Effects of spatial wind inhomogeneity and turbulence anisotropy on circulation in an elongated basin: a simplified analytical solution

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Abstract

The response of a closed basin to wind forcing has been studied extensively, but the role of turbulence anisotropy coupled with spatial wind inhomogeneity has never been considered explicitly. An analytical solution is presented for steady state hydrodynamics, considering the central part of a constant-depth, elongated basin with a laterally varying wind, and a homogeneous, yet anisotropic, eddy viscosity tensor. The solution is derived both for a single layer and for a two-layer basin, which is representative of a stratified lake with a well-developed thermocline. Since the focus is on the short-term barotropic reaction to wind forcing, which determines the type of lake circulation, baroclinic effects are neglected as a first approximation. In this case, the development of planimetric (depth-averaged) circulation superimposed on circulation in the vertical plane can be determined as a function of wind lateral variation and a few dimensionless parameters. The relevance of such an analytical solution is twofold. Firstly, the knowledge of the prevailing circulation can help in the choice of the best type of numerical model (three-dimensional vs. two-dimensional, depth- or lateral-averaged). Secondly, it shows the importance of correct estimates of both vertical and horizontal eddy viscosity, whereas the latter is not usually considered as an important parameter in lake hydrodynamics modelling.

Key words: Lake hydrodynamics, Wind-driven circulation, Turbulence anisotropy, Eddy viscosity, Analytical solution

1. Introduction

The response of a closed water body to external forcing, like wind stress on a lake surface, has been investigated extensively both in past and in recent literature. Several studies have been carried out for steady...
circulation in simplified cases and for different types of waves arising from the interaction with topography and Earth rotation [19, 10, 22, e.g.]. In particular, [35] provided an analytical solution for variable wind stress in the ocean. The two-dimensional motion in two- and three-layered narrow lakes has been analyzed in detail [7, 6, 17, e.g.] and some authors have tackled the case of three-dimensional simplified solutions [40, 32, 9]. More recently, the attempt to find an analytical description of lake circulation in simplified cases has been set aside, in favor of more complex simulations of real cases by means of numerical models. Nevertheless, we believe that the old-fashioned search of analytical solutions in simplified geometries is still relevant to understand the essential physics of the natural phenomena.

As a matter of fact, most of the topics might seem already well-known, in particular for wind-induced steady circulation in simplified basins. However, we believe that some aspects are still worth considering, for instance the role of lateral shear stresses on the response of the water body in terms of the prevailing circulation either in the vertical or in the horizontal plane. This subject has never been considered explicitly: Witten and Thomas [40] and Hunter and Hearn [9] studied the development of planimetric circulation, but they neglected the transversal exchange of momentum. The justification for the latter assumption might seem reasonable, since the vertical gradients of velocity are much stronger than the horizontal ones. On the other hand, turbulence anisotropy is typical of stratified flows, where the vertical eddy viscosity can be some order of magnitude smaller than the horizontal eddy viscosity; hence, vertical shear stresses may be comparable with the horizontal ones. Incidentally, we note that Hunter and Hearn [9] take into account bathymetric variations in a statistical fashion that is not compatible with considering lateral momentum exchange.

The suppression of vertical turbulence in stratified flows has been recognized for a long time [20, e.g.], but it is receiving considerable attention more recently due to its importance in geophysical fluid dynamics [43, 11, e.g.] and thanks to the recent developments of DNS models and laboratory analysis [29, 31, e.g.]. Although most of the three-dimensional models for lake circulation consider very different values for the eddy coefficients (either directly in Fickian turbulence closures or indirectly in more refined models), the consequences of this fact have not been discussed explicitly. Several papers deal with the estimate of vertical eddy viscosity in stratified flows (see for instance the review in [16]), but the calibration of the horizontal values is often related to the calculation’s grid spacing [33] or is estimated through field observations of diffusive processes, which usually retain the effects of dispersive mixing, also called “shear diffusion” [21]. Since the uncertainties in such estimates are very large, the effect of turbulence anisotropy on lake hydrodynamics deserves further study.

In this respect, Wang [38] points out the strong spatial and temporal variability of vertical and horizontal eddy viscosities. In particular, by applying two subgrid-scale modeling techniques – Smagorinsky’s model for the horizontal eddy viscosity and the Mellor-Yamada level-2 model for the vertical eddy viscosity – as turbulence closure conditions in numerical simulations, the anisotropy between vertical and horizontal eddy coefficients is shown: the amplitude of the order of $10^{-3}m^2/s$ for the vertical eddy coefficient is estimated versus a magnitude of the horizontal eddy viscosity varying in the range of $0.1 \div 1 m^2/s$ in the epilimnion.

Various factors can be considered important for the appearance of planimetric gradients of velocity in lake circulation, for example the finite dimension of the enclosed basin, bathymetric variations or wind field inhomogeneities [4, 36, 14]. In this contribution we focus on the last factor, assuming an elongated, constant-depth basin as a first approximation; this is a reasonable approximation for several real cases. Wind inhomogeneities are common both in large lakes [15, 30, 13, 28], where differences in wind velocities mainly depend on pressure gradients over variable topography, and in small lakes, where they are mostly due to local sheltering effects [26, 27]. In particular, topographic sheltering and wind fetch can play a key role also in basins characterized by a regular shape. For sake of precision, we stress that planimetric circulation can be generated in real lakes even by a spatially uniform wind where the boundaries are complex or the depth is not constant; nevertheless, we do not tackle this problem and hence the present analysis is restricted to a very simple basin geometry.

The flow field in real lakes can be very complex and is commonly studied by means of numerical models, which usually include the lateral exchange of momentum. Despite the hydrodynamics being inherently three-dimensional, in particular in those cases where thermal stratification is important and hence the vertical variability strongly affects the solution, it is common to study such a problem by means of depth-averaged...
models [26, 30, 3, 28, e.g.]. Depth-averaged models are much less time-consuming than three-dimensional models, but unfortunately they implicitly neglect that vertical circulation may be important.

In this paper we focus on the importance of turbulence anisotropy on the development of lake circulation. To this aim a simplified model is proposed to obtain an analytical solution in the case of a rectangular basin with constant depth, forced by a wind field directed along the main axis of the basin and characterized by a transversal variability (see also [37] for a preliminary analysis). The solution is sought for the steady conditions determined by a spatially variable wind forcing, neglecting the horizontal density gradients associated with the transport of temperature (baroclinic effects). Although the circulation of a thermally stratified lake can be affected by the differential accumulation of cold and warm water, it is the initial barotropic response to wind forcing that determines the flow field that is finally responsible for the onset of baroclinic circulation.

Several assumptions are introduced in order to find an explicit solution. The effect of Coriolis acceleration due to earth rotation is neglected and the turbulence stresses are closed within the framework of a Fickian diffusive model [39, e.g.]. For sake of simplicity, spatial variations of the eddy diffusivity tensor are neglected, and turbulence anisotropy is considered only between the horizontal and the vertical direction, where stratification is important. In the central part of the basin, considering the steady state and far from the longitudinal boundaries, the flow field is shown to be dependent only on wind stress lateral variability and a few dimensionless parameters.

We distinguish two cases: (i) a single-layer lake, which can show a certain degree of continuous stratification; and (ii) a two-layer lake, where the development of a strongly stratified thermocline allows one to separate the upper layer, epilimnion, from the lower layer, hypolimnion. In both cases, an analytical solution for the velocity profile is obtained. In addition to the wind stress inhomogeneity, we investigate the role of the anisotropy of the eddy viscosities and the role of the geometrical ratio between the transversal breadth and the depth of each layer. As a general result of our analysis, the role of horizontal viscosity compared to vertical viscosity seems to be more important than that usually ascribed by most authors.

The paper is organized as follows: the problem is stated in section 2 for the case of a rectangular basin; then the analytical solution is obtained in section 3 introducing some simplifying assumptions, considering both the case of a single layer system and that of a two-layer lake; and finally the results are shown in section 4 and discussed in section 5.

2. Formulation of the problem

The onset of steady wind stress in an enclosed basin initially determines the development of barotropic and baroclinic waves, which tend to decay with time leaving an aperiodic set-up of the interfaces [34] and a global steady circulation. In this paper we focus only on the latter response, assuming that the wind forcing maintains constant strength for a time long enough to let initial waves decay. Moreover, we assume that the stratification is not significantly affected during the time considered. In this sense, our analysis neglects the role of internal waves and upwelling, which may represent important phenomena in the dynamics of stratified basins [18, e.g.].

