Morphological equilibrium of short channels dissecting the tidal flats of coastal lagoons

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Abstract.

The equilibrium bed profile of tidal channels dissecting the tidal flats of coastal lagoons is studied within a rational one-dimensional framework. A general analytical solution is obtained which expresses the bed profile in terms of a modified longitudinal coordinate, accounting implicitly for channel convergence and adjacent shoals. For values of the relevant parameters typical of coastal lagoons, inertia and friction effects, as well as overtides, are shown to provide minor corrections to the equilibrium bed profile. The overall shape of the profile is also shown to be slightly affected by the equilibrium condition used in its derivation, consisting in a requirement on either residual erosion/deposition fluxes or maximum velocity. In particular, the asymptotic form of the analytical solution is common to both the equilibrium requirements, thus suggesting the existence of general morphological relationships relating the depth at the channel mouth or, alternatively, the length of the channel to the tidal amplitude, to the degree of channel convergence, to the critical velocity for erosion/deposition and to the extent of intertidal storage areas. The profile shape is affected as well: for instance nearly constant-depth channels tend to form when convergence is strong. The equilibrium configuration also implies that a power law relationship of the type proposed by O’Brien-Jarret-Marchi for tidal inlets, relating the channel cross section to the tidal prism, holds throughout the entire channel. Finally, the theoretical profile is shown to reasonably reproduce the bed configurations observed in a number of tidal channels surveyed within the Lagoon of Venice (Italy).

1. Introduction

The problem of determining the morphological features of tidal channels has recently received considerable attention [de Swart and Zimmermann, 2009]. Two major research lines can be identified when considering the equilibrium shape of these channels. The first is based on analytical or semi-analytical solutions obtained by suitably simplifying the relevant equations, and aims at deriving general results in order to capture the key features of the hydrodynamic field [Friedrichs and Aubrey, 1994; Lanzoni and Seminara, 1998; Savenije et al., 2008] and of morphodynamic processes shaping the channel bed [Friedrichs and Aubrey, 1996; Schuttselaars and de Swart, 2000; Prandle, 2003; Seminara et al., 2010]. The second relies on more or less detailed numerical models used to explore some forcing conditions and planform geometries [Lanzoni and Seminara, 2002; Pritchard et al., 2002; Tambroni et al., 2005; Todeschini et al., 2008].

In the case of a short (with respect to the tidal wavelength), frictionless tidal embayment of constant width, Schuttselaars and de Swart [1996] showed that, if the landward shoaling portion of the embayment is neglected, a morphodynamic equilibrium state is ensured by a linearly varying longitudinal profile. This profile is unique, is reached for arbitrary initial conditions and is characterized by a vanishing net transport in the entire embayment (i.e., by a spatially uniform bed shear stress distribution). Schuttselaars and de Swart [2000] subsequently extended the analysis to embayments with arbitrary lengths, by considering also the effects of a depth-dependent linearized friction. For a sinusoidal tidal forcing, they found that a unique equilibrium solution exists only for embayment shorter than the frictional length scale of the tidal wave, while no equilibrium is possible for larger embayments. The presence of external overtides, inducing a tidal asymmetry which contributes to net sediment transport, determines the possible existence of
multiple equilibria, even though for a range of the governing parameters of theoretical rather than practical relevance.

The role played by the landward shoaling portion of a tidal embayment was investigated by Friedrichs and Aubrey [1996], with particular reference to the morphology of tidally dominated intertidal flats. Under the usual assumption of a short frictionless embayment of constant width, they sought an equilibrium profile by requiring that the tidal velocity never exceeds the threshold for sediment motion throughout the entire tidal cycle, a more restrictive condition with respect to that imposed by Schutteelaars and de Swart [1996]. The resulting bed profile varies linearly only below mean sea level, while it assumes a concave configuration, described by a quarter of sine, near the landward embayment head, where the bed emerges above mean sea level.

Recently, Seminara et al. [2010], focusing on the morphology of tidal channels, generalized the problem accounting for longitudinal variations induced by planform channel convergence and for storage associated with tidal flats flanking the channel. By means of a perturbation analysis, they treated the landward region of the shoaling channel, subject to wetting and drying, as an inner boundary layer, thus obtaining a fully analytical solution valid throughout the entire channel. In the absence of sediment exchange with the sea, this solution was found to be associated with a ‘static’ equilibrium of the channel, implying a vanishing sediment transport at each instant of the tidal cycle. The analytical solution indicates that the concavity of the bed profile in the channel portions located below mean sea level arises from two major effects, namely channel convergence and longitudinal variations of the relative roughness. It also turns out that tidal flats affect significantly the hydrodynamics, leading to flow acceleration in the channel and causing shortening of the equilibrium channel length, the more so as the flats widen.

The assumptions of short channel length and negligible friction effects were relaxed by Prandle [2003], who investigated analytically the case of a ‘synchronous’ estuary (i.e., with constant tidal amplitude and, hence, ‘ideal’ according to the definition of Savenije et al. [2005]) dominated by convergence and friction. However, as the the length of the channel increases and the tidal wave is distorted, the solution gets more and more complicated. For this reason, Lanzi and Seminara [2002] and Todeschini et al. [2008], for sandy tidal channels, and Pritchard et al. [2002], for intertidal mud flats, tackled the problem numerically. In the case of sandy tidal channels, the numerical results indicate that a well defined equilibrium bed profile exists for given external parameters, independently of the initial conditions. In the absence of external sediment inputs, the system is in particular observed to evolve asymptotically towards a configuration ensuring a vanishing small net sediment discharge all along the channel, i.e., a nearly symmetrical distribution of the tidal flow during flood and ebh phases. This result incidentally supports the complementary approach put forward by Dronkers [1986] to assess morphological equilibrium in tidal channels on the basis of the analysis of tidal asymmetry. Pritchard et al. [2002] showed that even though the theoretical convex profile by Friedrichs and Aubrey [1996] approximates real bed configurations on intertidal mudflats well, the flats have a tendency to accumulate sediment over long times when realistic values of the seaward boundary concentration are considered. Tidal asymmetry (flow-or ebh-dominance) leads to a steeper flat, and ebh dominance can cause the flat to retreat landwards in the long term. This behavior was found to be strictly related to settling lag effects [Pritchard and Hogg, 2003], causing a hysteresis in the response of suspended mud concentration to current speeds and, hence, a not perfect symmetry between erosion and deposition fluxes determining a net import of sediment.

In this paper we focus on tidal channels which typically develop within tidal flats of micro- and meso-tidal coastal lagoons (barrier lagoon being uncommon under macro-tidal conditions). These channels are usually short enough (of the order of a few kilometer) to assume a quasi-static propagation of the tidal wave, and their dynamics is essentially controlled by the action of gravity, while friction and inertia play a minor role. Two main classes of such channels can be distinguished within coastal lagoons, depending on the location of their head [Ashley and Zeff, 1988]. The first corresponds to branching dead end channels, terminating in salt marshes typically fringing the larger channels or located in the innermost part of a lagoon. The bed of these channels tends to shoal landward, forming a region that periodically wets and dries during the tidal cycle: this is the configuration investigated by Seminara et al. [2010]. The second class of channels include the through-flowing channels connecting the lagoon to the sea, dissecting the much deeper tidal flats usually located close to a lagoon inlet or in large inner soundings (e.g., Figure 1). The head of these channels, usually located near to the divide of tidal sub-basins, tends to attain the elevation of mean spring low water level and, hence, is normally submerged during the whole tidal cycle. It is just this second specific, yet common, type of tidal channels that we are interested in. In particular, we here again try to clarify some fundamental questions related to the links between the long term behavior of tidal channels and the morphodynamics of costal lagoons.

The first issue concerns the type of equilibrium condition that likely determines the observed morphologic configurations. We will discuss how this condition possibly relates to the evolutionary trend exhibited by a given lagoon, considering either a vanishing tidally averaged residual flux resulting from erosion/deposition processes or the requirement that, throughout the tidal cycle, the maximum velocity (a measure of the intensity of sediment transport) never exceeds a given threshold, proportional to that for incipient sediment erosion.

