VERTICAL PROFILE CORRECTIONS FOR SUSPENDED SEDIMENT CONCENTRATION IN NON-UNIFORM FLOWS

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ABSTRACT: The theoretical prediction of the morphological evolution of river and tidal systems dominated by suspended load is often based on approximate two-dimensional models, where the vertical profiles of velocity and concentration are usually taken to coincide with equilibrium profiles valid for uniform flows. In this paper we try to determine to what extent typical variations of the basic flow, for instance those occurring in tidal channels, are likely to be adequately reproduced through the above assumption. Using a two-dimensional x-z numerical model, we determine the deviation of vertical concentration profiles from the equilibrium profiles in stationary non-uniform flows. We then compare the results of numerical simulations with those obtained through the analytical perturbation model recently proposed by Bolla Pittaluga and Seminara (2003a).

1 INTRODUCTION

The lower reaches of rivers and tidal environments like estuaries and lagoons are systems in which sediment transport mainly occurs as suspended load. The vertical concentration profiles, and hence the morphological computations with suspended load, are very sensitive to the choice of the reference value of concentration which is used in the bed boundary condition, as pointed out by van Rijn (1984). Under uniform conditions the vertical concentration profile can be represented through the well known Rousean distribution; on the other hand, changes in the boundary conditions can modify significantly such distribution, as in the case of the transition from one equilibrium state to another due to an abrupt change of the bed boundary condition, a problem which has been investigated by Hjelmfelt and Lenau (1970) among others. Deviation of the vertical profile from the local equilibrium profile, which is defined as the Rouse distribution corresponding to the local and instantaneous hydraulic conditions, can be fairly large under non-uniform conditions, since suspended load requires a relatively large adaptation length to respond to changing hydraulic conditions.

Few analytical formulations are presently available for the determination of the concentration field in non-uniform flows. In particular, Bolla Pittaluga and Seminara (2003a) have recently revisited the approximate solution proposed by Galappatti and Vreugdenhil (1985), which was aimed at deriving a suitable two-dimensional closure for sediment transport. Suspended sediment transport has been also analyzed recently through three-dimensional numerical models, like those proposed by Lin and Falconer (1996) and Wu et al. (2000). Both models have been applied to relatively simple cases; in fact, their use is limited by the computational time, since a three dimensional formulation requires a larger number of grid nodes, say 10−100 times

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the number of nodes of a two dimensional model applied to the same context. Hence, a three dimensional model can be hardly used to describe the long term morphodynamic behaviour of a river reach or an estuary, because it would require prohibitively long numerical simulations; this is the reason why an analytical formulation for the vertical concentration profile, to be included within two dimensional morphological models, would be highly desirable.

It is worth noticing that to achieve a reasonable accuracy in the prediction of the local morphological response of fluvial and tidal systems, the analytical solution must represent adequately the typical delay of suspended load with respect to the local bottom shear stress. This phase lag crucially affects bed stability, as suggested by recent results on bar formation with suspended load, which have been derived within the context of approximate linearized theories (Tubino et al. 1999; Seminara and Tubino 2001).

In this paper we compare the analytical asymptotic solution of Bolla Pittaluga and Seminara (2003a) with the results of a two dimensional $x$-$z$ numerical model. As a first step, we restrict our analysis to the case of spatially non-uniform flows; we then consider a steady flow over a sinusoidal bottom profile, for different values of the amplitude and wavelength. In section 3 we describe the numerical model, after the formulation of the problem which is given in the next section. An outline of the perturbation solution is given in section 4. Finally, section 5 is devoted to the comparison between numerical and analytical solutions.

2 FORMULATION OF THE PROBLEM

We consider a channel with varying bottom level and subject to steady boundary conditions. The problem is formulated in terms of longitudinal and vertical coordinates, $x^*$ and $z^*$ respectively (hereafter a superscript star will denote dimensional quantities). The lateral structure of the bed will be taken to be horizontal; however, the present approach could be potentially extended to the case of three dimensional flows.

The two dimensional momentum and continuity equations, assuming the hydrostatic approximation, read:

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + w^* \frac{\partial u^*}{\partial z^*} = -g \frac{\partial h^*}{\partial x^*} + \frac{\partial}{\partial z^*} \left( \nu_z \frac{\partial u^*}{\partial z^*} \right)$$

(1)
\[ \frac{\partial u^*}{\partial x^*} + \frac{\partial w^*}{\partial z^*} = 0 \]  

(2)

where \( u^* \) and \( w^* \) are the longitudinal and the vertical component of the velocity respectively, \( t^* \) is the time variable, \( h^* \) is the free surface elevation and \( \nu_z^* \) is the vertical component of the eddy viscosity.