In order to obtain an analytical solution we consider a simplified geometry for the water body, assuming that the bathymetry of a real lake could be approximated by a rectangular basin of constant depth $D_0^*$ (hereafter a superscript star denotes dimensional variables). We assume a coordinate system where the longitudinal axis $x^*$ is aligned with the wind direction, the lateral axis $y^*$ is orthogonal and the vertical axis $z^*$ is directed upward, starting from the bottom (Figure 1). The wind intensity varies only along the axis $y^*$ and its variations are responsible for the non-uniform response of the lake. The longitudinal dimension of the simplified basin is $L_x^*$ and the lateral dimension is $L_y^*$. The focus is on steady state circulation induced by a steady wind forcing. A rather strong geometrical simplification arises from the assumption that the longitudinal length of the lake is much larger than its width ($L_x^* \gg L_y^*$). This is not rare in real lakes, which can be significantly elongated, and allows us to consider a central part of the basin where longitudinal velocities are dominant.

Several additional assumptions are introduced to get a simpler solution; most of them are common to other studies regarding lake hydrodynamics. The turbulence-averaged Reynolds equations are adopted and
the turbulent fluxes are parameterized diffusively by means of a Fickian closure, with the introduction of an eddy viscosity tensor. The role of the Coriolis acceleration is not considered. The pressure is assumed hydrostatically distributed, as it is usually done in the case of shallow water; in fact, the hydrodynamic pressure is usually small in steady conditions and far from the steep shores [12] in relatively deep lakes as well. Finally, turbulence is considered anisotropic due to vertical density stratification.

With the above assumptions and in the case of stationary flow, the momentum equations, namely the Navier-Stokes equations averaged over turbulence fluctuations, can be written as follows:

\[
\begin{align*}
&u* \frac{\partial u*}{\partial x*} + v* \frac{\partial u*}{\partial y*} + w* \frac{\partial u*}{\partial z*} + (g* - g_0^*) \frac{\partial h*}{\partial x*} + g_0 \frac{\partial H^*}{\partial x*} + \frac{g*}{\rho_0} \frac{\partial}{\partial x*} \int_z h^* \rho' \, dz = \\
&\frac{\partial}{\partial x*} \left( \nu_x^* \frac{\partial u*}{\partial x*} \right) + \frac{\partial}{\partial y*} \left( \nu_y^* \frac{\partial u*}{\partial y*} \right) + \frac{\partial}{\partial z*} \left( \nu_z^* \frac{\partial u*}{\partial z*} \right), \\
&\frac{\partial v*}{\partial x*} + v^* \frac{\partial v*}{\partial y*} + w* \frac{\partial v*}{\partial z*} + (g* - g_0^*) \frac{\partial h*}{\partial y*} + g_0 \frac{\partial H^*}{\partial y*} + \frac{g*}{\rho_0} \frac{\partial}{\partial y*} \int_z h^* \rho' \, dz = \\
&\frac{\partial}{\partial x*} \left( \nu_x^* \frac{\partial v*}{\partial x*} \right) + \frac{\partial}{\partial y*} \left( \nu_y^* \frac{\partial v*}{\partial y*} \right) + \frac{\partial}{\partial z*} \left( \nu_z^* \frac{\partial v*}{\partial z*} \right),
\end{align*}
\]

where \((u*, v*, w*)\) are the components of the velocity vector along the axis directions \((x*, y*, z*)\), \(h^*\) is the top surface elevation, \((\nu_x^*, \nu_y^*, \nu_z^*)\) are the diagonal components of the eddy viscosity tensor, \(g^*\) is gravity acceleration. The water density \(\rho_0^*\) is taken as a reference, and \(\rho_0' = \rho^* - \rho_0^*\) is the vertical variation of the actual density \(\rho^*\) with respect to \(\rho_0^*\); in the spirit of the usual Boussinesq approximation, \(\rho_0'\) is assumed to be small and to play a role only in the baroclinic term. Moreover, equations (1)-(2) are written for a main layer considering the possibility that an overlying layer is present above it (see Figure 2). The overlying layer is characterized by the following variables: \(\rho_0^*\) is the density (supposed constant), \(H^*\) is the free surface elevation above reference; \(g_0^* = g^* \rho_0^*/\rho_0^*\) is the modified gravity acceleration acting on the main layer. If the fluid in the main layer is water with air above, \(\rho_0^* \ll \rho_0^*\) and \(g_0^* \simeq 0\) and the flow in the main layer can be safely considered as a free surface flow: this is the case of a single layer lake or the top layer of a two-layer system. In the case of lower layer of a two-layer water system (e.g. hypolimnion with epilimnion above), \(\rho_0^* \sim \rho_0^*\) and the additional terms, which are proportional to \(g_0^* \simeq g^*\), are not negligible. In addition to 1)-(2), the three-dimensional continuity equation reads

\[
\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0.
\]

Let us now refer to the single layer system; the extension to the two-layer case will be done in section 3.3. We consider a wind forcing directed in the \(x^*\) direction with lateral variations. Thus the wind shear stress \(\tau_w^*(y^*)\) is applied at the free surface in the following form:

\[
\rho^* \nu_{z^*} \frac{\partial u^*}{\partial z^*} = \tau_w^*, \quad \rho^* \nu_{z^*} \frac{\partial v^*}{\partial z^*} = 0 \quad (z^* = h^*),
\]

where \(\rho^* \simeq \rho_0^*\) is the density of water at the free surface. At the bottom boundary the following boundary conditions apply:

\[
\rho^* \nu_{z^*} \frac{\partial u^*}{\partial z^*} = \gamma_b^* (u^* - u_b^*), \quad \rho^* \nu_{z^*} \frac{\partial v^*}{\partial z^*} = \gamma_b^* (v^* - v_b^*) \quad (z^* = 0),
\]

where \(\rho^* \simeq \rho_0^*\) and a general slip condition has been used, being \(\gamma_b^*\) a drag coefficient and \((u_b^*, v_b^*)\) the velocity of the boundary. Moreover, the bed is impermeable and hence \(w^*|_{z^*=0} = 0\). The usual no-slip conditions \((u^* = 0, v^* = 0)\) are found from (5) for fixed boundary \((u_b^* = 0, v_b^* = 0)\) and \(\gamma_b^* \to \infty\).
At the other boundaries (lateral, longitudinal) we assume vanishing tangential stress and depth-averaged flux; in particular, at the lateral boundaries the following conditions hold for every $x^*$:

$$
\rho^* \nu^* \frac{\partial u^*}{\partial y^*} = 0, \quad \int_0^{h^*} u^* \, dz^* = 0 \quad (y^* = 0, L_y^*).
$$

The stationarity of the process requires further integral conditions to be satisfied, namely that the total discharge through every cross-section that divides the lake in two parts must be null; in particular, being interested in the cross-sections orthogonal to the longitudinal axis,

$$
\int_0^{L_y^*} \int_0^{h^*} u^* \, dz^* \, dy^* = 0.
$$

3. A simplified analytical solution

By excluding the longitudinal boundaries, where the circulation is closed in the horizontal ($v^* \neq 0$) and/or in the vertical plane ($w^* \neq 0$), as a first approximation we can assume that in the central part of the lake the longitudinal velocity is dominant and, consequently, $v^* \simeq 0$ and $w^* \simeq 0$. According to (2), the former condition is associated with negligible transversal gradient of the total pressure term $g^* h^* + g^*_s (H^* - h^*)$ in a significant part of the water body. As a consequence, we can assume that also the longitudinal gradient $g^* s_h + g^*_s s_s$, where

$$
s_h = \frac{\partial h^*}{\partial x^*}, \quad s_s = \frac{\partial (H^* - h^*)}{\partial x^*},
$$
does not laterally vary in the central part of the lake as a result of the wind forcing. Moreover, with the above assumptions the continuity equation (3) becomes $\partial u^*/\partial x^* = 0$, which simply suggests that the longitudinal velocity can be considered uniform along $x^*$ in the central region of the water basin. Exploiting such considerations, only the $x$-momentum equation is retained. This means that in the central part of the basin the main physical mechanism to be considered is the transmission of momentum from the superficial stress, which governs the distribution of the longitudinal velocity $u^*$ within the cross-section.