The second question we will address is associated with the typical scales and morphological relations suggested by the shape of the equilibrium channel profile. In particular, we will substantiate the along channel validity of the relations between the tidal prism and the cross sectional area and, considering the morphologic signatures emerging from observational evidence (e.g., the spatial distribution of the mean channel width to mean depth ratio) we will discuss the mutual influence of bed profile variations and channel planform configuration (usually exhibiting a certain degree of landward convergence).

The rest of the paper is organized as follow. Section 2 describes the main morphologic features of a few channels
dissecting the tidal flats of the Venice lagoon, a typical microtidal lagoon located in North of Italy. In Section 3 we set up the mathematical formulation to discuss the validity of the adopted simplifications. In Section 4, we determine analytically the flow field structure and the equilibrium profiles resulting from two different equilibrium requirements, discussing the morphological consequences of these results and providing also a comparison with field evidence. Finally, in Section 5 we report some conclusions and suggestions for future research.

2. Observational Evidence

We have analyzed the bathymetric and planform configurations of a number of tidal channels dissecting the tidal flats of the Venice Lagoon (Italy), a semi-diurnal (tidal period $T = 12.4$ h), micro-tidal (tidal amplitude $g^* \approx 0.5$ m) lagoon extending over an area of roughly $550$ km$^2$ (hereafter a superscript star will denote dimensional quantities).

The present morphology of the lagoon, characterized by networks of channels departing from three inlets and cutting into the intertidal areas, is the result of a morphodynamic evolution which has recently experienced a relatively rapid acceleration. In the last century, in fact, large areas of the lagoon have experienced a progressive degradation due to relative sea level rise (including subsidence) and erosion processes enhanced by human activities. We have then focused our attention on the northern part of the lagoon, the least disturbed, where tidal flats have an elevation ranging between $-0.5$ m and $-1.2$ m above mean sea level (MSL), with a mean value of $-0.75$ m [Carniello et al., 2009] close to the mean spring low water level. The bathymetric survey used to extract channel features was carried out in 2003 [Consorzio Venezia Nuova-Technital, 2007] using different techniques (multibeam, single beam, GPS, orthophoto restitution, direct topographic survey) in order to obtain precise results for each range of elevations. The standard error in bottom elevation data is $\pm 5$ cm for tidal flats and $\pm 10$ cm for tidal channel beds. The spatial resolution of the data on tidal flats is approximately $100 \times 80$ m, while the measuring grid is suitably thickened ($50 \times 10$ m) when surveying the cross sections of tidal channels. The minimum width of surveyed channels is about $30$ m.

Figure 1 shows the planforms of the various channels, determined on the basis of aerial photographs, and their location within the lagoon system. The channel length, $L^*$, and the channel width at the mouth, $B^*$, vary in the ranges $1.0 - 1.3$ km and $45 - 120$ m, respectively. The longitudinal bed profiles (Figure 2a) are generally characterized by an upward concavity while the planform funnelling shape is well approximated by an exponential law (Figure 2b). Figure 2c indicates that the width to depth ratio (the flow depth being computed with respect to mean sea level) varies in the range $15 \div 35$ and shows a certain degree of variability along the channel, although a clear general trend does not emerge. The channel beds are composed by very fine sand-silt sediment, with a mean grain size $d_{50} = 0.05$ mm [Maggiarotto et al. et al., 2008]. The critical bed shear stress for the erosion of this sediment, estimated from the Shields diagram, thus neglecting the possible effects of silty fractions, is $\tau^* \approx 0.15$ N/m$^2$, and corresponds to a critical velocity $U^*_c \approx 0.17$ m/s. This latter estimate has been obtained by expressing the friction coefficient through the Manning-Strickler relation and assuming the roughness coefficient $K_s^* = 35$ m$^{1/3}$s$^{-1}$ determined on the basis of the calibration of a two-dimensional hydrodynamic, finite element model of the Venice lagoon [D’Alpaos and Defina, 2007].

3. Formulation of the problem

3.1. Governing equations

Let us consider a straight tidal channel of length $L^*$ dissecting an idealized tidal flat of total width $B^*$. We assume that the cross-section of the channel can be described by an equivalent rectangular section of width $B^*$ and depth $D^*$. In the absence of fresh water upstream supply and of density- or wind-driven currents, the classical one-dimensional continuity and momentum equations governing the flow field throughout the channel are:

$$\frac{\partial U^*}{\partial t^*} + U^* \frac{\partial U^*}{\partial x^*} + g^* \frac{\partial H^*}{\partial x^*} + \frac{\tau^*}{\rho^* D^*} = F^*$$  \hspace{1cm} (1)

$$r^* \frac{\partial B^* D^*}{\partial t^*} + \frac{\partial B^* U^* D^*}{\partial x^*} = 0$$  \hspace{1cm} (2)

where $x^*$ is the longitudinal coordinate, pointing seaward and originating at the channel head, $t^*$ is time, $H^*$ is the local free surface elevation, $U^*$ the cross-sectionally averaged velocity, $g^*$ is gravity, $\rho^*$ is water density, and $\tau^* = \rho^* C^f |U^*| |U^*|$ is the cross-sectionally averaged bed shear stress. The coefficient $r^* = B^*/B^*$ multiplying the storage term in the continuity equation (2) accounts for storage effects of tidal flats flanking the channel, while their dynamic role is assumed to be small as typically occur in micro- and meso-tidal coastal lagoons [Friedrichs and Aubrey, 1994; Savenije, 2005; Seminara et al., 2010]. Note that at this stage of the analysis the channel width is assumed to vary arbitrarily along the channel, thus relaxing the hypothesis of exponential funnelling usually adopted in the study of tidal channels [Lanzoni and Seminara, 2002; Seminara et al., 2010, e.g.]. Finally, the term $F^*$ in (1), which is frequently neglected [Friedrichs and Aubrey, 1994, e.g.], accounts for the possible exchange of momentum between the channel and the tidal flats [Fagherazzi et al., 2003]. As discussed by

![Figure 2](image-url)
Dronkers [1964], this term can be relevant only during the ebb phase, when the water abandoning the tidal flats moves into the channel. It can be easily demonstrated that

$$\mathcal{F}^* = \frac{q_n^*}{B^* D^*} (U^*_m \cos \phi - U^*) = (U^* - U^*_m \cos \phi) \left( \frac{r_s - 1}{D^*} \right) \frac{\partial H^*}{\partial x^*},$$

(3)

where \(q_n^*\) is the discharge exchanged per unit channel length, \(U^*_m\) is the velocity of the flow entering the channel from the tidal flats, and \(\phi\) is the angle between this flow and the direction of the channel axis.

The boundary conditions associated with equations (1) and (2) are the tidal oscillation at the channel mouth and the flow discharge at the landward channel end. The latter can be specified observing that in the case here considered the channel head is not only submerged during the entire tidal cycle but it is also quite close to the divide of the subbasin drained by the tidal channel. We thus assume, with a good approximation, a vanishing flow discharge at the landward channel end:

$$U^* D^* B^* = 0 \quad (x^* = 0),$$

(4)

In order to determine how the erodible channel bed adapt to tidal currents we assume that the sediment is transported mainly in suspension and introduce the bed evolution equation [Pritchard et al., 2002, e.g.]

$$\frac{\partial \eta^*}{\partial t^*} = \frac{1}{1 - p} (Q^*_d - Q^*_e),$$

(5)

relating temporal variations of bed elevation \(\eta^*\) (= \(H^* - D^*\)) to erosion and deposition fluxes \(Q^*_d\) and \(Q^*_e\). Here \(p\) is void fraction of the sediment bed, while \(Q^*_d\) and \(Q^*_e\) are described through the classical Partheniades-Krone formulation, read [Dyer, 1986; D’Alpaos et al., 2006, e.g.]

$$Q^*_d = Q^*_o (\frac{|\tau^*|}{\tau^*_d} - 1), \quad |\tau^*| > \tau^*_d$$

(6)

$$Q^*_e = c_s u^*_w (1 - \frac{|\tau^*|}{\tau^*_e}), \quad |\tau^*| < \tau^*_e$$

(7)

where \(\tau^*_d\) and \(\tau^*_e\) are the threshold of the bed shear stress for sediment erosion and deposition, \(Q^*_o\) is a characteristic erosion rate, depending on sediment properties, \(u^*_w\) is the sediment settling velocity, and \(c_s\) is the sediment concentration at the bed.