The advection-diffusion equation for the suspended sediment in conservative form reads:

\[ \frac{\partial C^*}{\partial t^*} + \left( \frac{\partial u^*}{\partial x} C^* + \frac{\partial w^*}{\partial z} C^* \right) = \frac{\partial}{\partial z} \left( \sigma_z^* \frac{\partial C^*}{\partial z} \right) \]  

(3)

where \( C^* \) is the sediment concentration, \( w_s^* \) the settling velocity and \( \sigma_z^* \) the vertical component of the eddy diffusivity. Let us make the relevant physical quantities dimensionless as follows:

\[ t = \frac{t^*}{T_0^*}; \quad (u, w) = \left( \frac{u^*}{U_0^*}, \frac{w^*}{W_0^*} \right); \quad C = \frac{C^*}{C_R^*}; \quad x = \frac{x^*}{L_0^*}; \quad \nu_z = \frac{
u_z^*}{u_{f0}^* D_0^*}; \quad h = \frac{h^*}{D_0^*} \]  

(4)

where \( T_0^* \) is the characteristic time scale, \( D_0^* \) is the reference flow depth, \( U_0^* \) is the longitudinal velocity scale, \( W_0^* = \frac{D_0^* U_0^*}{L_0^*} \) is the scale for the vertical velocity, \( L_0^* \) is the longitudinal spatial scale, \( C_R^* \) is the scale for the sediment concentration and \( u_{f0}^* \) is the reference friction velocity. Using the above dimensionless variable, equations (1), (2) and (3) take the form:

\[ \frac{\alpha \partial u}{\partial t} + \beta \left( \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = -\frac{\beta \partial h}{F_0^2 \frac{\partial u}{\partial x}} + \frac{\partial}{\partial z} \left( \nu_z \frac{\partial u}{\partial z} \right) \]  

(5)

\[ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \]  

(6)

\[ \alpha \frac{\partial C}{\partial t} + \beta \left( \frac{\partial u C}{\partial x} + \frac{\partial w C}{\partial z} \right) - \kappa Z_0 \frac{\partial C}{\partial z} = \frac{\partial}{\partial z} \left( \sigma_z \frac{\partial C}{\partial z} \right) \]  

(7)

having introduced the Froude number \( F_0 = \frac{U_0^*}{\sqrt{g D_0^*}} \), the friction factor \( C_{f0} = \left( \frac{u_{f0}^*}{U_0^*} \right)^2 \) and the parameters:

\[ \alpha = \frac{D_0^*}{T_0^* U_0^* \sqrt{C_{f0}}}; \quad \beta = \frac{D_0^*}{L_0^* \sqrt{C_{f0}}}; \quad Z_0 = \frac{w_{s0}^*}{\kappa u_{f0}^*} \]  

(8)

where \( \kappa \) is the Von Karman constant and \( Z_0 \) is known as Rouse number.

The boundary conditions associated with the system (5-6) are the no-slip condition at the bottom (\( z = z_0 \)):

\[ u = w = 0, \]  

(9)

the kinematic condition and vanishing stress at the free surface (\( z = h \)):

\[ h, \partial_x \left( uh_{,x} - w \right) = 0, \quad u_{,z} + \gamma^2 w_{,x} = 0; \]  

(10)

where \( \gamma = \frac{D_0^*}{L_0^*} \), which represents the ratio between the vertical and the longitudinal scales, is usually very small.

Equation (6) may be integrated over the flow depth \( D = D^*/D_0^* \) using the above boundary conditions, to obtain:

\[ \frac{\partial D}{\partial t} + \frac{\beta}{\alpha} \frac{\partial}{\partial x} \int_{z_0}^h u \partial z = 0 \]  

(11)
where \(z_0\) is the conventional value of the reference elevation for the no-slip condition in uniform flow.

The boundary conditions to be imposed for the solution of (7) are the vanishing sediment flux through the free surface and the gradient boundary condition at the reference level \(a\). They read

\[
C \left( \kappa Z_0 - \beta w + \beta u \frac{\partial h}{\partial x} \right) + \sigma_z \frac{\partial C}{\partial z} = 0 \quad (z = h)
\]

\[
C_e \left( \kappa Z_0 - \beta w + \beta u \frac{\partial \eta}{\partial x} \right) + \sigma_z \frac{\partial C}{\partial z} = 0 \quad (z = \eta + aD)
\]

where \(\eta\) is the local bed elevation, scaled by \(D_0^*\), and \(C_e = C_e^*/C_R\) is the dimensionless sediment concentration at the reference level \(a\) (evaluated using van Rijn 1984 for instance), under equilibrium conditions.