Different characteristic scales are involved in the problem. We select the basin depth $D_0^*$, its lateral width $L_y^*$, the average wind stress $\tau_{w0}^*$, and the reference value of the vertical eddy viscosity $\nu_{z0}^*$, in order to introduce the following dimensionless variables:

$$
y = \frac{y^*}{L_y^*}, \quad z = \frac{z^*}{D_0^*}, \quad u = \frac{u^*}{U_0^*}, \quad h = \frac{h^*}{D_0^*},
$$

$$
\tau = \frac{\tau^*}{\tau_{w0}^*}, \quad (\nu_y, \nu_z) = \left(\frac{\nu_{y0}}{\nu_{z0}}, \frac{\nu_{z0}}{\nu_{z0}}\right),
$$

where $\tau^*$ is the generic shear stress, and the scale of velocity

$$
U_0^* = \frac{\tau_{w0}^* D_0^*}{\rho_0^* \nu_{z0}^*}
$$
is chosen on the basis of the free surface dynamic condition (4). The dimensionless parameters

$$
\delta = \frac{D_0^*}{L_y^*}, \quad F^2 = \frac{U_0^* \nu_{z0}^*}{g^* D_0^2}, \quad \frac{\tau_{w0}^*}{g^* D_0^2}, \quad g_s = \frac{g_s^*}{g^*}, \quad \Delta \rho = \frac{\Delta \rho^*}{\rho_0^*},
$$

where $\Delta \rho^*$ is the initial density variation along the depth of the main layer, can be used to write the dimensionless form of equation (1),

$$
\frac{s_h}{F^2} \left[1 + \Delta \rho (R_1 + R_2)\right] + g_s \frac{s_s}{F^2} = \delta^2 \frac{\partial}{\partial y} \left(\nu_y \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z} \left(\nu_z \frac{\partial u}{\partial z}\right),
$$

(12)
where the information deriving from the continuity equation allowed for the simplification of the longitudinal advective and diffusive terms. The term multiplying $g_s$ derives from the presence of the overlying layer, which can be neglected if the fluid above is air with respect to water ($g_s \sim 0$); on the contrary, it gives a contribution in the case of two water layers having similar density ($g_s \simeq 1$). The term multiplying $\Delta \rho$ (which is the order of magnitude of the relative density variation within the main layer, typically $\sim O(10^{-3})$ for the normal temperature range in lakes) represents the baroclinic term. The definitions of $R_1$ and $R_2$ (which vary along the depth) and their order of magnitude are reported in Appendix A: $R_1 \sim O(1)$ always and therefore this term (multiplied by $\Delta \rho$) can always be neglected, whereas the conditions for which the term $R_2$ is not negligible are discussed in Section 4.5. In particular, $R_2 = 0$ under the assumption of self-similarity of the density profiles (i.e. density variations have the same vertical structure everywhere). Keeping in mind that it is not the vertical stratification itself (term $R_1$) that is important, but only the horizontal density gradients that can develop when temperature is unevenly transported by currents (term $R_2$), we can derive a simplified solution under the above-mentioned assumptions; such a solution can be seen as the barotropic response of an initially vertically stratified (without horizontal variations) water body to an inhomogeneous wind stress.

In order to obtain a fully analytical solution further assumptions about turbulence are to be exploited. As a matter of fact, there are strong difficulties in determining turbulence intensity in lakes. One of the most common choices is to adopt a constant value of the horizontal eddy viscosity and a vertical eddy viscosity varying along the depth, as a function of thermal stratification, since the presence of strong density gradients inhibits turbulence [8, 31]. Therefore, the order of magnitude of $\nu_z^* \eta$ can be significantly smaller than that of the horizontal eddy viscosity $\nu_y^*$. In the present simplified model constant values of both horizontal and vertical eddy viscosity are adopted ($\nu_y^* = \nu_y^0$, $\nu_z^* = \nu_z^0$); thus $\nu_z = 1$ and $\nu_y = \eta$ are considered in each layer, where the turbulence anisotropy is introduced as

$$\eta = \frac{\nu_y^0}{\nu_z^0}.$$  \hspace{1cm} (13)

In multi-layered basins, where the different layers are separated by shallow regions with strong temperature gradients (thermocline), this assumption is adopted independently for each layer. In the following, we present the solutions for the single-layer case and the two-layer case; the latter case is the common schematization for lakes with a well developed thermocline.

With the above assumption of spatially constant yet anisotropic eddy viscosity, and neglecting the baroclinic term, from (12) we finally end up in a Poisson equation for the velocity $u$,

$$\frac{\partial^2 u}{\partial z^2} + \alpha^2 \frac{\partial^2 u}{\partial y^2} = \theta,$$  \hspace{1cm} (14)

which is valid in each layer; the corresponding boundary conditions read

$$\frac{\partial u}{\partial z} \bigg|_{z=h} = \tau_w, \quad \frac{\partial u}{\partial z} \bigg|_{z=0} = \frac{u - u_b}{\kappa}, \quad \frac{\partial u}{\partial y} \bigg|_{y=0,1} = 0.$$  \hspace{1cm} (15)

In the above differential system three dimensionless parameters have been introduced:

$$\alpha = \delta \sqrt{\eta} = \frac{D^0_y}{L^y} \sqrt{\frac{\nu_y^0}{\nu_z^0}}, \quad \kappa = \frac{\rho_y^0 \nu_z^0}{\gamma^0 D^0_y}, \quad \theta = \frac{s_h + g_s s_s}{F^2}.$$  \hspace{1cm} (16)

It is worthwhile to note that $\alpha$, $\theta$ and $\kappa$ are constant, while the forcing $\tau_w$ varies along $y$. The no-slip condition at the bottom is recovered when $u_b = 0$ and $\kappa = 0$. The parameter $\theta$ (representing the free surface slope) is not independent, but can be determined by means of the dimensionless integral condition

$$\int_0^1 \int_0^b u \, dz \, dy = 0.$$  \hspace{1cm} (17)
Since the main object of our analysis is the central part of the elongated basin, the water depth is approximately equal to the reference depth $D_0^*$. Thus, even taking the slope $s_h$ into account, we can safely assume $h = 1$ in the equations (15a) and (17).

A simple solution can be obtained considering a sinusoidal lateral variation of the wind stress as

$$
\tau_w^* = \tau_{w0}^* + \tau_{wn}^* \cos \left( n\pi \frac{y}{L_y} \right),
$$

where $\tau_{w0}^*$ is the scale considered in (9), and $n$ is the lateral Fourier mode having amplitude $\tau_{wn}^*$. For $n = 1$ there is wind amplification on one side and reduction on the opposite, while $n = 2$ (having $\tau_{w2}^*$ a sign opposite to $\tau_{w0}^*$) is the case of sheltering effects near the shores of the lake and stronger wind in the central part. It is important to point out that, given the linearity of the problem, any lateral distribution of stress can be taken into account simply considering the components of its Fourier development. Therefore, looking for a simple solution within the above dimensionless framework, the wind stress (18) can be written using a single Fourier mode as

$$
\tau_w = 1 + \phi_w \cos (n\pi y),
$$

where the excess ratio at the surface is

$$
\phi_w = \frac{\tau_{wn}^*}{\tau_{w0}^*},
$$

3.1. Plane-flow solution

Let us first consider the case where the lateral momentum exchange is neglected ($\nu_y = 0$): a plane flow develops in the vertical $x$-$z$ reference system. The governing equation (14) becomes

$$
\frac{\partial^2 u}{\partial z^2} = \theta,
$$

and can be solved by expressing the velocity as $u(y,z) = \tau_w(y) u_0(z)$, where the vertical structure $u_0$ is the same for every $y$. The boundary conditions (15a-b) do not change and the integral condition (17) can be applied in the simplified form $\int_0^1 u_0 \, dz = 0$. Then the plane-flow solution reads

$$
u_0 = \frac{1}{2(1+3\kappa)} \left[ z \left( \frac{3}{2} z - 1 \right) + \kappa \left( 3z^2 - 1 \right) + u_{b0} \left( 3z^2 - 6z + 2 \right) \right],
$$

where $u_{b0}$ is the slip velocity. In the case of no-slip ($\kappa = u_b = 0$), equation (22) reduces to the well-known solution [6, e.g.]

$$
\nu_0^{(ns)} = \left( \frac{3}{2} z - 1 \right) \frac{z}{2},
$$

having bottom shear stress $\tau|_{z=0} = -1/2$ and $\theta = 3/2$, so the free surface inclination (considering water with air above, i.e. $g_s \sim 0$) results $s_h = 3F^2/2 = 3\tau_{w0}^*/(\rho g^* D_0^*)$.

It is worth noting that the solution (22) is valid also when the lateral momentum exchange is retained ($\nu_y \neq 0$) in the case of a uniform wind distribution, without lateral variations in the forcing ($\phi_w = 0$). In fact, symmetry reasons associated with the assumed boundary conditions suggest that the velocity does not change in the lateral direction, and hence lateral shear stress is not present.