Although the existence of a threshold \(\tau^*_e\) for erosion is widely accepted, the physical meaning of the threshold for deposition \(\tau^*_d\) is less clear. For instance, Winterwerp [2006] suggests that the process of deposition can be better represented by the settling flux \(u^*_w c_s\). Moreover, Maa et al. [2008] noted that the erosion threshold varies with depth during the tidal cycle, i.e. \(Q^*_o = Q^*_o(t^*)\). Because of these uncertainties in the description of the process and the difficulties in the experimental determination of the critical deposition stress [Sanford and Maa, 2005; Pritchard and Hogg, 2003], while keeping the structure of the Partheniades-Krone formulation, as a first approximation we assume that the net flux \(Q^*_d - Q^*_e\) varies continuously by assuming \(\tau^*_d \approx \tau^*_e\) and \(Q^*_o \approx u^*_w c_s\). Expressing the bed shear stress in terms of the threshold velocity for erosion \((\tau^*_e = \rho^* C_f U^*_e)^2\), we eventually write

$$Q^*_d - Q^*_e \approx Q^*_o \left( \frac{U^*_e^2}{U^*_m^2} - 1 \right)$$

(8)

The morphologic equilibrium condition resulting from (5) and (8) is determined by requiring that net variations of bed level over a given tidal cycle vanish. Denoting by angle brackets the tidal average of a generic function \(f\),

$$\langle f \rangle = \frac{1}{T} \int_{0}^{T} f \, dt^*,$$

(9)

such equilibrium condition implies that, on average, erosion must balance deposition, namely:

$$\langle \frac{\partial \eta^*}{\partial t^*} \rangle = \frac{1}{1 - p} \left( \langle Q^*_d \rangle - \langle Q^*_e \rangle \right) = 0$$

(10)

In other words the net suspended sediment flux must either vanish or, in the presence of a possible sediment import/export associated with the long term lagoon evolution, keep constant along the channel [Lanzoni and Seminara, 2002; Tambroni et al., 2005; Todeschini et al., 2008]. An alternative, more restrictive equilibrium condition will be discussed in Section 4.2.

### 3.2 Dimensionless formulation

In order to derive an analytical solution of the problem, it is useful to cast the relevant equations in dimensionless form. Following the approach suggested by Toffolon et al. [2006], we consider the external scales (i.e., not resulting from morphological processes) given by the amplitude \(a^\omega\) and the frequency \(\omega^\pi/T^\pi\) of the main harmonic constituent of the forcing tide, the critical velocity for sediment erosion \(U^*_e = U^*_c\), and the width of the channel mouth \(B^*_m\). The longitudinal flow field variations are consequently characterized through the intrinsic length scale:

$$L^* = \frac{U^*_c}{\omega^\pi}.$$ 

(11)

The dimensionless form of equations (1) and (2), governing the flow field, then read:

$$\alpha \left( \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} \right) + \frac{\partial H}{\partial t} + \chi \frac{D}{D^*} = \alpha \mathcal{F},$$

(12)

$$r_s B \frac{\partial H}{\partial t} + \frac{\partial BU D}{\partial x} = 0,$$

(13)

where \(\mathcal{F} = (U - U_m \cos \phi)(r_s - 1) D^{-1} \partial H/\partial t\), and \(t^* = t \omega^\pi\), \(\{x, L\} = \{x^*, L^*\}/L^*_1\), \(B = B^*/B^*_m\), \(\{H, D, \eta\} = \{H^*, D^*, \eta^*\}/a^\omega\), \(U = U^*_c/U^*_c\), 

$$\tau = \frac{\tau^* \rho^* C_f U^*_e^2}{\rho^\omega}, \quad \alpha = \frac{U^*_e^2}{\rho^\omega a^\omega}, \quad \chi = C_f \frac{U^*_e^3}{g^\omega a^\omega^2 \omega^\pi}.$$ 

(16)

It is immediately recognized that for channels dissecting the tidal flats of coastal lagoons \(L^*_1\) is usually much smaller than the frictionless tidal wavelength \(L^*_T = T^\pi \sqrt{g^\omega D^*}\), and the dimensionless parameters \(\alpha\) and \(\chi\), weighting inertia and friction with respect to gravitational effects in the momentum equation, are typically small. For example, for the channels reported in Figure 1 it results that \(L^*_1/L^*_T \sim 4 \cdot 10^{-3}, \alpha \sim 5 \cdot 10^{-3}\), and \(\chi \sim 4 \cdot 10^{-2}\).

The dimensionless form of the tidally averaged bed evolution equation

\[\mathcal{F}^* = \frac{q_n^*}{B^* D^*} (U^*_m \cos \phi - U^*) = (U^* - U^*_m \cos \phi) \left( \frac{r_s - 1}{D^*} \right) \frac{\partial H^*}{\partial x^*},\]

\[\langle f \rangle = \frac{1}{T} \int_{0}^{T} f \, dt^*,\]

\[\langle \frac{\partial \eta^*}{\partial t^*} \rangle = \frac{1}{1 - p} \left( \langle Q^*_d \rangle - \langle Q^*_e \rangle \right) = 0\]

\[L^* = \frac{U^*_c}{\omega^\pi}\]

\[\alpha \left( \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} \right) + \frac{\partial H}{\partial t} + \chi \frac{D}{D^*} = \alpha \mathcal{F},\]

\[r_s B \frac{\partial H}{\partial t} + \frac{\partial BU D}{\partial x} = 0\]

\[\tau = \frac{\tau^* \rho^* C_f U^*_e^2}{\rho^\omega}, \quad \alpha = \frac{U^*_e^2}{\rho^\omega a^\omega}, \quad \chi = C_f \frac{U^*_e^3}{g^\omega a^\omega^2 \omega^\pi}\]

\[\{x, L\} = \{x^*, L^*\}/L^*_1\]

\[B = B^*/B^*_m\]

\[\{H, D, \eta\} = \{H^*, D^*, \eta^*\}/a^\omega\]

\[U = U^*_c/U^*_c\]

\[\tau = \frac{\tau^* \rho^* C_f U^*_e^2}{\rho^\omega}, \quad \alpha = \frac{U^*_e^2}{\rho^\omega a^\omega}, \quad \chi = C_f \frac{U^*_e^3}{g^\omega a^\omega^2 \omega^\pi}\]

\[L^*_1/L^*_T \sim 4 \cdot 10^{-3}, \alpha \sim 5 \cdot 10^{-3}\], and \(\chi \sim 4 \cdot 10^{-2}\).
\[ \frac{\partial \eta}{\partial t_m} = 1 - (U^2), \]

(17)

points out the existence of a slow morphodynamic time \( t_m = t' / T_m \), where \( T_m = (1 - \rho) \alpha / Q_{m0} \) is the geomorphic time scale. Since the tidal period \( T \) is much shorter than \( T_m \), tidal oscillations can be regarded as high-frequency perturbations of a bed profile \( \eta(x, t, t_m) \). The long term, much slower evolution of this profile is described by (17) and is caused by residual (i.e., tidally averaged) sediment fluxes which are usually very small. Relatively intense net sediment fluxes likely occur only if the channel is very far from its equilibrium configuration, e.g., as a consequence of dredging activities, impulsive input of large amounts of sediment, impact of heavy storms.