3 NUMERICAL SOLUTION

The solution for the flow field and the concentration profile is obtained numerically; in order to achieve a reasonable accuracy near the bottom, where velocity and concentration gradients attain fairly large values, we adopt a logarithmic transformation for the vertical coordinate

\[
\xi = \ln \left( \zeta \right), \quad \zeta = \left( \frac{z - \eta}{D} \right)
\]

such that the vertical domain falls in the range

\[
\xi \in \left[ \ln \left( \frac{z_0 - \eta}{D} \right), 0 \right].
\]

The spatial mesh adopted in the numerical solution consists of elements of length \(\Delta x\) and height \(\Delta z\); the center of each box is numbered by indices \(i\) and \(k\). The discrete values of the velocity components are defined on the sides of the numerical grid (see right part of Figure 1), the concentration at the center of each computational cell, the surface elevation is defined at the center of each water column.

Equation (5) is solved using an Eulerian-Lagrangian scheme in the form proposed by Casulli and Cattani (1994), which has been modified to account for the no-slip condition at the bed. The flow continuity equation (11) is solved through a finite-volume scheme. The solution of the transport equation (7) with the boundary conditions (12) and (13) is obtained, following Vignoli and Tubino (2002), by splitting the concentration into two parts \(C = C_0 + C_1\) where \(C_0\) represents a Rouse-type vertical profile of concentration, which is the distribution that the concentration would attain at equilibrium with the local shear stress, while the second component \(C_1\) quantifies the delay of sediment concentration with respect to shear stress due to the advection and diffusion processes (while \(C_0\) is in phase with the local shear stress). In order to increase the accuracy of the computation, \(C_0\) is evaluated analytically, while only \(C_1\) is computed numerically, using a second order finite volume scheme.

4 ANALYTICAL SOLUTION

Following the approach recently pursued in Bolla Pittaluga and Seminara (2003a), it is possible to provide analytical solutions of the vertical concentration profiles under the assumption
of slowly varying conditions. In fact, as shown by the above authors, the parameters \( \alpha \) and \( \beta \) appearing in the differential equation (7) are typically small for slowly varying flows like tidal flows (Lanzoni and Seminara, 2002) and flood waves. Hence, focusing our attention to the case of steady flows in the \( x-z \) plane, a formal perturbation solution of the differential equation (7), associated with the boundary conditions (12, 13), can be obtained by expanding the concentration \( C \) in powers of the small parameter \( \delta \) in the form

\[
C = C_0 + \delta C_1 + O(\delta^2)
\] (16)

being

\[
\delta = \frac{U_* D_0^*}{w_s' L_0^*} = \frac{\beta}{\kappa Z}
\] (17)

with \( Z \) the local Rouse number.

By substituting the latter expansion into the differential problem (7, 12, 13) and equating likewise powers of \( \delta \), a sequence of differential problems at the various orders of approximation are found.

At the leading order the classical Rouse type concentration profile is found

\[
C_0 = C_e (\vartheta, d_s, R_p, a) f (\zeta, Z, a)
\] (18)

where \( \vartheta \) is the local effective Shield stress when dunes are present; \( \vartheta \) can be evaluated in term of the local Shields stress

\[
\theta = \frac{|\tau^*|}{(\rho_s - \rho) g d_s^*},
\] (19)

where \( \tau^* \) is the local bed stress vector, \( \rho_s \) and \( \rho \) are the sediment and the water density respectively, \( d_s^* \) is the sediment diameter, here assumed to be uniform and \( d_s = d_s^*/D_0^* \). Finally \( R_p \) is a function of \( d_s^* \) defined as follows

\[
R_p = \sqrt{\frac{g \Delta d_s^3}{\nu}}, \quad \Delta = \frac{\rho_s - \rho}{\rho}
\] (20)

where \( \nu \approx 10^{-6} m^2/s \) is the kinematic viscosity of the fresh water. The function \( f (\zeta, Z, a) \) depends only on the closure relationship employed for the eddy diffusivity \( \sigma_z \). Alternatively equation (18) can also be expressed in terms of the depth averaged concentration \( \overline{C}_0 \) in the form

\[
C_0 = \overline{C}_0(x,t) \phi_0(\zeta, Z, a)
\] (21)

with

\[
\phi_0(\zeta, Z, a) = \frac{f(\zeta, Z, a)}{I(Z, a)}, \quad I(Z, a) = \frac{1}{(1-a)} \int_a^1 f(\zeta, Z, a) d\zeta
\] (22)