3.2. One-layer solution

As a second step of the analysis, we consider a single layer with a generic slip condition at the bottom and a sinusoidal wind forcing (19). In this case the complete solution reads (the details of the procedure are explained in Appendix B.1)

$$
u = \nu_0 + \hat{u}_1 \cos(n\pi y),
$$

7
where \( u_0 \) is the plane-flow solution (22) and

\[
\tilde{u}_1 = \frac{\phi_w \sinh(\chi z) + \kappa \chi \cosh(\chi z) + \tilde{w}_1 \chi \cosh(\chi(z - 1))}{\chi [\cosh(\chi) + \kappa \chi \sinh(\chi)]},
\]

having introduced the parameter \( \chi = n \pi \alpha \), where \( \alpha \) is the main parameter of the system, together with \( \kappa \); both parameters are defined in (16).

The depth-averaged velocity can be easily calculated as \( U(y) = \int_0^1 u \, dz \) and reads

\[
U = \left[ \phi_w \frac{\cosh(\chi) - 1 + \kappa \chi \sinh(\chi)}{\chi} + \tilde{w}_1 \sinh(\chi) \right] \frac{\cos(n \pi y)}{\chi [\cosh(\chi) - 1 + \kappa \chi \sinh(\chi)]}.
\]

In the case of no-slip at the bottom (\( \kappa = u_b = 0 \)), solutions (24) and (26) reduce to

\[
u^{(ns)} = \left( \frac{3}{2} z - 1 \right) \frac{z}{2} + \phi_w \cos(n \pi y) \sinh(n \pi \alpha z) \]

\[
U^{(ns)} = \phi_w \cos(n \pi y) \frac{1 - \text{sech}(n \pi \alpha)}{n^2 \pi^2 \alpha^2}.
\]

### 3.3. Two-layer solution

The solution for a two-layer system of immiscible fluids can be derived by assigning a suitable condition on the shear stress exchanged among the layers. In real lakes, the upper layer corresponds to epilimnion (hereafter denoted with subscript \( E \)) and the lower layer corresponds to hypolimnion (subscript \( H \)). We assume that these two layers have different densities, due to their different temperature, and are separated by a sharp interface (subscript \( T \), namely thermocline).

The system is forced by the wind in the upper layer \( E \) and the shear stress is transmitted through the interface to the lower layer according to a relationship of the type (5), where \( u_b = u_H|_{z=1} \) is the velocity at the top of the lower layer \( H \). For sake of simplicity we assume no-slip conditions at the bottom of the lower layer \( H \); the consideration of slip conditions is straightforward, but the solution is more complicated.

The details about the derivation of the solution are given in Appendix B.2; here it suffices to say that four dimensionless parameters arise:

\[
\alpha_E = \frac{D_E}{L_E}, \quad \alpha_H = \frac{D_H}{L_H}, \quad \psi = \frac{\rho_E \nu_E D_E}{\rho_H \nu_H D_H}, \quad \kappa_T = \frac{\rho_E \nu_E}{\gamma \rho_H D_H},
\]

where the dimensional scales of the two layers have been considered separately: \( D_E, \nu_E, \nu_H, \gamma \) for the epilimnion; \( D_H, \rho_H, \nu_H, \gamma_H \) for the hypolimnion. The parameters \( \alpha_E \) and \( \alpha_H \) (and consequently \( \chi_E = n \pi \alpha_E \), \( \chi_H = n \pi \alpha_H \)) are defined in analogy with (16), \( \psi \) represents the ratio between the significant scales of the upper layer and those of the lower layer, and \( \kappa_T \) the efficiency of shear stress transfer across the thermocline.

The dimensionless solution in the upper layer (epilimnion) becomes

\[
u_E = \frac{z(3z - 2) + (3z^2 - 1)(\psi + 4 \kappa_T) / 2 + \phi_w \cos(n \pi y) \sinh(\chi_E z) [\Psi_0 \chi_E \cos(\chi_E z)]}{\chi_E \Psi_1},
\]

where \( \Psi_0 \) and \( \Psi_1 \) are functions of the parameters introduced in (29) and reported in Appendix B.2. In order to compare the solution in the lower layer \( H \) with that in the upper layer \( E \), it is useful to rewrite the former in dimensionless form using the same velocity scale as in the upper layer. After some algebra, the modified dimensionless velocity in the lower layer (hypolimnion) reads

\[
\tilde{u}_H = \frac{u_H}{U_E} = \psi \left[ \sigma \frac{z(3z - 2)}{4} + \phi_w \cos(n \pi y) \sinh(\chi_H z) \chi_H \cosh(\chi_H z) \Psi_1 \right].
\]

Note that (31) is still defined between \( z = 0 \) and \( z = 1 \), but \( u_H^* \) is made dimensionless using the epilimnetic velocity scale \( U_E \).
The layer-averaged velocities corresponding to (30) and (31) are the following:

\[
U_E = \int_0^1 u_E \, dz = \phi_w \cos(n\pi y) \frac{\psi_1 - 1}{\chi_{E}^2}.
\]  

(32)

\[
\tilde{U}_H = \int_0^1 \tilde{u}_H \, dz = \phi_w \cos(n\pi y) \frac{\psi [1 - \text{sech}(\chi_H)]}{\chi_{H}^2}. 
\]  

(33)

3.4. The role of thermocline

The analysis of the two-layer solution allows for interesting considerations. As a first annotation, it is easy to see from (15) that for \( \kappa_T = 0 \) the velocity field is continuous through the interface between the two layers: this means a strong momentum flux (shear transmission) between the two layers. On the contrary, \( \kappa_T \to \infty \) corresponds to impose a vanishing stress at the interface.

An estimate of the value of \( \kappa_T \) in lake thermocline can be obtained by considering that the assumption of an interface between layer \( E \) (epilimnion) and \( H \) (hypolimnion) is a simplification of reality, where usually a strongly stratified layer with finite thickness (thermocline, or metalimnion) exists. Thus the stress at the interface (in dimensional form) can be approximated as

\[
\tau^* = \rho^* \nu^* z_T \left( u^*_E \big|_{z=0} - u^*_H \big|_{z=1} \right) D^*_T, 
\]  

(34)

where \( \nu^*_T \) is the vertical viscosity in the thermocline (usually much smaller than in the rest of the lake) and \( D^*_T \) is the thickness of the thermocline, having a reference density \( \rho^*_T \). Recalling (29d) and comparing (34) with the adaptation of (5) to the two-layer case (see also equation (65) in Appendix B.2), the dimensionless parameter reads

\[
\kappa_T \simeq \frac{\rho^*_E \nu^*_T D^*_T}{\rho^*_T \nu^*_T D^*_E}, 
\]  

(35)

which shows a clear analogy with (29c), where the parameter \( \psi \) describes the difference between the epilimnion \( E \) and the hypolimnion \( H \), while \( \kappa_T \) is defined between the epilimnion and the thermocline.

We can consider some asymptotic limits of the general solution for the velocity in the upper layer. When \( \kappa_T = 0 \) and \( \psi = 0 \), equation (30) reduces to the single-layer, no-slip solution (27) because there is strong shear at the interface and a vanishing velocity in the lower layer. This can occur when the eddy viscosity in a thin thermocline is comparable to that in a thick epilimnion, and when also the hypolimnion thickness is relatively small, in association with the bottom no-slip condition.

At the other limit, we find that when \( \kappa_T \to \infty \) (very small value of vertical viscosity in the thermocline), the solution (30) becomes

\[
u_E = \frac{3z^2 - 1}{6} + \phi_w \cos(n\pi y) \frac{\cosh(\chi_E z)}{\chi_{E}^2} \sinh(\chi_E), 
\]  

(36)

corresponding to a vanishing stress at the bottom of the layer. Such condition is typical of a strongly stratified thermocline.

4. Results

The analytical solutions suggest that the prevalence of vertical or planimetric circulation depends on wind inhomogeneity, quantified by the parameter \( \phi_w \) (wind excess ratio), and on a few parameters describing the lake geometry and the turbulence anisotropy (namely, the single parameter \( \alpha \) for one-layer systems with no-slip conditions, and \( \alpha_E, \alpha_H, \psi \) and \( \kappa_T \) for two-layer systems). With regard to the two-layer scheme, in the following analysis of the resulting circulation, we will focus mostly on what happens in the epilimnion, where the strongest velocities develop.

In order to have a quantitative measure of the predominance of vertical vs. planimetric motion in the water body, we introduce a new variable obtained by suitably averaging the flow field over the whole cross-section. Firstly, we consider the vertically averaged velocity \( \bar{U} \), calculated by means of (26) for the one-layer
solution, or $U_E$ given by (32) for epilimnetic velocity in the two-layer solution. For sake of simplicity, in this paragraph we refer only to the former case, implicitly extending the results to the epilimnion layer in the latter case.

