbf{I}], \quad \dot{\boldsymbol{\chi}}^{eT} = \frac{1+\nu}{E\,\ell_e^2}\dot{\boldsymbol{\mu}}, \tag{2.7}$$

where E denotes the elastic Young modulus, ν the Poisson ratio and ℓ_e is the elastic length scale introduced in [5] in order to divide the deformation curvature rate tensor into its elastic part $\dot{\boldsymbol{\chi}}^e$ and plastic part $\dot{\boldsymbol{\chi}}^p$, being $\ell_e < \ell$.

The fundamental relationships of the constitutive model are briefly summarised below.

- Yield condition

$$f(\Sigma, Y) = \Sigma - Y = 0, \tag{2.8}$$

  where Σ is the overall effective stress defined as

$$\Sigma^2 = 3\,(\boldsymbol{\sigma}_{\text{dev}} \cdot \boldsymbol{\sigma}_{\text{dev}} + \ell^{-2}\boldsymbol{\mu} \cdot \boldsymbol{\mu})/2, \tag{2.9}$$

  and Y denotes the uniaxial flow stress defining isotropic hardening behaviour. Linear isotropic hardening is introduced below.

- Associative flow rule

$$\dot{\boldsymbol{\varepsilon}}^p = \Lambda\frac{\partial f}{\partial \boldsymbol{\sigma}} = \frac{3\Lambda}{2\,\Sigma}\,\boldsymbol{\sigma}_{\text{dev}}, \quad \dot{\boldsymbol{\chi}}^{pT} = \Lambda\frac{\partial f}{\partial \boldsymbol{\mu}} = \frac{3\Lambda}{2\,\Sigma\,\ell^2}\,\boldsymbol{\mu}, \tag{2.10}$$

  where Λ is the plastic multiplier.

- Linear isotropic hardening rule

$$\dot{Y} = \Lambda H, \tag{2.11}$$

where $H = \alpha E/(1-\alpha)$ is the hardening modulus and $\alpha = E_t/E$ $(0 < \alpha < 1)$ is the ratio between the tangent modulus and the Young modulus for a bilinear stress-strain curve obtained by a uniaxial tension test.

- Prager consistency condition

$$\dot{f} = 0 \quad \text{or equivalently} \quad \dot{\Sigma} = \dot{Y}, \tag{2.12}$$

which gives the non-negative plastic multiplier Λ as

$$\Lambda = \begin{cases} \langle \dot{\Sigma} \rangle / H & \text{if} \quad f(\Sigma, Y) = 0 \\ 0 & \text{if} \quad f(\Sigma, Y) < 0, \end{cases} \tag{2.13}$$

where $\langle x \rangle = (|x| + x)/2$ and

$$\dot{\Sigma} = 3\,(\boldsymbol{\sigma}_{\text{dev}} \cdot \dot{\boldsymbol{\sigma}} + \ell^{-2} \boldsymbol{\mu} \cdot \dot{\boldsymbol{\mu}})/(2\,\Sigma). \tag{2.14}$$

From (2.7) the elastic-plastic incremental constitutive equations for the stress and couple-stress tensors, $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$, turn out to be

$$\dot{\boldsymbol{\varepsilon}} = \frac{1}{E}\,[(1+\nu)\,\dot{\boldsymbol{\sigma}} - \nu(\text{tr}\dot{\boldsymbol{\sigma}})\,\mathbf{I}] + \frac{3}{2\,\Sigma}\,\Lambda\,\boldsymbol{\sigma}_{\text{dev}}, \tag{2.15}$$

$$\ell^2 \dot{\boldsymbol{\chi}}^T = \frac{1+\nu}{E\,\xi^2}\,\dot{\boldsymbol{\mu}} + \frac{3}{2\,\Sigma}\,\Lambda\,\boldsymbol{\mu}, \tag{2.16}$$

where $\xi = \ell_e/\ell < 1$ is a non-dimensional parameter. Equations (2.15)-(2.16) hold when the yield condition (2.8) is satisfied. Otherwise, the incremental constitutive relationship reduces to the couple-stress isotropic elasticity, recovered when $\Lambda = 0$. Note that strain gradient effects occur also for a purely elastic response. However, their magnitude may be made arbitrarily small by choosing a sufficiently small ξ. Finally, it can be observed that the resulting constitutive equations (2.15)-(2.16) represent a generalization of the widely used J_2-flow theory of plasticity, and reduce to that model when the strain gradients are vanishing small.