4. Results

4.1. Flow field solution

In order to find an analytical solution for the flow field, we take advantage of the fact that the parameters \( \alpha \) and \( \chi \) are usually small in coastal lagoons, setting up a rational perturbative model (Appendix A). At the leading order of approximation, \( O(\alpha^2, \chi^2) \), inertia and friction can be neglected in the momentum equation (12). This implies a quasi-static propagation of the tide along the channel: \( H_0 \) depends only on \( t \) and is determined by the water elevation prescribed at the channel mouth [Friedrichs and Aubrey, 1996; Schutte-laars and de Swart, 1999; Seminara et al., 2010].

Integrating the continuity equation (13) and imposing the landward boundary condition (4), we find

\[ U_0 = \frac{\xi}{(H_0 - \eta_0)} \frac{\partial H_0}{\partial t}, \]

(18)

where

\[ \xi(x) = \frac{1}{B} \int_0^x r_s B dx' \]

(19)

is a modified longitudinal variable which implicitly accounts for channel width variations. A few relevant values of the flow velocity characterizing a generic tidal cycle are readily determined from equation (18): the maximum velocity \( U_{max}(x) = \max(|U_0|) \), the mean square root, \( U_{rms} = \sqrt{\langle U_0^2 \rangle} \), and the absolute value \( U_{av} = \langle |U_0| \rangle \) of the tidally averaged velocity.

Assuming that the semidiurnal tide can be described by its first harmonic \( M_2 \) (the effects of possible external over-tides are analyzed in Appendix B) it results that:

\[ H_0 = \cos t. \]

(20)

and, further assuming that the bed is always wet (i.e. \( \eta_0 < -1 \)),

\[ U_{rms} = \xi \frac{\eta_0}{\sqrt{\eta_0^2 - 1}}. \]

(21)

\[ U_{max} = \frac{\xi}{\sqrt{\eta_0^2 - 1}}. \]

(22)

\[ U_{av} = \frac{\xi}{\pi} \ln \frac{\eta_0 - 1}{\eta_0 + 1}, \]

(23)

where (21) and (23) are found by integrating over one tidal cycle the square and the absolute value of (18), while (22) results from equating to zero the time derivative of \( U_0 \). It then turns out that when the tidal channel is relatively deep compared to the tidal amplitude, so that the velocity is approximately sinusoidal, \( U_{max} \) occurs at time \( t = \arccos(1/\eta_0) \). Moreover, \( U_{max} / U_{rms} = \sqrt{2} \) and \( U_{av} / U_{rms} = 2\sqrt{2}/\pi \).

Although the first order, \( O(\alpha) \) and \( O(\chi) \), corrections to \( H_0 \) and \( U_0 \) associated with inertia and friction effects can be easily determined, we defer their derivation and discussion to Appendix A. These contributions, in fact, do not alter appreciably the structure of the equilibrium channel profile we are interested in.

4.2. Bed equilibrium profile

The equilibrium bed profile is easily obtained from (17) by setting \( \partial \eta / \partial t_m = 0 \) and, hence, \( U^2 = 1 \). Further assuming a sinusoidal tidal forcing (21) gives

\[ \eta_0 = -\frac{1 + \xi^2}{\sqrt{1 + 2\xi^2}}. \]

(24)

This profile attains the value \( \eta_0 = -1 \) at the landward channel head, where the flow depth vanishes only at low tide (Figure 3a) and for small \( \xi \) tends to a flat configuration with negative concavity, i.e., \( \eta_0 = -1 - \xi^4/2 + O(\xi^6) \). Conversely,
for large enough values of \( \xi (\xi \gg 1) \) the bed elevation tends to vary linearly with \( \xi \),

\[
\eta_0 = -\frac{\xi}{\sqrt{2}} + O(\xi^{-1}),
\]

thus recovering asymptotically (far enough from the channel head) the linear relationship between \( \eta_0 \) and \( \xi \) embodied by the analytical solutions obtained by Friedrichs and Aubrey [1996]; Schutteelaars and de Swart [1996]; Seminara et al. [2010]. Importantly, we note that the influence of the planform shape of the channel and storage effects are implicitly accounted for through the modified longitudinal coordinate \( \xi \). Moreover, owing to the choice of the intrinsic quantity \( L^* \) as longitudinal length scale, the channel bed shape does not depend on the actual channel length \( L^* \), which simply determines in terms of \( x \) (and hence of \( \xi \)) where the channel ends.

If we assume that, as illustrated in Figure 2b, channel width variations can be approximated by the exponential law

\[
B = \exp\left(\frac{x - L}{L_b}\right)
\]

and that \( r_s \) is constant, it turns out that

\[
\xi = r_s L_b \left[1 - \exp\left(\frac{x}{L_b}\right)\right],
\]

with \( L_b = L^*/L^* \) the convergence length. Therefore, as channel funneling increases (smaller \( L_b \)) the bed profile tends to attain an upward concavity (Figure 3b), leading to a nearly constant-depth channel. We will discuss more in detail in Section 4.4 the possible relation between altimetric and planimetric configurations of tidal channels at equilibrium.

Equation (17) indicates that a dynamic equilibrium (i.e., a condition in which both erosion and deposition occur, but they balance each other) may exist when a certain amount of sediment (originating either within the lagoon or offshore) is imported/exported through the channel mouth. This result, however, relies on the assumption, subtended by (17), that not only settling and scour lag effects, but also the actual value of the suspended sediment concentration provide higher order corrections to the equilibrium profile. In other words, in the case of short channels the form of the equilibrium profile is essentially controlled by the temporal symmetry of the flow field, a result in agreement with the indications provided by the numerical solution of the complete equations [Lanzoni and Seminara, 2002; Tambroni et al., 2005; Todeschini et al., 2008]. It is also important to note that the existence of a dynamic equilibrium characterized by a relatively small, spatially constant, residual sediment flux is compatible not only with higher order contributions to the channel profile provided, e.g., by the settling and scour lag effects [Pritchard and Hogg, 2003], but also with the fact that, in practice, equilibrium can be attained only asymptotically. Indeed, as suggested by numerical models, residual sediment fluxes progressively reduce as equilibrium is approached. On the other hand, on the long term embedded by this asymptotic process, tidal channels morphology is also conditioned by the morphodynamic evolution experienced by the entire lagoon which, in turn, strongly depends on possible variations of external forcing (e.g., owing to relative sea level variations induced by subsidence and eustatism, or to the intense pressure exerted by human activities on coastal environments). This suggests that real channels can never achieve equilibrium rigorously, but they rather attain a near-equilibrium state, which can be reasonably approximated by the dynamic equilibrium condition derived requiring that \( (U^*)^2 = 1 \) in (17).

A more restrictive equilibrium condition, denoted as static equilibrium by Seminara et al. [2010], is the one whereby no sediment is transported at any instant of the tidal cycle. This condition implies that \( \partial \eta_0/\partial t = 0 \) at any time and requires that the maximum velocity is constant along the channel and equal to the threshold for sediment motion. Clearly, no instantaneous or residual sediment flux and, hence, no long term import/export of sediment, are in principle admitted by static equilibrium. On the other hand, Friedrichs [1995] showed that the observed spring tide peak shear stress (and, therefore, the maximum velocity) keeps nearly constant along many tidal channels, attaining a value \( \geq \tau^*_e \) just necessary to maintain a zero gradient in net along channel sediment transport. Such experimental evidence suggests that the static equilibrium condition can be generalized by setting \( U_{rms} = \kappa U^*_c \), with the proportionality coefficient \( \kappa \) depending on the amount of sediment imported/exported on the long term through the channel mouth.

Here we assume that \( U_{max}^* \) equals the maximum velocity of a sinusoidal function with quadratic average \( U_c^* \), setting \( \kappa = \sqrt{2} \). This choice leads to an equilibrium profile,

\[
\eta_0 = -\sqrt{1 + \frac{\xi^2}{2}},
\]

which, as shown in Figure 4, is fully coherent with that prescribed by (24). In both cases the bed tends to vary linearly as \( \eta_0 = -\xi/\sqrt{2} \) for large enough values of \( \xi \). This trend is also in accordance with the linear dependence, in terms of the modified variable \( \xi \), prescribed by the solutions obtained by Friedrichs and Aubrey [1996] and Seminara et al. [2010] for wetting and drying tidal flats and dead end channels, respectively. Thus, far from the channel head the two cases (through-flow or dead-end channel) cannot be distinguished.