A small value of $U$ means that the circulation occurs mainly in the vertical plane. However, we can have quantitative information only if we compare it with a suitable scale. For this purpose, we use the average of the absolute values of $u$, namely

$$\langle U \rangle(y) = \int_0^1 |u(y, z)| \, dz ,$$

which gives an idea about the intensity of the flow. Thus, the ratio

$$\varphi(y) = \frac{|U(y)|}{\langle U \rangle(y)} ,$$

defined in the range $0 \div 1$, measures the prevailing circulation in each lateral position $y$. It is clear that in the case $\varphi = 1$ the vertical profile of velocity never changes its sign. We can have a single parameter describing the behavior of the whole cross-section by averaging $\varphi$ along the lateral direction:

$$\omega = \int_0^1 \varphi(y) \, dy .$$

The parameter $\omega$, which we may call circulation number, is defined in the range $0 \div 1$, being equal to 1 when the circulation is dominantly planimetric and vanishing for perfectly vertical circulation.

4.1. One-layer model

In this section we consider the case of a single layer with no-slip condition, represented by the analytical solution (27). Two different factors affect the governing parameter $\alpha$, defined in (16): the geometrical ratio $\delta$ between the water depth and the width of the basin and the anisotropy ratio $\eta$ between the horizontal and the vertical eddy viscosity. The former ratio can be easily determined (also through a simplified equivalent geometry in real lakes) and can vary from $10^3$ for very deep basins to $10^{-3}$ for shallow lakes. On the other hand, the latter ratio is much more difficult to estimate when stratification occurs. In fact, in the case of homogeneous water (constant density) turbulence is almost isotropic, but in a thermally stratified fluid there are large uncertainties in empirical laws to estimate the reduction of turbulence intensity in the vertical direction.

Two limit cases can be identified. (i) Deep and stratified lakes ($D_0^* \sim L_y^*$, $\nu_z^* \ll \nu_y^*$), for which $\alpha \sim 1$ or larger; for $\alpha \to \infty$ the velocity profile (27) tends to that of the plane flow $u_{0ns}^{(ns)}$ given by (23). (ii) Shallow and homogeneous lakes ($D_0^* \ll L_y^*$, $\nu_z^* \sim \nu_y^*$), for which $\alpha \ll 1$; for $\alpha \to 0$ the solution reads

$$u^{(ns,sh)} = \left( \frac{3}{2} - 1 \right) \frac{z}{2} + \phi_w z \cos(n \pi y) .$$

An example of the flow field of the longitudinal velocity for $\alpha = 0.3$ is shown in Figure 3, where the first mode of transversal variation is considered ($n = 1$): for positive values of the wind excess ratio $\phi_w$, there is an increase of stress in the region $y \in (0, 1/2)$ and a reduction for $y \in (1/2, 1)$. Thus the velocity profiles in the excess side ($y = 0$) are deformed towards larger positive value, while on the opposite side ($y = 1$) the inversion of the flow is enhanced.

The case of Figure 3 ($\alpha = 0.3$) is characterized by a moderate anisotropy (for instance $\delta = 0.03$ and $\eta = 10^2$). A different kind of circulation develops when the anisotropy of turbulence is larger: in fact, increasing $\alpha$ with the same value of wind excess $\phi_w$ enhances the development of vertical circulation because the lateral transfer of momentum is stronger than the vertical one. A example of the circulation occurring with $\alpha = 3$ is shown in Figure 4, corresponding to a strong anisotropy ($\eta = 10^4$) with the same $\delta$ as in the previous case, or to a narrow lake ($\delta = 0.3$) with the same anisotropy.
It is interesting to evaluate the condition for which the wind inhomogeneity causes the inversion of the velocity at the free surface. We can study such a critical condition by imposing \( u|_{z=1, y=1} = 0 \) in equation (27), which gives the condition

\[
\phi_w = r \frac{n \pi \alpha}{\tanh(n \pi \alpha)}, \quad \text{where} \quad r = \frac{1}{4} (-1)^{n+1},
\]

whose behavior can be described as follows: for vanishing values of \( \alpha \) the critical ratio is approximately constant \((\phi_w = r)\), while for \( \alpha \) larger than about 1/\( n \) it increases linearly \((\phi_w = r n \pi \alpha)\). Thus, if \( \alpha \) is small (shallow lakes) an inhomogeneity of the wind \(|\phi_w| > 0.25 \) \((n = 1)\) suffices to give rise to an inverse circulation appearing at the free surface. At the other limit (deep stratified lakes), the critical ratio grows with \( \alpha \), which means that deeper lakes are more likely to develop vertical circulation instead of planimetric one. We note that, generally, when the stratification becomes strong, a thermocline develops and the system is more correctly reproduced by a two-layer scheme. Assuming a maximum variation of the wind stress \( \phi_w = \pm 1 \), the threshold value of the parameter \( \alpha \) to have inversion of the surface velocity (for the limit of large \( \alpha \)) is \( \alpha \approx 1.27 \) for \( n = 1 \), and \( \alpha \approx 0.64 \) for \( n = 2 \). Above those values, the surface velocity cannot change sign in response to a wind forcing having constant sign.

The overall behavior of the water motion is summarized in Figure 5 in the case of \( n = 1 \): the contour plot of the circulation number \( \omega \) shows the predominance of vertical circulation \((\omega \to 0)\) for small value of \( \phi_w \) and large values of \( \alpha \), and the predominance of planimetric circulation \((\omega \to 1)\) for strong wind variation (large \( \phi_w \)) and shallow basins (small \( \alpha \)). In fact, when the lateral variation of wind forcing increases, the velocity vertical profile tends to become monotonic, hence reducing the importance of the vertical circulation.

### 4.2. Two-layer model

Although the single layer model allows for a strong simplification of the flow description, the two-layer system can represent a better approximation of the real behavior of a lake. In this case, the parameters \( \alpha_E, \alpha_H \) and \( \psi \) can be estimated from (29) using the scale of the upper layer (epilimnion) and the lower layer (hypolimnion). We have already seen in section 3.4 that a crucial role is played by the thermocline, which separates the two layers. Its effect is described by the parameter \( \kappa_T \), whose estimate is given by (35).

When the stratification is strong, \( \kappa_T \gg 1 \), the simplified solution (36) can be used and the behavior of the epilimnion, in response to the wind forcing given by \( \phi_w \), can be considered function of the parameter \( \alpha_E \) only. On the contrary, when \( \kappa_T \sim O(1) \) or smaller, the other parameters (\( \psi, \alpha_H \)) become important. An example of the flow field in an intermediate situation is given in Figure 6.

In general, the overall behavior of a two-layer system cannot be sketched in a simple way as for the single layer with no-slip at the bottom (Figure 5). However, we can look at the behavior of the epilimnion in the asymptotic case of large \( \kappa_T \) predicted by (36). The contour plot of the circulation number \( \omega \) in such case is shown in Figure 7: the planimetric circulation is strongly favored with respect to the one-layer solution because of the reduced friction at the interface and the reduction of the layer thickness. In particular, if \( \alpha_E \) is sufficiently small (smaller than about 0.1 \( \div \) 0.3), \( \omega \) is close to 1 also for relatively small values of the wind inhomogeneities (small \( \phi_w \)). Thus, lakes with a strong thermocline are supposed to easily respond to spatially variable wind forcing by developing planimetric circulation. In the case of weak thermocline (relatively small values of \( \kappa_T \)), the behavior will be somehow intermediate between the limits described by Figure 5 and Figure 7; then, the circulation number \( \omega \) has to be estimated using the complete solution (30).

The possibility of inversion of surface velocity can be estimated for the two-layer solution in analogy with the one-layer solution. Imposing \( u_E|_{z=1, y=1} = 0 \) in equation (30) gives

\[
\phi_w = \frac{(1 + \psi + 4 \kappa_T)}{(4 + 3(\psi + 4 \kappa_T))} \frac{\chi E \Psi_1}{[\sinh(\chi E) + \Psi_0 \chi E \cosh(\chi E)]},
\]

For \( \kappa_T \to \infty \), the condition (42) simplifies in

\[
\phi_w = \frac{1}{3} \chi E \tanh(\chi E) (-1)^{n+1},
\]
which, for \( \alpha_E \to 0 \) (\( \alpha_E \leq 0.16 \, \text{m}^{-1} \) for an error of 10%), tends to
\[
\phi_w = \frac{(m \pi \alpha_E)^2}{3} (-1)^{n+1}.
\] (44)

The threshold value of \( \alpha_E \) for the inversion of surface velocity can be obtained from (43). Assuming \( \phi_w = \pm 1 \), \( \alpha_E \approx 0.96 \) (\( n = 1 \)), \( \alpha_E \approx 0.48 \) (\( n = 2 \)). We can also estimate the minimum wind variation to have inversion: for small \( \alpha_E \) and \( n = 1 \), (44) gives \( \phi_w \approx 3.3\alpha_E^2 \). For \( \alpha_E = 0.2 \), for instance, it is sufficient a variation in wind intensity of the order \( \phi_w \approx 0.13 \) to produce surface velocity inversion associated with remarkable planimetric circulation.

