Are scaling laws on strength of solids related to mechanics or to geometry?

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One of the largest controversial issues of the materials science community is the interpretation of scaling laws on material strength. In spite of the prevailing view, which considers mechanics as the real cause of such effects, here, we propose a different argument, purely based on geometry. Thus, as happened for relativity, geometry could again hold an unexpected and fundamental role.

Natural structures are often hierarchical, for example, with a sponge-like topology. Self-similar topologies can be mathematically well described by fractal geometry[1]. Let us consider objects of a specified family, for example, cracks and fragments[2] or dislocation cells[3], having a size larger than a given value; self-similar structures imply an inverse proportionality between the number of objects and their size raised to a positive real number D, the so-called fractal exponent (Table 1, fractal law). Interestingly, similar laws can be found in nature with respect to time, not only to space, that is, they describe the distribution of the number of events (earthquakes, for example) having a duration larger than a given value.

By fragment-size analysis it is experimentally possible to observe that, in most cases involving the crushing of three-dimensional objects[2], such an exponent mostly lies between the values of 2 and 3, with rare exceptions, and typically it is close to the intermediate value of 2.5 (Fig. 1). For example[4], experimentalists have measured a value of D equal to 2.13 for disaggregated gneiss, 2.50 for broken coal, 2.61 for sandy clays, 2.82 for terrace sands and gravels and 2.88 for glacial till. Fractal exponent values outside (that is below or above) this interval were measured only in a few cases such as artificially crushed quartz (1.89) or ash and pumice (3.54). A value of D between 2 and 3 corresponds to a comminution in which the smallest fragments provide the main contribution to the creation of the surface of fragments, whereas the largest ones contribute to defining their volume; this could be the reason for its common appearance. But similar results have also been observed in dislocation cell structures[3]: for [100]-oriented Cu single crystals, D was observed in the range between ~2.6 and ~2.8, for different applied stresses. Thus the natural topology, tending to maximize the entropy, often evolves towards a fractal set with values of D in a restricted range.

In spite of this strong experimental evidence, scaling laws on the strength of solids usually ignore geometry and are studied by applying mechanics (see ref. 5): in particular, fracture mechanics for elasto–brittle materials or strain-gradient plasticity for elasto–plastic ones. Here, we show that this prevailing view could be unjustified, presenting a possible alternative

Table 1 Scaling laws as derived according to a pure geometrical argument. Fractal quantities with anomalous physical dimensions would give a more appropriate, that is, size-independent, description of phenomena.

Fractal law	N = number of objects with size larger than r; D = fractal exponent	$N \propto r^{-D}$
Energy scaling	W = dissipated energy after the fragmentation of a solid of volume V	$W \propto V^{D/3}$; $W \propto V^{2/3}$ for $D < 2$; $W \propto V$ for $D > 3$
Energy density scaling	$\Phi = W/V$	$\Phi \propto V^{(D-3)/3}$
Fractal energy density	$\Phi^* = W/V^{D/3}$	$[\Phi^*] = [N/m^{D-1}]$
Strength scaling	Strength σ_C of a solid of size $R \propto \sqrt[3]{V}$	$\sigma_C \propto R^{(D-3)/2}$
Fractal strength	$\sigma_C^* = \sigma_C R^{(3-D)/2}$	$[\sigma_C^*] = [N/m^{(D+1)/2}]$
Complex strength scaling	R_0 = constant	$\sigma_C \propto \sqrt{1 + R_0/R}$

interpretation of size effects, purely based on geometry. We show that scaling laws (in general — see Box 1 for a biological application — although herein we refer in particular to strength of solids) are connected to the geometrical and multiscale character of the domain in which the energy exchanges occur.

THE ROLE OF GEOMETRY IN ENERGY SCALING

Let us start by considering the dissipative processes, for example, fracture or plasticity, locally arising at the interfaces of self-similar objects, such as cracks and fragments or dislocation cells, that is, scaling in size as given in Table 1 (fractal law). On computing by integration the total energy dissipated over the sample volume, it is simple to demonstrate that such energy dissipation takes place not over a euclidean domain but on a fractal one, which is always included within euclidean surfaces and volumes[2] (Table 1, energy scaling). The self-similar geometry in which the energy dissipation occurs is summarized by the fractal exponent D. Thus, according to fractal geometry, the energy dissipation density has to scale and in a specified form (Table 1, energy density scaling). This result has been experimentally confirmed[6] by evaluating the energy density dissipated during compression of quasi-brittle solids (for example, rock and concrete) having different size-scales, also monitored by the acoustic emission technique[7]. According to such a scaling, the common definitions of energy dissipations, per unit surface or per unit volume, become naive in the light of the geometrical argument. Only an energy per unit fractal volume, that we could call 'fractal energy density', seems to represent a physical, that is, scale-independent, quantity. This parameter — the constant of proportionality in the energy density scaling — has to be measured with anomalous, that is, non-integer, physical dimensions (Table 1, fractal energy density), as a consequence of the geometrical nature of the domain in which the energy dissipations occur. Note that only for $D = 3$ can the energy be considered per unit volume, as in plasticity, whereas for $D = 2$, it becomes per unit surface, as considered in fracture mechanics. Thus, even if integer physical dimensions should be considered as exceptions, unfortunately the Community often persists in considering them as the normality. To go beyond this oversimplification would represent — in our opinion — a fundamental step, comparable to that played by the introduction of rational numbers in algebra. A better understanding of scaling laws requires the abandonment of the idea of only integer physical dimensions.