The slightly different bed elevations observed close to channel head reflect the different behavior of the flow field near the landward boundary illustrated in Figure 5. When a dynamic equilibrium condition on the flux (i.e., \( U_{rms} = 1 \)) is imposed, the velocity strongly deviates from a sinusoidal trend near the channel head (see Figure 5a and also Table 1),

**Figure 4.** Comparison between the dimensionless bed profiles obtained: imposing a constant quadratic averaged velocity, (24), dashed thick black line; imposing a constant maximum velocity, (28), continuous red line; derived by Friedrichs and Aubrey [1996] and extended by Seminara et al. [2010], continuous thick black line; linear, dash-dot magenta line.
where it becomes singular at low tide \( U \to \xi \sin t / \left( \cos t + 1 \right) \) and \( U_{max} \to \infty \) for \( \xi \to 0 \). Conversely, the velocity varies sinusoidally towards the channel mouth and \( U_{max} \sim \sqrt{2} \).

This latter condition is obviously verified throughout the entire channel when imposing a threshold on the maximum flow velocity (Figure 5b). The along channel variations of the tidally averaged velocity \( U_{av} \), on the other hand, are quite similar for the two bed profiles. It is also worthwhile to note that even though the landward portion of the equilibrium profiles obtained by Friedrichs and Aubrey [1996] and Seminara et al. [2010] \( (\eta_0 = - \sin(\xi/\sqrt{2}) \) for \( \xi < 0 \) emerges during low tide, its concave shape is very similar to that exhibited by the equilibrium configurations (24) and (28). Figure 5 shows the equilibrium bed profile described by (24) represents the leading order equilibrium solution of the morphodynamic problem set by equations (12), (13), and (17). It can be demonstrated (Appendix A) that, for a sinusoidal tidal forcing, the \( O(\chi) \) frictional correction vanishes while inertia causes an \( O(\alpha) \) correction such that

\[
\eta = \eta_0 + O(\chi^2, \alpha),
\]

In particular, local inertia determines a correction of order

\[
\frac{\partial H^*}{\partial x^*} \sim \frac{a \omega^*}{U^*_c}
\]

with respect to the horizontal water surface elevation characterizing a quasi-steady propagation of the tidal wave. For typical values of the relevant parameters this correction is of order \( O(10^{-3}) \), thus confirming that the leading order flow field solution and the equilibrium profile (24) resulting from it captures most of the features of the channel morphology.

Some other simplifications introduced in the present analysis may play a minor role in shaping the bed profile. We have seen that the term \( \mathcal{F}^* \) on the right hand side of equation (1), accounting for momentum exchange between the channel and the tidal flats, provides a contribution of order \( O(\alpha) \). Since this contribution is in principle significant only during the ebb phase, it could cause a \( O(\alpha) \) asymmetry of the velocity and, as a consequence, a first order modification of the bed profile. However, we note that in the case of channels dissecting the tidal flats of a (micro-, meso-tidal) coastal lagoon, the tidal platform is usually submerged during the whole tidal cycle, thus ensuring a relatively small bed roughness. As discussed by Lawrence et al. [2004] on the basis of two-dimensional numerical simulations, a low flow resistance leads to streamlines joining the channel at an angle much smaller than \( \pi/2 \). This fact implies that the term \( \mathcal{F} \propto (U - U_{in} \cos(\phi)) \) likely provides a contribution to the equilibrium profile at an order of approximation higher than \( O(\alpha) \).

Another possible limitation of the present analysis comes form the approximations embodied by equation (8) (i.e., that \( Q_m \sim c_w \omega^* \) and \( \tau^*_v \sim \tau^*_\omega \) ). These approximations are supported by the sediment properties reported by Roberts et al. [2000], covering most of the regimes observed in northwestern European macro-tidal flats. Indeed, the erosion rate constant \( Q_m \sim 5 \cdot 10^{-5} \frac{kg}{m^2/s} \) and the product \( c_w \omega^* \sim 0.02-0.2 \frac{kg}{m^3/s} \) are of the same order of magnitude, as well as the thresholds \( \tau^*_v \sim 0.2 \frac{N/m^2} \) and \( \tau^*_\omega \sim 0.1 \frac{N/m^2} \) for sediment erosion and deposition. Even though the above simplifications do not allow us to account for settling and scour lag effects, nevertheless, they have been already demonstrated to provide a good approximation to the equilibrium condition for the cross-shore profile of tidal flats. The numerical simulations carried out by Pritchard and Hogg [2003] considering a full description of sediment erosion and deposition, in fact, suggest that the criterion of uniform peak velocity across a flat, originally postulated by Friedrichs and Aubrey [1996], provides a good approximation to the equilibrium bed configuration. Lag effects due to non-uniformities in the tidal excursion in fact tend to compete with, and partially cancel out, those due to variations in the fluid depth.

Finally, some deviations from the profile given by (24) are also to be expected when considering the effects of over-tides. Under the assumption that over-tide contribution to

---

### Table 1

<table>
<thead>
<tr>
<th>Vanishing Residual Exchange</th>
<th>Maximum Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta_0 = -\sin(\xi/\sqrt{2}) ) for ( \xi &lt; 0 )</td>
<td>( \eta = \eta_0 + O(\chi^2, \alpha) )</td>
</tr>
</tbody>
</table>

*Figure 5.* Along channel variations of the maximum velocity \( U_{max} \) (thick continuous line), of the quadratic mean \( U_{rms} \) (dashed line) and of the tidally averaged velocity \( U_{av} \) (continuous line) for the equilibrium profile obtained assuming either (a) a vanishing residual sediment exchange with the bed \( (\eta_{rms} = 1) \) or (b) a constant maximum velocity.

*Table 1.* Equilibrium bed profile \( (\eta_0) \), and relative flow field features: longitudinal velocity distribution \( U_0 \), maximum velocity \( U_{max} \), mean square root velocity \( U_{rms} \), tidally averaged velocity \( U_{av} \) resulting by imposing either a vanishing residual sediment exchange with the bed \( (\eta_{rms} = 1) \) or a constant maximum velocity \( U_{max} \).
the principal tide component is small enough to neglect non-linear interactions, it can be easily demonstrated (Appendix B) that the equilibrium bed profile tends to distort slightly compared to the configuration (24), the deviations being strictly related to the overtide phase shift.

4.3. Typical scales in tidal channel morphology

The fact that the equilibrium bed condition (31) asymptotically to the common profile \( \eta_0 = -\xi/\sqrt{2} \) allows us to determine some typical quantities characterizing tidal channel morphology, namely the mean cross-section depth at the channel mouth, \( D_m \), a typical slope of the channel bed, defined as \( S = D_m/L^* \), and the maximum velocity at the mouth, \( U_m^* \), providing a representative measure of the velocity scale within the channel [Friedrichs and Aubrey, 1994; Toffolon et al., 2006].