At the following order of approximation the contribution of the spatial non-uniformity of the flow field on the vertical profile of the sediment concentration is obtained. In fact we can write

\[
\frac{\partial C_0}{\partial x} = \phi_0(\zeta) \frac{\partial \overline{C}_0}{\partial x} + \overline{C}_0 \frac{\partial \phi_0}{\partial x}
\] (23)

The solution for \( C_1 \) can then be written in the form

\[
C_1 = D \left( \frac{\partial \overline{C}_0}{\partial x} C_{11} + \overline{C}_0 C_{12} \right)
\] (24)
where the functions $C_{1j}$ ($j = 1, 2$) are the solutions of the boundary value problems:

$$\frac{1}{kZ_0D} \left[ \frac{d}{d\zeta} \left( \sigma_z \frac{dC_{1j}}{d\zeta} \right) \right] + \frac{dC_{1j}}{d\zeta} = p_j (\zeta) ,$$

$$\frac{\sigma_z}{kZ_0D} \frac{dC_{1j}}{d\zeta} + C_{1j} = 0 \quad (\zeta = 1) ,$$

$$\frac{dC_{1j}}{d\zeta} = 0 \quad (\zeta = a) ,$$

with the following forcing terms

$$p_1 = u \phi_0 , \quad p_2 = u \frac{\partial \phi_0}{\partial x}$$

A self similar logarithmic structure of the velocity $u$, appropriate to slowly varying flows, has been assumed in the forcing terms appearing in equation (28).

Note also that Bolla Pittaluga and Seminara (2003a) assumed that $\phi_{0x}$ is negligible and, consequently, they neglected the second term in the right hand side of the equation (24). As it will be shown later, it does not seem that the effect of the neglected contribution is crucial, as argued by the above authors.

### 4.1 Closures Relationships

Closure assumptions for the eddy viscosity $\nu_z$ and diffusivity $\sigma_z$ are obtained assuming that the slowing varying character of the flow field both in space and in time leads to a sequence of equilibrium states. Hence, we write:

$$\nu_z = \sqrt{\frac{C_f}{C_f^o}} |U| D N (\zeta) , \quad \sigma_z = \sqrt{\frac{C_f}{C_f^o}} |U| D P (\zeta) ,$$

where $U$ is the depth averaged velocity vector and $N(\zeta)$ and $P(\zeta)$ are the vertical distributions of $\nu_z$ and $\sigma_z$ at equilibrium; they are evaluated through the relationships proposed by Dean (1974) and McTigue (1981), respectively. In particular, the latter formulation reads:

$$P(\zeta) = \begin{cases} 
0.35 \zeta, & (\zeta < 0.314) \\
0.11, & (0.314 \leq \zeta \leq 1)
\end{cases}$$

Calculation are performed employing van Rijn (1984) relationships for the reference level $a$ and the reference concentration $C^*_e$:

$$C^*_e = 0.015 \frac{d_s}{a} \left( \frac{\vartheta - \theta_{cr}}{\theta_{cr}} \right)^{3/2} R_p^{-0.2}$$

where $\theta_{cr}$ is the critical value for sediment mobilization and the reference level $a$ is the maximum between 0.01 and the equivalent roughness (van Rijn, 1984). The concentration $C^*_e$ evaluated with the reference values of the hydrodynamic variables $U^*_0$ and $D^*_0$ is assumed as a suitable scale $C^*_R$ in equations (4). Finally, the particles fall velocity $w^*_s$ is evaluated using the relationship proposed by Parker (1978).
5 RESULTS AND DISCUSSION

As a first step of the analysis, the comparison between numerical and analytical solutions is made under stationary conditions, using a sinusoidal longitudinal bed profile

\[ \eta^* = \eta_0^* \sin \left( \frac{2\pi x^*}{L_b^*} \right) \]  

(32)

with wavelength \( L_b^* \) ranging from 2500\( m \) to 10000\( m \) and a mean flow depth ranging from 5\( m \) to 15\( m \). Since the wavelength is assigned, the length scale \( L_0^* \) to be introduced in the scalings (4) is intrinsically defined as equal to \( L_b^* \). In the numerical model periodic boundary conditions are imposed for the flow field and for the concentration field in the initial and in the final sections of the channel. Comparisons between the two models are performed computing the depth-integrated solution with the numerical model and then introducing the variables needed in the asymptotic model, i.e. \( C_0, x_0, C_0, D \) and \( \phi_{0,x} \), in order to determine the analytical solution.