3. ASYMPTOTIC CRACK-TIP FIELDS

Equations (2.3)-(2.6) together with the constitutive incremental equations (2.11), (2.15) and (2.16) form a system of first order PDEs that governs the problem of mode III crack propagation. The solution is sought in a separable variable form, by considering the most singular terms in the asymptotic expansion of near-tip fields. In particular, within the zone of radius ℓ about the crack tip, namely, where strain gradients are supposed to be dominant, the velocity, stress and couple-stress asymptotic fields are assumed in the following form

$$v_3(r,\theta) = V\left(\frac{r}{R}\right)^s w(\theta), \qquad \sigma_{\alpha 3}(r,\theta) = E\left(\frac{r}{R}\right)^s s_\alpha(\theta),$$
$$\tau_{\alpha 3}(r,\theta) = E\,\frac{\ell^2}{r^2}\left(\frac{r}{R}\right)^s t_\alpha(\theta),\ \mu_{\alpha\beta}(r,\theta) = E\,\frac{\ell^2}{r}\left(\frac{r}{R}\right)^s M_{\alpha\beta}(\theta), \tag{3.1}$$

where $w(\theta)$, $s_\alpha(\theta)$, $t_\alpha(\theta)$, $M_{\alpha\beta}(\theta)$ are scalar functions and the greek indices α and β stand for r and θ. The exponent s defines the radial dependence of the symmetric stress and velocity fields. Moreover, R denotes a characteristic dimension of the plastic zone which defines the amplitude of the leading order asymptotic fields. Note that the condition $M_{\theta\theta} = -M_{rr}$ holds true since $\boldsymbol{\mu}$, and thus $\mathbf{M}$, must be deviatoric. According to the stress and couple-stress crack-tip fields $(3.1)_{1,4}$ the overall effective stress and flow stress fields near the crack tip assume the following asymptotic representations

$$\Sigma(r,\theta) = E\,\frac{\ell}{r}\left(\frac{r}{R}\right)^s \Gamma(\theta), \quad Y(r,\theta) = E\,\frac{\ell}{r}\left(\frac{r}{R}\right)^s \gamma(\theta), \tag{3.2}$$

respectively. From (2.9) and $(3.1)_{2,4}$ the function Γ may be defined as

$$\Gamma = (1.5\ \mathbf{M}\cdot\mathbf{M})^{1/2}, \tag{3.3}$$

and thus the leading asymptotic term of the effective stress is given by the sole contribution of the couple-stress field.

By using the steady-state derivative (2.1), the rates of the fields $\boldsymbol{\sigma}$, $\boldsymbol{\mu}$ and Y in $(3.1)_{2,4}$ and $(3.2)_2$ can be written in the form

$$\dot{\sigma}_{\alpha 3}(r,\theta) = E\,\frac{V}{r}\left(\frac{r}{R}\right)^s h_\alpha(\theta), \quad \dot{\mu}_{\alpha\beta}(r,\theta) = EV\frac{\ell^2}{r^2}\left(\frac{r}{R}\right)^s H_{\alpha\beta}(\theta), \tag{3.4}$$

$$\dot{Y}(r,\theta) = EV\frac{\ell}{r^2}\left(\frac{r}{R}\right)^s \kappa(\theta). \tag{3.5}$$

The constitutive relations (2.15)-(2.16) imply that the strain and deformation curvature rates must have the same radial dependence assumed for the stress and couple-stress rates in (3.4). Therefore, the strain and deformation curvature rates assume the following asymptotic representations

$$\dot{\varepsilon}_{\alpha 3}(r,\theta) = \frac{V}{r}\left(\frac{r}{R}\right)^s d_\alpha(\theta), \quad \dot{\chi}_{\alpha\beta}(r,\theta) = \frac{V}{r^2}\left(\frac{r}{R}\right)^s X_{\alpha\beta}(\theta). \tag{3.6}$$

By substituting the above expressions in the governing equations a system of nine first order ODEs is attained. Appropriate boundary conditions may be obtained from the skew-symmetric character of the problem along the plane $\theta = 0$, and by imposing null tractions at the crack flanks through a specialisation of (2.2), namely

$$t_{\theta 3} + \mu_{\theta\theta,r}/2 = 0, \quad \mu_{\theta r} = 0. \tag{3.7}$$

An eigenvalue problem is therefore formulated in term of the exponent s. Details of the procedure are given in [6, 9].