Assuming, for the sake of simplicity, that channel width varies exponentially and the storage ratio \( r_s \) is constant, we obtain:

\[
S = r_s \frac{a^* \omega^*}{U^*} \Lambda, \quad \Lambda = 1 - \exp\left( -\frac{-\lambda}{\lambda} \right), \tag{31}
\]

with \( \lambda = L^*/L_t^* \) the ratio between channel and convergence lengths. Figure 6 shows that two limit cases exist, depending on \( \Lambda \). The first is the weak convergence/constant width case, for which \( \Lambda \to 0 \), \( \Lambda \to 1 \) and hence:

\[
D_m^* \approx \frac{r_s a^* \omega^*}{U^*} L^* \tag{32}
\]

The depth at the channel mouth is proportional to the channel length and the bed profile tends to be linear. On the contrary, when the convergence is strong enough so that \( \lambda \to \infty \), then \( \Lambda \to \lambda^{-1} \) and it turns out that:

\[
D_m^* \approx \frac{r_s a^* \omega^*}{U^*} L_t^*, \tag{33}
\]

The depth at the channel mouth depends linearly on the convergence length, thus implying that channel slope tends to decrease as convergence increases. The maximum velocity at the channel mouth always satisfies the equilibrium condition \( U_m^* = \sqrt{2} U^* \). However, it can also be rewritten as \( U_m^* = L^*/r_s a^* \omega^*/D_m^* \) and \( U_m^* = L_t^*/r_s a^* \omega^*/D_m^* \) in the limit of weakly convergent and strongly convergent channels, thus recovering the velocity scales indicated by a number of authors [Lanzoni and Seminara, 1998; Toffolon et al., 2006; Savenije et al., 2008, e.g.] in the analysis of tidal channel hydrodynamics.

Alternatively, following Seminara et al. [2010], relationship (31a) can be used to calculate the equilibrium channel length, given the depth \( D_m \) at the inlet:

\[
L = -L_b \ln \left( 1 - \frac{\sqrt{2} D_m}{r_s L_b} \right), \tag{34}
\]

It is readily seen that a solution exists only if \( L_b > L_{b_{\text{min}}} = \sqrt{2} D_m/r_s \). This implies that for a given \( L_b \) there is a maximum depth that can occur at the channel mouth, given by (35), i.e. \( D_{\text{max}} = r_s L_b/\sqrt{2} \) in dimensionless form. Equation (34) can also be used to analyze the relation between the length and the shape of a tidal channel, depending on the values attained by the critical velocity \( U_m^* \) and the convergence length \( L_t^* \). In fact, taking \( L_t^* = 1 \) as the value of the modified coordinate discriminating between the landward flat portion and the seaward linear region of the equilibrium profile (24) (see Figure 3a), it is possible to define the threshold length

\[
L_t^* = -L_b^* \ln \left( 1 - \frac{U_m^*}{r_s \omega^* L_b^*} \right), \tag{35}
\]

which reduces to \( L_t^* = U_m^*/\omega^* (= L_t^*) \) in the case of a constant-width channel with \( r_s = 1 \). It is immediately observed that high values of the critical velocity (associated with coarser sediment) and strong convergence favor the formation of a flat equilibrium profile (i.e., \( L_t^* \) is large and this increases the chance that \( L_t^* < L_t^* \)). On the contrary, for weak convergence and lower values of the critical velocity (finer material) the seaward linear reach tends to prevail in determining the bed profile (\( L_t^* \) is small and, hence, it is more likely that \( L_t^* > L_t^* \)). Note that this latter condition is compatible with the short channel assumption (\( L_t^* \) \( \ll \) \( L_t^* \)) used in the derivation of (24). Indeed, for weakly convergent channels the ratio

\[
\frac{L_t^*}{L_{\omega^*}} \sim \frac{U_m^*}{2 \pi \sqrt{g D_m^*}}, \tag{36}
\]

is typically \( O(10^{-2}) \).

4.4. Relationship between bed profile and planform configuration

The equilibrium bed profiles (24) and (28) are univocally defined by the modified longitudinal coordinate \( \xi \) and, therefore, are in principle valid for any along-channel distributions of the section width and of the geometry of intertidal storage areas flanking the channel. These distributions, however, are strictly related to the hydrodynamic field. Channel width variations, in fact, are determined by the complex interplay between the flow field which establishes in the channel and the erosion/deposition processes that control bank stability. The geometry of storage areas, on the other hand, is strictly related to the form of the channel watershed which, in turn, is determined by the overall structure of the tidal channel network.

In order to analyze how the bed profile could affect the planform channel shape, we observe that the usual width-to-depth ratio, \( \beta = B^*/a^* \), can be rewritten as

\[
\beta = B^* a^* / \Lambda, \tag{37}
\]
where the expression $B = -B_{0} / r_{0}$ relates width variations to the equilibrium bed profile expressed in dimensionless form. Differentiating this latter expression yields the implicit differential relationship:

$$\frac{dB}{dx} = - \left( B_{e} \frac{d\eta}{dx} + \eta \frac{dB}{dx} \right) \left( 1 + \frac{B}{\eta \frac{d\xi}{dx}} \right)^{-1},$$  \hspace{1cm} (38)$$

involving the three unknown variables $B$, $r_{x}$, and $B_{e}$. In order to make some progress in the analysis, we next assume that $B$ varies moderately along the channel (i.e., $dB/dx \ll 1$, $B \approx B_{0} = \text{const}$), as suggested by the data reported in Figure 2c. In the limit of large enough $\xi$, the bed profile is described by (25) and equation (38) becomes:

$$\frac{dB}{dx} \simeq \frac{B_{0} r_{x}}{2 \sqrt{2}},$$  \hspace{1cm} (39)$$

This relationship can be used to determine $B(x)$ when the storage ratio $r_{x}$ is known or, vice versa, $r_{x}(x)$ if $B$ is prescribed. Let us first assume that $r_{x} = r_{x0}$ is constant. Solving (39) with the seaward boundary condition $B(L) = 1$ gives

$$B(x) = 1 - \frac{B_{0} r_{x0}}{2 \sqrt{2}} (L - x),$$  \hspace{1cm} (40)$$

indicating that the width should vary linearly. On the other hand, assigning the total storage width, $B_{T} = r_{x} B$, leads to the relationship:

$$\frac{dB}{dx} \simeq \frac{B_{0} B_{T}}{2 \sqrt{3} B},$$  \hspace{1cm} (41)$$

which, integrated under the assumption of constant $B_{T}$, yields

$$B(x) = \sqrt{1 - \frac{B_{0} B_{T}}{2 \sqrt{2}} (L - x)},$$  \hspace{1cm} (42)$$

Both the hypotheses of constant $r_{x}$ and constant $B_{T}$ then result in along channel distributions of $B$ which strongly deviate (especially the linear one) from the exponential trend suggested by observational evidence (Figure 2b). On the other hand, assuming for the channel width an exponential law of the form (26), equation (39) implies that the storage ratio should also vary exponentially:

$$r_{x}(x) = 2 \sqrt{2} \frac{B(x)}{B_{0} L_{b}},$$  \hspace{1cm} (43)$$

and, therefore, $B_{T}(x) \propto B(x)^{2} = e^{2(L-x)/L_{b}}$. Also in this case, the resulting trend does not exactly conform to the more or less elongated leaf shape usually exhibited by tidal watersheds (see, e.g., Figure 3b of Rinaldo et al. [1999]).

The difficulty to relate the equilibrium profile to the actual channel and watershed planforms has many possible explanations. First of all, it is likely that along channel variations of $B$, although moderate, cannot be completely neglected in (38), as postulated in the above discussion. Moreover, the effective value of $B_{T}$ pertaining to a given channel section is possibly influenced by the actual structure, not considered in (12) and (13), of the flow field establishing over the tidal flats flanking the channel. Finally, we observe that the actual planform configuration and the watershed shape of a given channel are also affected by the confinements with other minor tidal channels (e.g., channel #1 in Figure 7).