4.3. Higher modes of wind variation

It is also worth considering the effect of the lateral mode of variation of the wind stress. In the case of the first mode (\( n = 1 \)), widely discussed above, the variation is representative of sheltering effects produced by the topography surrounding the lake, for instance where a higher mountain is present on one side of the lake. On the contrary, the second lateral mode (\( n = 2 \)) accounts for the case of wind velocity reduction close to the shores, for instance because of the presence of vegetation that increases the local roughness (in this case \( \phi_w \) is negative). An example of the response to the latter type of forcing (\( n = 2 \), \( \phi_w = -0.4 \)) is shown in Figure 8 for a single layer (a) and a two-layer system (b, c). Furthermore, by assuming sinusoidal lateral variation, any lateral distribution of wind stress can be reproduced by means of a Fourier series.

4.4. Typical scales and validity of the solution

Typical dimensional values of the quantities involved in the problem can be estimated as follows (for sake of simplicity reference is made to the single-layer case). Knowing that the wind stress depends on the square of the wind velocity [41, e.g.],
\[
\tau_w^{*} = c_w U_w^{*2},
\] (45)
where \( c_w \) is the product of the drag coefficient and air density, we assume that the eddy viscosity scale can be approximated as a typical wall turbulence,
\[
\nu_0^{*} \approx U_0^{*} D_0^{*}.
\] (46)
The horizontal eddy viscosity is of the same order of magnitude (\( \nu_0^{*} \approx \nu_0^{*} \)), whereas the vertical eddy viscosity is reduced because of stratification (\( \nu_0^{*} \approx \nu_0^{*}/\eta \)). The anisotropy \( \eta \), defined by (13), can be estimated as a function of the local Richardson number [20]. Therefore, by substituting (45) and (46) in (10), it is easy to find the scale of velocity in terms of the anisotropy \( \eta \):
\[
U_0^{*} = \sqrt{\frac{\tau_w^{*} \eta}{\rho_0^{*} \nu_0^{*}}} U_w^{*},
\] (47)
By analogy, the scale of the vertical eddy viscosity can be obtained from (10) using (45) and (47),
\[
\nu_0^{*} = \frac{c_w^{*}}{\rho_0^{*} \eta} D_0^{*} U_w^{*}.
\] (48)

Accurate estimates require numerical simulations with suitable subgrid turbulence models. For example, using a wind velocity \( U_w^{*} = 4 \, \text{m/s} \) and \( c_w^{*} = 2.2 \times 10^{-3} \, \text{kgm}^{-3} \), Wang [38] finds \( \nu_0^{*} \approx 10^{-3} \, \text{m}^{2} \text{s}^{-1} \) and \( \nu_0^{*} \approx 10^{-1} \, \text{m}^{2} \text{s}^{-1} \) in the epilimnion (\( \sim 10 \, \text{m} \)) for a typical stratification in lake Constance. Wang also looks for the typical velocities occurring with prescribed values of the vertical eddy viscosity (keeping the horizontal equal to 0.2 \( \text{m}^{2} \text{s}^{-1} \)): \( \sim 0.06 \, \text{m/s} \) for \( \nu_0^{*} = 5 \times 10^{-3} \, \text{m}^{2} \text{s}^{-1} \), and \( \sim 0.13 \, \text{m/s} \) for \( \nu_0^{*} = 1 \times 10^{-3} \, \text{m}^{2} \text{s}^{-1} \). Equation (47) gives \( U_0^{*} = 0.037 \, \text{m/s} \) and \( U_0^{*} = 0.083 \, \text{m/s} \) in the two cases, respectively. The order of magnitude of the approximate estimate is the same as the reference values in the numerical simulations by Wang. Moreover, the effect of turbulence anisotropy \( \eta \) is reproduced correctly: the ratio between the two
4.5. Baroclinic effects

The assumption that the baroclinic terms are negligible can be questionable when density variations occur. In Appendix A we show that the self-similar vertical stratification does not affect the analytical solution. On the contrary, the spatial variations of the shape of density profiles can generate relevant baroclinic effects. In fact, the circulation developing in a real lake tends to accumulate fluid with different temperature in different parts of the basin. Lighter water accumulates downwind, in particular in the region with higher stress if the wind has a lateral variation.

In general, the onset of baroclinic effects depends on the prevailing circulation. When a planimetric flow field develops initially (small $\alpha$ and large $\phi_w$), the transport of temperature takes place primarily in horizontal layers. Such transport determines transversal gradients of the surface elevation ($\partial h^*/\partial y^* \neq 0$), longitudinal and transversal gradients of the isopycnals, and hence baroclinic effects. As a consequence, the flow field changes, although the nature of the overall circulation remains approximately the same.

The situation is different when vertical circulation prevails (large $\alpha$ and small $\phi_w$). The flow field tends to accumulate lighter fluid in the downwind region in the upper part of the basin because the velocity $u > 0$ everywhere; on the contrary, heavier fluid is transported upwind in the lower part where $u < 0$. As a consequence, the density profiles strongly deviate from self-similarity and a longitudinal baroclinic gradient appears. Moreover, barotropic and baroclinic effects tend to cancel each other close to the bottom and the

\[
\mathbf{T}_* = \frac{D_0^*}{\nu_{*0}} = \frac{D_0^*}{U_w^*} \sqrt{\frac{\nu_{*,0}^*}{c_w^*}}. \tag{49}
\]

In the case of isotropic turbulence ($\eta = 1$), this scale corresponds to the estimate proposed by Spigel and Imberger [34, eq. 7]. Note that the time needed to reach the steady state can be large, depending on wind forcing (larger time scale for weaker wind), on the momentum transfer by means of vertical eddy viscosity (larger time scale for stronger anisotropy) and on the geometry of the basin (larger time scale for deeper lakes).

Several assumptions have been introduced in the derivation of the analytical solution: dimensional analysis can also give some hints about the limits of validity of the solution. For instance, the case of a compact lake could limit the validity of some simplifications related to the elongated shape of the basin. In fact, if the length $L_x^*$ is short, it is not possible to isolate a central region where only the longitudinal velocity is prevalent, and the longitudinal variations of velocity become relevant. Thus, the effect of the neglected advective term $u^* \partial h^*/\partial x^*$ starts competing with the barotropic term $g^* \partial h^*/\partial x^*$ and the simplified analytical solution is not justified. From dimensional considerations, deriving $s_h = \theta F^2$ from (16c) as an estimate for the surface slope (assuming $\theta \sim O(1)$) and exploiting the scales (47) and (48) introduced above, it follows that the advective non-linearity is negligible when

\[
L_x^* > D_0^* \eta. \tag{50}
\]

Such condition is easily satisfied for isotropic turbulence, whereas it may require a long basin for large anisotropy $\eta$. The condition (50) is independent of wind velocity, since both the gravitational and the inertial term are proportional to $U_w^*$. In fact, the former depends on $\tau_{*,0}^*$, and hence on $U_w^*/2$ through (45), and the latter on $U_0^*2$, which is linearly related to $U_w^*/2$ through (47).

4.5. Baroclinic effects

The simulated values of velocity is approximately 2.2, and the ratio predicted by (47) is 2.23. Equation (48) can be used to find an estimate of the vertical eddy viscosity as a function of the anisotropy $\eta$, which depends on the vertical density gradient. For instance, using $D_0^* = 10 m$, $U_w^* = 4 m/s$ and $c_w^*$ as in [38], we find $\nu_{*,0}^* = 6 \cdot 10^{-3} m^2 s^{-1}$ ($U_0^* = 0.6 m/s$) for $\eta = 10^2$, and $\nu_{*,0}^* = 6 \cdot 10^{-4} m^2 s^{-1}$ ($U_0^* = 0.6 m/s$) for $\eta = 10^4$. For a weaker wind, $U_w^* = 1 m/s$, $\nu_{*,0}^* = 1.5 \cdot 10^{-3} m^2 s^{-1}$ ($U_0^* = 0.15 m/s$) for $\eta = 10^2$, and $\nu_{*,0}^* = 1.5 \cdot 10^{-4} m^2 s^{-1}$ ($U_0^* = 0.15 m/s$) for $\eta = 10^4$.