4. RESULTS

The results here reported are for $\nu = 0.3$. The variation of the exponent s with the ratio $\xi = \ell_e/\ell$ is plotted in Fig. 1a for $\alpha = 0.01, 0.1, 0.45$. For the same crack growth problem the classical J_2-flow theory predicts an extremely weak stress singularity [4], $s \sim -\alpha^{0.5}$. The results here obtained for couple-stress elastoplasticity show that the exponent s of the stress fields ranges between 0.5 and 0.94, (depending on α) so that, even if the symmetric stress components are not singular at the crack tip, the couple-stress and the skew-symmetric stress fields display strong singularities $(3.1)_{2,4}$, respectively of order $s-1$ and $s-2$. In particular, when the ratio ξ is reduced, the exponent s tends to decrease and for vanishing elastic characteristic length the exponent of the stress fields s approaches the value 0.5, correspondingly the couple-stress and the skew-symmetric stress fields display the singularities -0.5 and -1.5, respectively. As ξ tends to zero the limit value for s is found to be almost independent of the hardening coefficient α, so that the strength of the stress singularity increases with respect to the classical J_2-flow theory also for low hardening, mainly due to the contribution of the non-symmetric stress components. Although the skew-symmetric stresses dominate the asymptotic solution, only the couple-stress field contributes to the effective stress Σ and to the strain-energy density, so that the high stress singularity does not violate the boundedness of the flux of energy toward the crack tip.

The variation of the elastic unloading and reloading angles, θ_1 and θ_2, with ξ is then reported in Fig. 1b. Sectors $0 < \theta < \theta_1$ and –possible– $\theta_2 < \theta < \pi$ are plastic zones, while in the intermediate region $\theta_1 < \theta < \min\{\theta_2, \pi\}$, elastic unloading occurs. As ξ tends to vanish, an elastic unloading sector starts at $\theta_1 = 95.6^\circ$ and extends up to the crack flanks for each α. For $\xi = 0.1, 0.35$ a plastic reloading sector appears for $\alpha = 0.01, 0.1$ respectively and the elastic unloading sector rapidly reduces in size and tends to vanish for $\xi = 0.18, 0.6$ for the two cases. For higher values of ξ the crack-tip zone is fully plastic. As noted in [5], the considered constitutive model may give reasonable predictions for small values of ξ, namely $\xi \ll 1$, in view of the fact that the magnitude of the couple stress in the elastic sector turns out to be proportional to ξ. As the strain gra-

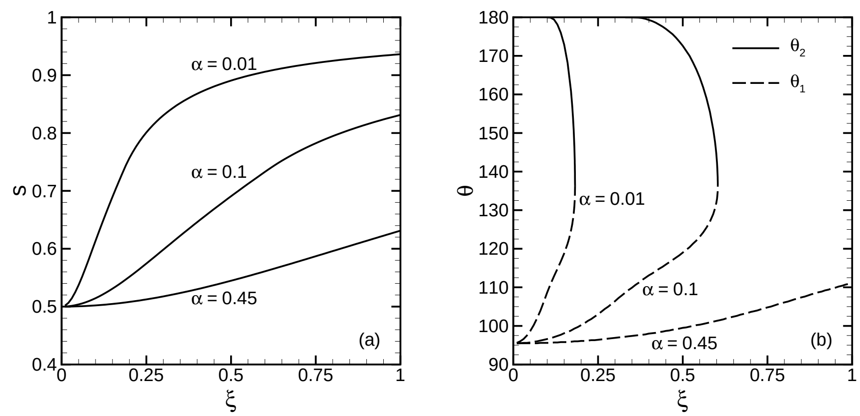

Figure 1 Exponent of the stress fields (a), elastic unloading and plastic reloading angles (b) as functions of ratio ξ, for $\nu = 0.3$.

dient effects are associated with the occurrence of geometrically necessary dislocations, they scarcely influence the elastic behaviour. Therefore, the results obtained for small values of ξ are expected to be more realistic.