In any case, it is worthwhile to note that the present equilibrium profile obey the classical O’Brien-Jarrett-Marchi law [D’Alpaos et al., 2009] which has been observed to hold not only for tidal inlets but also for sheltered sections of tidal channels [Friedrichs, 1995; D’Alpaos et al., 2010]. Indeed, recalling (18) and (19), it is easily obtained that the dimensionless tidal prism reads:

$$P = \int_{0}^{x} U(H - \eta_{0}) B \, d\xi = 2 \left[ B_{e} \frac{r_{x} B}{2 \sqrt{2}} \right],$$  \hspace{1cm} (44)$$

Returning to dimensional quantities and recalling (24) we obtain

$$P^{*} = \psi(\xi) \frac{2 u_{c}}{\omega^* \sqrt{\gamma_{f}}},$$  \hspace{1cm} (45)$$

where, as discussed in D’Alpaos et al. [2009], the exponent $\gamma$ and the proportionality coefficient $k^{*}$ depend on the relation used to estimate the friction coefficient. For example, adopting the usual Manning-Strickler relation yields $\gamma = 6/7$ and $k^{*} = (\omega^{*} \gamma^{1/2} B_{e}^{1/5})/(2 k^{*}_{s} u_{c}^{*})$, with $k^{*}_{s}$ the Strickler roughness coefficient. Importantly, the configuration of the bed profile affects the channel area - tidal prism relation through the multiplicative term $\psi(\xi)$ which, for large enough $\xi$, tends to attain the constant value $\sqrt{2}$.

4.5. Comparison with field data

We have seen that the equilibrium bed profile of a short tidal channel is essentially controlled by: 1) the tidal forcing, quantified by $\omega^{*}$, $\gamma^{*}$; 2) the critical velocity $U_{cr}$, related to the nature of the bed sediment and, possibly, to the relatively small net fluxes associated with the long-term morphodynamic trend of the overall lagoon; 3) the degree of channel funnelling, measured by the convergence length $L_{b}$ if an exponential decay holds; 4) the storage ratio, $r_{x}$, depending on the geometry of tidal flats flanking the channel. Although $\omega^{*}$, $\gamma^{*}$, $L_{b}$ can be easily determined when considering real tidal channels, much more uncertainties are entailed by the estimation of $r_{x}$ and, to a less extent, of $U_{cr}$. The along channel distribution of $r_{x}$, in fact, is strongly affected not only by the shape of the watershed pertaining to a given channel, dictated by the complex form of the channel network dissecting the lagoon, but also by the possible presence of strips of salt marshes flinging the channel and by the curved planform often characterizing tidal channels (e.g., channel #2 of Figure 7). Given all these difficulties, for each channel reported in Figure 1 we have assumed a constant value of $B_{e}$ (reported in Table 2), chosen to ensure the matching of computed and observed profiles. Moreover, we have considered a constant critical velocity, equal to the value $U_{cr}^{*} = 0.17 \text{m/s}$ estimated in Section 2 neglecting the possible effects of silty fractions. Note that this value is quite close to those (denoted as $U_{calc}$ in Table 2) provided by a
Table 2. Characteristics of the channels of Figure 1.

<table>
<thead>
<tr>
<th>channel</th>
<th>( L_0^* ) [m]</th>
<th>( B_0^* ) [m]</th>
<th>( L_i^* ) [m]</th>
<th>( D_{e0}^* ) [m]</th>
<th>( B_{e0}^* ) [m]</th>
<th>( U_{c0}^* ) [m/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>2480</td>
<td>121</td>
<td>2900</td>
<td>4.0</td>
<td>5.8</td>
<td>0.15±0.27</td>
</tr>
<tr>
<td>#2</td>
<td>2040</td>
<td>109</td>
<td>1300</td>
<td>2.8</td>
<td>381</td>
<td>0.17±0.20</td>
</tr>
<tr>
<td>#3</td>
<td>1610</td>
<td>45</td>
<td>1270</td>
<td>2.6</td>
<td>218</td>
<td>0.16±0.18</td>
</tr>
<tr>
<td>#4</td>
<td>1950</td>
<td>63</td>
<td>1830</td>
<td>3.8</td>
<td>376</td>
<td>0.18±0.22</td>
</tr>
<tr>
<td>#5</td>
<td>1240</td>
<td>51</td>
<td>1170</td>
<td>4.2</td>
<td>407</td>
<td>0.16±0.20</td>
</tr>
<tr>
<td>#6</td>
<td>2950</td>
<td>57</td>
<td>5360</td>
<td>3.5</td>
<td>285</td>
<td>0.10±0.12</td>
</tr>
<tr>
<td>#7</td>
<td>1050</td>
<td>95</td>
<td>1120</td>
<td>5.1</td>
<td>954</td>
<td>0.15±0.18</td>
</tr>
</tbody>
</table>

Figure 7. Comparison between observed (circles) and computed (continuous line) bed profiles. The approximate shape of the watershed contributing to channel #1 is also reported.

Although the present analytical treatment of tidal channel equilibrium seems to capture reasonably the morphological shapes exhibited by real tidal channels, various issues surely merit further attention in a near future. The existence of a well defined relation between the bed profile and the channel width is strictly associated with the equilibrium section of tidal channels, a problem that is not yet clarified. The geometrical features of watershed shape and their role on channel configuration need to be further addressed. Finally, settling and scour lags, external overides, local inertia and friction, momentum exchange with the flats, are all deemed to provide higher order corrections which, however, surely deserve further research.

Appendix A: Frictional and inertial effects

In this Appendix we evaluate the first order corrections of the morphodynamic problem set by equations (12), (13), and (17), neglecting the contribution of the momentum exchange term \( \alpha \mathcal{F} \) as discussed in Section 4.2. Substituting the expansions

\[
\{H, \eta, U\} = \{H_0, \eta_0, U_0\} + \chi \{H_{11}, \eta_{11}, U_{11}\} + \alpha \{H_{12}, \eta_{12}, U_{12}\} + O(\chi^2, \alpha^2, \chi \alpha) \tag{A1}
\]
In this case, in fact, and, therefore, the velocity is anti-symmetrical (quantities (summarized in Table 3) allows us to demonstrate that also \( V_{11} \) is anti-symmetrical, it turns out that \( V_{11} \) is symmetrical, as one can easily see from (A6).

A simple analysis of the symmetry properties of the various quantities (summarized in Table 3) allows us to demonstrate that \( \eta_{11} \) vanishes in the presence of a sinusoidal forcing tide. In this case, in fact,

\[
U_0 = \frac{\xi \sin t}{\cos t - \eta_0}, \quad D_0 = \cos t - \eta_0,
\]

and, therefore, the velocity is anti-symmetrical \((U_0(-t) = -U_0(t))\) while the depth is symmetrical \((D_0(-t) = D_0(t))\) with respect to time. According to (A5), \( H_{11} \) is anti-symmetrical because it results from a symmetrical quantity, \(|U_0|/D_0\), multiplied by an anti-symmetrical one, \( U_0 \). Observing that also \( V_{11} \) is anti-symmetrical, it turns out that the tidally averaged terms on the right hand side of (A8) are identically vanishing, owing to their anti-symmetrical nature. Therefore, since \( U_0^2/D_0 \) is a non-vanishing quantity, it results that must identically be \( \eta_{11} = 0 \). Note that also the velocity \( U_{11} \) is symmetrical, as one can easily see from (A6).

Let us now analyze the inertial correction. The \( O(\alpha) \) equations read

\[
\frac{\partial H_{12}}{\partial x} + \frac{\partial U_0}{\partial t} + U_0 \frac{\partial U_0}{\partial x} = 0,
\]

\[
\ell B \frac{\partial H_{12}}{\partial t} + \frac{\partial}{\partial x} [B (U_{12} D_0 + U_0 D_{12})] = 0,
\]

Integrating (A10) and (A11) along \( x \) and imposing the boundary conditions

\[
[H_{12}]_{x=L} = 0, \quad [U_{12} D_0 + U_0 D_{12}]_{x=0} = 0
\]

yields

\[
H_{12} = \int_x^L \frac{\partial U_0}{\partial t} dx + \frac{1}{2} \left[ \frac{U_0^2}{L} \right]_x
\]

\[
U_{12} = \frac{1}{D_0} \left[ U_0 (H_{12} - \eta_{12}) + \frac{1}{B} \frac{\partial V_{12}}{\partial t} \right],
\]

where

\[
V_{12} = \int_0^x r_s B H_{12} dx
\]

Imposing the equilibrium condition \( \langle U_0 U_{12} \rangle = 0 \) leads to

\[
\langle U_0^2 \rangle_{\eta_{12}} = \frac{U_0^2}{D_0} H_{12} + \frac{1}{D_0} \frac{U_0}{B} \frac{\partial V_{12}}{\partial t}.
\]

For a sinusoidal tide, it is easily demonstrated that \( H_{12} \) and \( V_{12} \) are symmetrical functions of time (see Table 3) and, therefore, \( \eta_{12} \neq 0 \). We do not report here the expression for such a correction which generally depends on the degree of channel convergence.