THE SCALING OF STRENGTH

Considering a characteristic structural size, defined as the cubic root of the volume, and assuming for the sake of simplicity that the dissipated energy density is proportional to the square of the strength (for details see ref. 8), implies a geometrical scaling on material strength (Table 1, strength scaling). Note that to obtain such a scaling law no micromechanical models have been introduced, but again only a geometrical argument. Such competition between surfaces and volumes, is, in our opinion, the real geometrical foundation of the 'universality' observed in the scaling laws of different fields (for example, mechanics and biology[9], see Box 1). Only on considering euclidean volumes for energy dissipation domains, that is, $D = 3$, would one deduce that the material strength is size-independent, as assumed in classical plasticity; the opposite case yields a scaling also deduced by (linear elastic) fracture mechanics, using the common assumption that crack length is proportional to the characteristic structural size.

Furthermore, in spite of its common acceptance, the physical dimensions of the stress, being based on euclidean geometry, could be misleading. In fact, considering the scaling of strength, it is evident that the real, that is, size-independent, physical property, that we could call 'fractal strength', is represented by the constant of proportionality. This parameter has to be measured with anomalous non-integer physical dimensions, as a consequence of the geometrical nature of the domain in which the energy dissipations occur (Table 1, fractal strength). Note that only for $D = 3$ has the fractal strength the physical dimensions of a classical stress, the critical parameter in plasticity; the opposite case corresponding to $D = 2$, can be measured with the same physical dimensions of the stress-intensity factor, the critical parameter in fracture.

In addition, assuming that at small size-scales surface effects are predominant ($D = 2$), whereas at large size scales volume effects prevail ($D = 3$), the previous considerations would yield a size-effect law of the form in Table 1 (complex strength

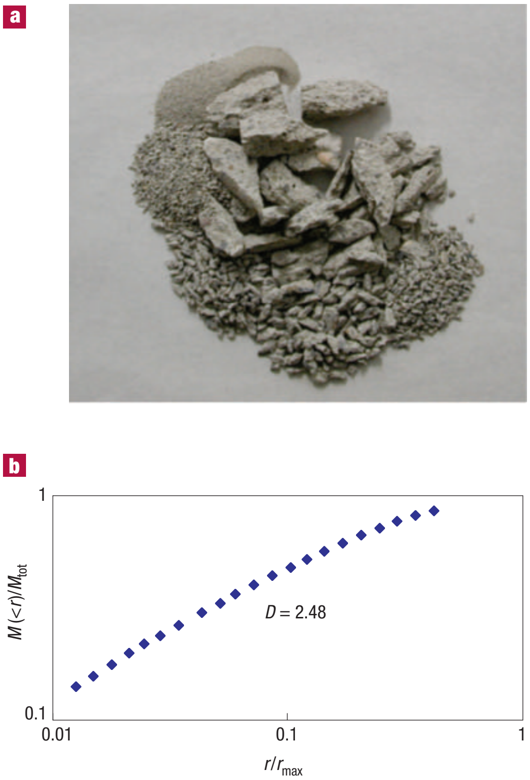

Figure 1 **Fragment analysis. a, Often nature produces geometrical self-similar fragments, to maximize the entropy. These fragments were obtained by compression of a quasi-brittle material (concrete), and have a *D* value of around 2. b, Fragments of concrete obtained under artificial fragmentation (drilling), were analysed by computing the cumulative mass of particles with size smaller than *r* (divided by the total fragment mass) versus the same size *r* (divided by the size of the largest particle). Geometrical self-similar size distribution would appear as a straight line in this bilogarithmic diagram; from the slope, the fractal dimension can be deduced, here found to be close to 2.5.**

scaling). The CEB (European) concrete design code and the German standards code DIN adopted this scaling law, an unequivocal sign of its effectiveness[5]. This size-effect law was first proposed[10] for fracture strength in 1994 and later reproposed in plasticity for hardness versus indentation depth[11]. On fitting the experimental data with such a law on indentation tests we find very good coefficients of correlations (R^2): 0.991 for single crystal and 0.995 for polycrystalline copper. Thus, experiments on both brittle and plastic materials seem to confirm the fractal argument.

BOX 1 BIOLOGY'S SCALING LAWS

We may observe that nutrition is locally a surface process, occurring as dissipation. Thus, if the energy dissipated is replaced by the energy B spent on growth, and the mass (or volume) of fragments by the mass M of biological cells, we would expect $B \propto M^{D/3}$, with the limit cases of $B \propto M^{2/3}$ and $B \propto M$. We note that in the literature $B \propto M^{3/4}$ is extensively assumed to fit an enormous number of biological cell growth processes[9]: for example, basal metabolic rate versus mass of mammals. Thus, no biological model is required to justify this scaling but simply the same geometrical argument applied in mechanics.

Finally, for a complete parallel of biology and mechanics, we expect a deviation from the 3/4 exponent towards 2/3 for small mammals (surface dominates at small scale) in the energy growth versus mass law. This deviation, nowadays still considered unjustified and unexpected, has been experimentally observed[9]. It is clear that we would expect deviation towards the unitary exponent (where volume dominates) for very large mammals such as whales or dinosaurs.

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WHAT OF THE FUTURE?

Owing to the increasing interest in scaling laws for material strength[12], we suggest reconsidering the role of geometry, and that it alone may be considered responsible for the observed size effects in some materials. Further, in spite of the prevailing views, pure geometry and competition between surfaces and volumes may explain the scaling laws observed in different fields, such as mechanics and biology. As in the past, the crucial role of the scaling laws on strength of solids was represented by the importance of predicting the strength of large-sized structures such as buildings, starting from experiments on material strength at the more familiar human size-scale, in the future the crucial role could become the prediction of the strength of nanostructures, as emphasized by quantized fracture mechanics[13]. Thus, nanomechanics is going to require a better understanding of such scaling laws, and the multiscale geometry of nature could represent the key.

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