In Fig. 2 the angular distributions of the asymptotic crack-tip fields for $\alpha = 0.01$, which are meaningful for $r < \ell$, are plotted for $\xi = 0.15$ ($\theta_1 = 118.9°, \theta_2 = 172.8°$). All functions are normalised by condition $s_\theta(0) = 1$. In detail, Fig. 2a shows that the symmetric stress components s_r and s_θ largely increase within the elastic unloading sector and are almost vanishing within the plastic loading sector. The angular variation of the out-of-plane velocity is also reported in Fig. 2a. In Fig. 2b the angular variations of the couple-stress field $\mathbf{M}$ and the current flow stress γ are shown. Note that the current flow stress, which is given by the single contribution of the couple-stress field, rapidly increases at the crack flanks as is usual for crack propagation problems in elastic-plastic materials displaying linear isotropic hardening [4]. The angular variations of the skew-symmetric stress components are plotted in Fig. 2c. Note that the shear stress t_θ ahead of the crack tip is negative and, thus, opposite to its counterpart in the mode III solution for classical J_2-flow theory [4] and also the radial shear stress component t_r displays negative values in a small sector ahead of the crack tip. The reason for the switch in the shear directions is unclear but certainly related to the strain gradient effects for the antiplane problem, since it agrees with the findings of Zhang et al. [10] obtained for a stationary mode III crack in couple-stress elasticity. Finally,

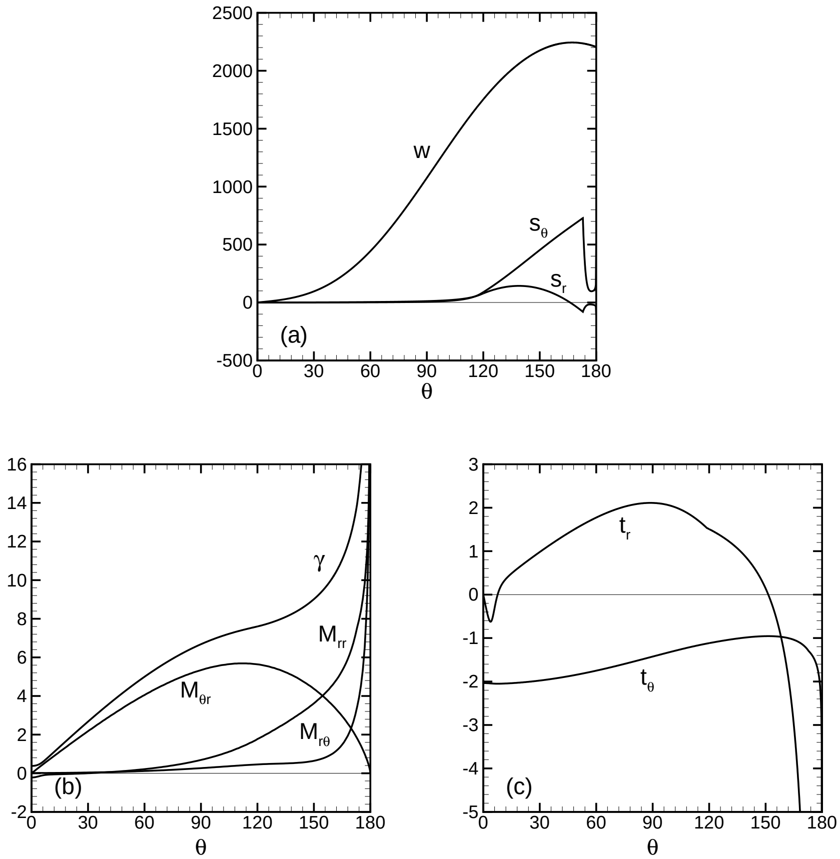

Figure 2 Angular variations of the asymptotic crack-tip fields of velocity and symmetric stress (*a*), couple-stress (*b*) and skew-symmetric stress (*c*) for $\alpha = 0.01$, $\xi = 0.15$ and $\nu = 0.3$.

note that t_θ and M_{rr} tend to large but opposite values on the crack faces, as required by $(3.7)_1$.

In conclusion, the obtained results show that the use of the strain gradient plasticity model for the analysis of the stress field in the vicinity of a propagating crack tip gives more realistic predictions on the level of traction ahead of the crack tip, allowing the detailed mechanisms by which fracture may grow and propagate in ductile metals to be understood in more depth.

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