An important question is concerned with the time scale necessary to reach the hydrodynamic steady state. A first approximation can be obtained by considering the vertical diffusion of momentum, which is affected by the vertical eddy viscosity $\nu_{*,0}^*$. Introducing the estimate (48), the time scale reads

\[
T = \frac{D_0^*}{\nu_{*,0}^*} = \frac{D_0^*}{U_w^*} \sqrt{\frac{\nu_{*,0}^*}{c_w^*}}. \tag{49}
\]
velocity tends to vanish there. Then, the flow field is appreciable only in the upper region of the basin, giving rise to the preliminary conditions for the development of the thermocline.

The baroclinic terms take time to produce visible effects, since the fluid with different temperature must be transported and accumulated. At the beginning the circulation develops as in the barotropic case, then it changes because of the growth of horizontal density gradients and, asymptotically, it would tend again towards the barotropic solution when the complete mixing is realized. The temporal scale for the vertical mixing process depends on stratification and wind shear stress. [34] propose an estimate that is linearly proportional to the Richardson number in the case of strong stratification, and proportional to its square root for weak stratification; such time scale for mixing is always significantly larger than the time for hydrodynamic adaptation.

Finally, it is worth noting that the baroclinic terms in the momentum equation are relevant only for weak winds or deep lakes, as it is demonstrated in Appendix A. In fact, stronger winds produce higher velocities in the lake, the order of magnitude the momentum diffusion terms increases and also the surface slope becomes steeper. On the other hand, the deformation of the density profiles has an upper limit given by the initial difference of temperature (the limit being approximately proportional to $\Delta \rho^* D^*/L^*_x$), and the baroclinic term cannot balance the other terms in the governing equation (1).

5. Discussion

The derivation of the analytical solutions for the flow field in response to wind inhomogeneity has allowed us to identify the governing parameters. We have shown that a very simple solution can be obtained when the elongated basin can be described as a single layer. That is the case of spring or autumn conditions, when the winter or summer stratification disappears and the mixed water body reacts to wind entirely. On the contrary, the solution derived in the case of a two-layer system can be used when the presence of the thermocline separates two quasi-homogeneous layers, epilimnion and hypolimnion, with a significant difference in density between them. In each layer of our model it is possible to retain a mild vertical temperature gradient determining a density profile and a reduced vertical eddy viscosity $\nu^*_z$, approximately constant. In the case of strong temperature gradient in a thin layer, our simplified model considers such layer as a thermocline, which is not solved explicitly but governs the transmission of shear stress among the two adjacent layers.

As a matter of fact, the lake response to the wind is not constant during the year, and can be reproduced using different models and different values of the characteristic parameters. Figure 9 clearly shows how the circulation within the lake can change from a single layer situation (Figure 9a), where the temperature is assumed constant along the vertical, to a two-layer system. The density gradient in the thermocline, responsible for suppression of momentum exchange and represented by the parameter $\kappa_T$, can influence the intensity of the circulation and the involvement of the hypolimnion in the motion (Figure 9b). When the stratification in the thermocline becomes stronger, only the upper layer of the lake is affected by the wind (Figure 9c); wind energy is transmitted to a smaller water mass and hence velocities are larger. The parameter $\alpha_E$ (defined by (29a) for the epilimnion) is influenced by two opposite effects: on the one hand, the depth $D^*_E$ of the epilimnion is smaller than the total depth $D^*$ of the homogeneous lake, and then $\alpha_E$ would tend to be smaller than $\alpha$ defined by (16) for the total single-layer lake; on the other hand, the ratio between horizontal and vertical eddy viscosities increases because a weak continuous stratification can be usually seen also in the epilimnion. Hence, $\alpha_E$ can become larger than $\alpha$ characteristic of the homogeneous single layer.

In general, the parameter $\alpha_E$ is strongly influenced by the physical dimensions of the lake. In stratified conditions the range of variation of the epilimnion thickness is limited, since it is related to the depth of penetration of the solar radiation. On the contrary, the horizontal dimension can change of several orders of magnitude considering small alpine lakes in narrow valleys or great lakes. Hence $\alpha_E$ tends to become very small as far as the horizontal dimension of the basin grows.

Concerning the second factor affecting the magnitude of $\alpha_E$ (and $\alpha$ as well), i.e. the turbulence anisotropy, it is hard to find reliable and complete values of both the horizontal and the vertical eddy viscosity in literature. It is worthwhile to note that the available estimates of the horizontal viscosity ($\nu^*_x$, $\nu^*_y$) must be inferred
from the observed horizontal diffusivity [23, 21, 25]. In most cases they are not very accurate and usually depend on the scale of observation, since they are evaluated by means of tracers diffusion, which accounts also for dispersive effects (i.e. associated with large scale velocity gradients) [21]. Similarly, the vertical viscosity is estimated by analogy with diffusivity using different methods, like the heat budget method, the analysis of temperature microstructure profiles, the diffusion of tracers [5, 42]. Field measurements span a large spectrum of values for both the horizontal and the vertical coefficients: the former can vary from $10^{-2}$ to $10^2 \, m^2/s$, the latter from $10^{-6}$ to $10^{-2} \, m^2/s$. As a consequence of the differences in geometry and turbulence characteristics, we have ascertained a large variability of $\alpha$, which may vary for instance from $10^{-3}$ (lake Michigan, data from [2]) to 0.7 (lakes Kinneret and Biwa, data from [24] and [1]).

It is important to note that the horizontal eddy viscosity is not usually considered as a crucial parameter for numerical simulations [38]. Contradicting such generally assumed belief, the present analysis attributes relevance to the anisotropy of the eddy coefficients when wind forcing inhomogeneity occurs.

In fact, the horizontal viscosity $\nu_h^*$ plays a crucial role in determining the solution, since it is responsible for the lateral transfer of momentum. This physical mechanism reduces the transversal gradient of velocity and hence tends to support the vertical circulation instead of the planimetric one, which is obviously associated with horizontal gradients. On the other hand, larger values of the vertical viscosity $\nu_v^*$ tend to reduce the magnitude of the velocity and consequently the intensity of the transversal gradient originated by a lateral variation of the applied surface stress. Thus the solution is determined by the balance between the vertical and the horizontal fluxes of momentum, as the structure of the parameter $\alpha$ suggests.

In a two-layer system, the role of the hypolimnion on the epilimnetic motion $u_E$, given by (30), is described by the parameter $\alpha_H$ (characteristic of the hypolimnion itself), and by $\psi$ and $\kappa_T$, both representing the interaction among different layers according to (29) and (35). For small values of the hypolimnetic parameter $\alpha_H$ (and hence of $\chi_H$), $u_E$ is independent of it (since $\tanh(\chi_H)/\chi_H \to 1$ in (71), see Appendix B.2), and depends only on the sum of $\psi$ and $\kappa_T$. These two values are interrelated: an increase of $\kappa_T$ reflects into an increase of $\psi$ because turbulence is not efficiently transferred to the hypolimnion through the strongly stratified thermocline.

Finally, it is necessary to discuss the choice of the bottom boundary condition: the no-slip condition with a constant eddy viscosity coefficient is probably not adequate to reproduce the real situation at the bottom. However, a detailed discussion of this issue is beyond the scope of this paper, where we have considered the whole spectrum of possibilities ($\kappa$ between 0 and $\infty$). Nevertheless, such condition plays a minor role for epilimnetic currents in two-layer systems with a well developed thermocline, since in many cases the hypolimnion is not significantly involved in water motion.

6. Conclusions

In the present contribution a simplified model has been proposed to identify the main parameters influencing the response of a lake to spatial inhomogeneities of wind forcing. An analytical solution has been derived in the case of an elongated rectangular basin with constant depth and a laterally varying wind stress. It has been found that a few dimensionless parameters are able to represent the effects of the major aspects of the problem (geometrical features of the basin, turbulence anisotropy associated with thermal stratification, thermocline strength).

To determine the prevailing motion in lakes, two types of circulation compete in a simplified geometry: in the vertical plane, as in the case of a uniform wind forcing on the surface, and in the planimetric plane, caused by a spatial wind inhomogeneity. Wind shear stress variations usually occur because of sheltering effects due to the surrounding morphology or because of wind reduction associated with vegetation at the lake’s edge. In this work we have assumed a sinusoidal lateral variation, considering the first and second mode as being representative of typical sheltering.