Nevertheless, it is interesting to estimate the magnitude of the correction of the free surface slope due to inertial effects, \( \partial H_{12}/\partial x \). According to (A13), we can distinguish the contribution due to local acceleration

\[
\frac{\partial U_0}{\partial t} = \frac{(1 - \eta_0 \cos t) \xi}{(\cos t - \eta_0)^2},
\]

\[
\text{Figure 8.} \quad \text{The tidal-averaged magnitude of the inertial terms in momentum equation, as estimated through (A19), continuous line, and (A20), dashed line. The limit } 2\sqrt{2}/\pi \text{ of the former curve is shown by a dash-dot line.}
\]

Table 3. Symmetry properties (with respect to time) of the terms in the expansion (A1).

<table>
<thead>
<tr>
<th>Terms</th>
<th>O(1)</th>
<th>O(\chi)</th>
<th>O(\alpha)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H_0, D_0, U_0, \eta_0, U_0^0, U_0 x )</td>
<td>( H_{11}, \eta_{11}, V_{11} )</td>
<td>( H_{12}, V_{12} )</td>
<td></td>
</tr>
</tbody>
</table>
and that given by convective acceleration
\[
U_0 \frac{\partial U_0}{\partial x} = \frac{(\sin t)\xi}{(\cos t - \eta_0)^2} \left( \cos t - \eta_0 + \xi \frac{\partial \eta_0}{\partial \xi} \right) \frac{\partial \xi}{\partial x}, \tag{A18}
\]

The importance of these two contributions to the first order correction of the water surface slope can be analyzed by considering the tidal average of their absolute values, namely
\[
\left| \frac{\partial U_0}{\partial t} \right| = \pi \left| \frac{-4 \xi \tan \Psi}{\eta_0 - 1 + (\tan \Psi)^2} \right|, \tag{A19}
\]
\[
\left| U_0 \frac{\partial U_0}{\partial x} \right| = I \left( \xi, \eta_0, \frac{\partial \eta_0}{\partial x}, \frac{\partial \xi}{\partial x} \right), \tag{A20}
\]
where \( \Psi = 0.5 \arccos (\eta_0^{-1}) \), while \( I \) is a complicated function of \( \xi \) and \( \eta_0 \), not reported here. The along channel variations of these quantities are shown in Figure 8, for the equilibrium bed profile \( (24) \) and \( r_s = 1 \). Only the correction provided by local inertia gives a non vanishing first order contribution to water surface slope in the seaward portion of the channel. Conversely, both corrections increase towards the channel head, where the problem become singular as a consequence of the fact that water depth vanishes at low tide.

Appendix B: Effect of overtides

The effect of overtides can be explicitly calculated when they are small enough to neglect non-linear interactions. Here we assume that the tidal forcing at the channel mouth is given by a purely sinusoidal semi-diurnal forcing \( (M_2) \) and a much smaller second harmonic \( (M_4) \),
\[
H = \cos(t) + \epsilon \cos(2t + \phi), \tag{B1}
\]
Here \( \epsilon \ll 1 \) is the ratio between the amplitudes of \( M_4 \) and \( M_2 \), while \( \phi \) is the phase of the overtide.

Neglecting inertial and frictional terms (\( \alpha = \chi = 0 \)) and introducing the linearization
\[
\{ \eta, U \} = \{ \eta_0, U_0 \} + \epsilon \{ \eta_1, U_1 \}, \tag{B2}
\]
the \( O(\epsilon) \) correction of the base flow provided by (A9) is
\[
U_1 = \xi \left[ \frac{\sin(2t + \phi)}{\cos(t) - \eta_0} + \frac{\eta - \cos(2t + \phi)}{(\cos(t) - \eta_0)^2} \sin(t) \right], \tag{B3}
\]
The equilibrium condition requires that \( \langle U_0 U_1 \rangle = 0 \), and yields
\[
p_1 \eta_1 + p_{uc} \cos(\phi) + p_{uo} \sin(\phi) = 0, \tag{B4}
\]
where
\[
p_1 = \frac{1}{2 \eta_0^2 - 1}^{3/2}, \quad p_{uo} = 0,
\]
\[
p_{uc} = -2 \eta_0 - \frac{\eta_0^3 (2 \eta_0^2 - 3) + 3/2}{(\eta_0^3 - 1)^{3/2}}. \tag{B5}
\]
The first order correction of the bed profile induced by the overtide then results
\[
\eta_1 = \frac{p_{uc}}{p_1} \cos(\phi), \tag{B6}
\]
and clearly depends only on the cosine of the overtide phase. In particular, \( \eta_1 = 0 \) for \( \phi = \pm \pi/2 \) (equal high and low water levels), while \( \eta_1 \) is maximum for \( \phi = 0 \) (high water level larger than low water level) or \( \phi = \pi \) (high water level smaller than low water level). The ratio \( \eta_1/\cos(\phi) \) is always positive and tends to 1 and to 3/2, at the channel head \( (\eta_0 = -1) \) and for very large depths \( (\eta_0 \to \infty) \), respectively.

This means that the considered overtide leads to an increase in equilibrium bed level falling in the range \( \epsilon \cos(\phi) \) (landward part) and \( 1.5 \epsilon \cos(\phi) \) (deep part) for \( -\pi/2 < \phi < \pi/2 \); conversely, the equilibrium bed level decrease for all the other values of \( \phi \). Note that, owing to the quasi-static propagation of the tide, the positive/negative vertical shift of the bed is a direct consequence of tidal amplitude variations associated with the \( M_4 \) overtide. Similar calculations can be carried out for higher harmonics.

Notations (Dimensional variables are denoted by a * superscript)

\( \rho^* \) water density,
\( g^* \) gravity acceleration,
\( \omega^* \) tidal angular frequency,
\( t^*, t \) temporal coordinate,
\( x^*, x \) longitudinal coordinate,
\( \xi^* \) intrinsic longitudinal coordinate,
\( L^*, L \) channel length,
\( L_0^*, L_0 \) intrinsic length scale,
\( B^*, B \) channel width,
\( B_T^*, B_T \) total width (storage area and channel),
\( r_s \) storage ratio,
\( \alpha^* \) tidal amplitude,
\( H^*, H \) water surface elevation,
\( D^*, D \) water depth,
\( B_m^*, D_m^* \) width and depth at the mouth,
\( \beta \) width-to-depth ratio,
\( \Omega^* \) channel cross-section area,
\( P^*, P \) tidal prism,
\( \eta^*, \eta \) bed elevation,
\( U^*, U \) cross-section averaged velocity,
\( U_{*c}^* \) critical velocity,
\( U_{Tc}^* \) critical friction velocity,
\( \tau^* \) bed shear stress,
\( \tau_{*c}^*, \tau_{*c} \) critical bed shear stress,
\( \tau_{*d}^*, \tau_{*d} \) erosion/deposition thresholds,
\( Q_{a}^* \) erosion/deposition fluxes,
\( Q_{0}^* \) reference erosion rate,
\( \bar{p} \) bed sediment void fraction,
\( w_{*}^* \) sediment settling velocity,
\( c_b \) near-bed sediment concentration,
\( d_{s0} \) sediment mean diameter,
\( k_s^* \) Manning-Strickler coefficient,
\( C_f^* \) dimensionless Chézy coefficient,
\( \mathcal{F}^* \), \( \mathcal{F} \) flat-to-channel momentum exchange,
\( \alpha \) inertial parameter,
\( \chi \) friction parameter,
\( U_0, H_0, \eta_0 \) base flow variables.

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References


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