Figure 2: Corrections \( \delta C_1 \) of the vertical concentration profiles in different longitudinal positions: \( L_b^* = 10 km, \eta_0^* = 0.5 m, D_0^* = 5 m; R_p = 10, \delta = 0.023 \) (left), \( R_p = 4, \delta = 0.042 \) (right). Dotted line: numerical solution; continuous line: analytical solution; dashed line: analytical solution assuming \( \phi_{0,x} = 0 \).

Figure 3: Corrections \( \delta C_1 \) of the vertical concentration profiles in different longitudinal positions: \( L_b^* = 2.5 km, \eta_0^* = 1.5 m, D_0^* = 10 m, R_p = 4, \delta = 0.37 \) (left); \( L_b^* = 5 km, \eta_0^* = 0.5 m, D_0^* = 5 m, R_p = 4, \delta = 0.085 \) (right). Dotted line: numerical solution; continuous line: analytical solution; dashed line: analytical solution assuming \( \phi_{0,x} = 0 \).
A comparison between the two solutions at different cross sections is reported in Figures 2 and 3, for different values of the parameter $\delta$ and of the amplitude $\eta_0$ of the sinusoidal bed profile. The perturbation approach is able to capture the main qualitative trend of the vertical corrections and, as it is typical of the asymptotic approaches, it tends towards the numerical solution for small values of $\delta$. However, it appears that the analytical model is affected by the local conditions in a stronger way; this fact results in a faster adaptation of the analytical model to the local conditions with respect to the numerical model.

It is interesting to consider also the deviation $\delta q_{s1}$ of the suspended sediment transport from the value $q_{s0}$ which would be attained under the assumption of equilibrium with the local hydrodynamic conditions:

$$ q_{s1} = D \int_a^1 C_1 u d\zeta $$

A comparison between the analytical and the numerical solution shows that the two curves are very close for small values of $\delta$ (see Figure 4). For larger values of $\delta$ (Figure 5), the analytical approach captures the order of magnitude of the correction, but the phase lag between the two solutions becomes significant.
Such result is reported in a synthetic fashion in Figure 6, which also shows the relative error of the amplitude of the analytical solution $\delta q_{s1}$ with respect to the numerical one. For values of $\delta$ smaller than 0.2 the analytical model reproduces the amplitude of the perturbations of the suspended load in a fairly good way. For larger values of $\delta$, only the order of magnitude is captured. Similarly, the phase lag between the maximum values of the analytical and the numerical solutions grows with $\delta$; notice that $\phi=0.25$ represents a phase lag of a quarter of the wavelength.

One may argue that the differences between the two solutions could also be due to the assumption of a logarithmic structure for the velocity profile, which is not required in the numerical model. However, in the slowly varying context analyzed herein, it has been found that the deviations from the logarithmic velocity profile are negligible.

Figure 6: Difference of amplitude $A=(q_{s1,\text{max}}^{\text{an}}-q_{s1,\text{max}}^{\text{num}})/q_{s1,\text{max}}^{\text{num}}$ (left) and phase lag $\phi=(x_{\text{max}}^{\text{num}}-x_{\text{max}}^{\text{an}})/L_b$ (right) between the analytical (an) and the numerical (num) results, as a function of $\delta$.

The comparison performed between the numerical model (Vignoli and Tubino, 2002) and the analytical model (Bolla Pittaluga and Seminara, 2003a) suggests that the latter can be a valuable tool for small values of $\delta$. The definition of the range of applicability of such a model is important because it could be incorporated in a depth averaged model in order to evaluate the suspended sediment flux. For instance, rather than using a three dimensional approach, the analytical model can be satisfactorily employed to investigate whether the presence of suspended load has any effect on the nature of bar instability; this challenging investigation has recently been renewed by Federici and Seminara (2003) in the case of bed load, neglecting the effect of suspended load. Unfortunately, when the wavelength is shorter or the sediment diameter decreases (smaller values of $R_p$) or the flow depth becomes larger, the analytical model may introduce remarkable approximations. We notice that the bed configuration can be influenced by several factors, like the meandering pattern or the artificial presence of river regulation works, which can also be characterized by lengths even smaller than those considered herein, with the same values of the other parameters.

The case of unsteady flows, like those occurring in estuaries and tidal channels, can be tackled in similar fashion, but they are more complex, both for the definition of the basic flow and the identification of the factors to take into account in the perturbation solution. Moreover, the definition of the length scale $L_0^*$ is not straightforward as in the case described herein, since it depends on the hydrodynamic behaviour of the tidal channel. However, Bolla Pittaluga and Seminara (2003b) showed that the morphodynamics of tidal channels is only slightly affected by non-equilibrium effects due to the marked slowly varying character of the flow field.
The analysis of the range of applicability of the analytical model in tidal flows is still matter of investigation and will be tackled in the near future.

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