Two different models have been considered: the single layer model, representing a mixed lake, and the two-layer model, resulting from the presence of a well developed thermocline. In the former case, the single parameter $\alpha$, given by (16), governs the hydrodynamic response: a shallow lake with isotropic turbulence (small $\alpha$) tends to develop planimetric circulation, whereas deep lakes with turbulence anisotropy (large
A. Negligibility of the baroclinic term

The magnitude of the baroclinic term can be compared with that of the barotropic term, \( g^* \partial h^*/\partial x^* \), in the momentum equation (1) by means of dimensional analysis. Introducing a boundary-fitted coordinate \( \zeta = z^*/h^* \) and expressing the density variation as \( \rho^* = \Delta \rho^* r(\zeta) \) in terms of its vertical range \( \Delta \rho^* \) within the layer, the baroclinic term reads

\[
\frac{g^*}{\rho_0^*} \frac{\partial}{\partial x^*} \int_{z^*}^{h^*} \rho^* dz^* = g^* \frac{\partial h^*}{\partial x^*} \frac{\Delta \rho^*}{\rho_0^*} \int_{\zeta}^{1} r d\zeta + \frac{g^* h^* \Delta \rho^*}{L_x^* \rho_0^*} \int_{\zeta}^{1} \frac{\partial r}{\partial x} d\zeta, \tag{51}
\]

where \( x = x^*/L_x^* \). The first term on the right hand side of (51), which derives from the dependence of \( \zeta \) on \( x^* \) through \( h^* \), is always negligible with respect to the barotropic term \( g^* \partial h^*/\partial x^* \) because \( \Delta \rho^*/\rho_0^* \ll 1 \) (typically \( \sim O(10^{-3}) \)). On the contrary, the last term depends on the longitudinal variation of density and can compete with the barotropic term. In fact, using the scale \( D_0^* \) for the surface elevation \( h^* \), the whole baroclinic term can be rewritten as

\[
g^* \frac{\partial h^*}{\partial x^*} \frac{\Delta \rho^*}{\rho_0^*} (R_1 + R_2), \quad R_1 = \int_{\zeta}^{1} r d\zeta + \zeta r, \quad R_2 = \frac{1}{s_h} \frac{D_0^*}{L_x^*} \int_{\zeta}^{1} \frac{\partial r}{\partial x} d\zeta \tag{52}
\]

Although \( R_1 \sim O(1) \) and then gives a vanishing contribution, the order of magnitude of \( R_2 \) may be comparable with \( (\Delta \rho^*/\rho_0^*)^{-1} \) if \( s_h \) is much smaller than \( D_0^*/L_x^* \). It is also important to note that for self-similar profiles \( \partial r/\partial x = 0 \) and hence \( R_2 = 0 \).
An estimate of the importance of the term $R_2$ in the momentum equation (12) can be assessed by comparing its order of magnitude,

$$\frac{s_h \Delta \rho}{F^2} R_2 \sim \frac{g^* D_0^* \Delta \rho^*}{\tau_{u0}^* L_z^*},$$

(53)

with the vertical momentum flux, $\partial (\nu \partial u/\partial z)/\partial z \sim O(1)$. Comparing the two quantities, it is clear that the relevance of the baroclinic term decreases quadratically as the wind velocity $U_w^*$ increases ($\tau_{u0}^* \propto U_w^* \alpha$) or the depth $D_0^*$ decreases.

The effect of the term associated with the presence of an overlying layer, which represents the tilting of the interface in a two-layer system and has the order of magnitude

$$\frac{g^* s_s \Delta \rho^*}{\tau_{u0}^* L_z^*},$$

(54)

(where $D_s^*$ is the scale for the thickness $H_s^* - h_s^*$ of the overlying layer), is included in the definition of the parameter $\theta$ and thus determined by means of the steady state integral condition.

B. Details of the solution procedure

B.1. One layer

The linearity of the differential problem suggests that we can split the solution into two parts in the form

$$u = u_0(z) + u_1(y, z),$$

(55)

which can be substituted into (14), (15), and (17) to give

$$\left[\frac{\partial^2 u_0}{\partial z^2} - \theta\right] + \left[\frac{\partial^2 u_1}{\partial z^2} + \alpha^2 \frac{\partial^2 u_1}{\partial y^2}\right] = 0,$$

(56)

$$\left[\frac{\partial u_0}{\partial z} - 1\right] + \left[\frac{\partial u_1}{\partial z} - \phi_w \cos(n \pi y)\right] = 0 \quad (z = 1),$$

(57)

$$\left[\frac{\partial u_0}{\partial z} - \frac{u_0 - u_{b0}}{\kappa}\right] + \left[\frac{\partial u_1}{\partial z} - \frac{u_1 - u_{b1}}{\kappa}\right] = 0 \quad (z = 0),$$

(58)

$$\frac{\partial u_1}{\partial y} = 0 \quad (y = 0, 1),$$

(59)

$$\int_0^1 u_0 \, dz + \int_1^1 \int_0^1 u_1 \, dz \, dy = 0.$$ 

(60)

At the bottom boundary condition (58) the slip velocity has been divided as $u_b = u_{b0} + u_{b1}(y)$.

Having separated the original differential system into two subsystems, it is easy to see that the problem for $u_0$ is exactly the same of the plane flow and gives the solution (22). On the other hand, the problem for $u_1$ admits a solution in the form

$$u_1 = a_0 + \left[a_1 \exp(kz) + a_2 \exp(-kz)\right] \sin\left(\frac{ky}{\alpha}\right) +$$

$$+ \left[a_3 \exp(kz) + a_4 \exp(-kz)\right] \cos\left(\frac{ky}{\alpha}\right),$$

(61)

where $a_j$ ($j=0,4$) are integration constants and $k$ is the wavenumber to be determined. The lateral boundary conditions (59) determine the lateral structure: the condition at $y = 0$ requires that $a_3 = a_4 = 0$, while the satisfaction of the condition at $y = 1$ needs that $k = n \pi \alpha$, with $n$ integer. Moreover, the integral condition (60) gives $a_0 = 0$. These conditions are compatible with the lateral structure (19) of the wind forcing; the
solubility of the problem imposes a sinusoidal lateral structure also for \( u_{b1} \). Thus, the surface and bottom boundary conditions, (57) and (58), can be used to determine \( a_3 \) and \( a_4 \), which completely determine the solution. Introducing

\[
\begin{align*}
  u_1(y) &= \tilde{u}_1 \cos (n\pi y), \\
  u_{b1}(y) &= \tilde{u}_{b1} \cos (n\pi y),
\end{align*}
\]

(62)

it is easy to find the solution (24)-(25).

As the procedure suggests, analytical solutions can be easily found for each Fourier component of the lateral distributions of the wind forcing. On the other hand, the solution (24), though considering only a single mode, is able to delineate the role of the main parameter governing the phenomenon.

Finally, the average surface slope \( s_h \) can be expressed from (16) in the general form

\[
s_h = \frac{\theta T^2 - g_s \frac{\partial H^*}{\partial y}}{1 - g_s},
\]

(63)

and specified using the condition on \( \theta \) derived as in the case of plane flow

\[
\theta = \frac{3}{2} \left( \frac{1}{2} + 2u_{b0} \right).
\]

(64)

B.2. Two layers

Since the dimensional scales are different in the two layers, it is necessary to consider the dimensional version of the shear stress at the interface

\[
\tau^*_t = \gamma^*_E \left( u^*_E |_{z=0} - u^*_H |_{z=1} \right),
\]

(65)

which can be decomposed into two terms as follows

\[
\tau^*_t = \tau^*_{T0} [1 + \phi_T \cos(n\pi y)],
\]

(66)

where the excess ratio at the interface \( \phi_T \) is introduced by analogy with the excess ratio at the free surface \( \phi_w \) defined in (20).

Firstly, we consider the lower layer, adapting the dimensionless solution to the corresponding scales \((D^*_0 = D^*_E, \rho^*_0 = \rho^*_E, \nu^*_0 = \nu^*_E, U^*_0 = U^*_E) = (\tau^*_{T0} D^*_E)/(\rho^*_E \nu^*_E)\). Assuming no-slip conditions at the bottom, the dimensional velocity (27) at the top of the layer \((z = 1)\) reads

\[
u^*_H |_{z=1} = \frac{\tau^*_{T0} D^*_H}{\rho^*_H \nu^*_E} \left[ \frac{1}{4} + \phi_T \frac{\tanh(\chi_H)}{\chi_H} \cos(n\pi y) \right].
\]

(67)

Analogously, in the upper layer \((D^*_0 = D^*_E, \rho^*_0 = \rho^*_E, \nu^*_0 = \nu^*_E, U^*_0 = U^*_E) = (\tau^*_{w0} D^*_E)/(\rho^*_E \nu^*_E)\); moreover \( \kappa = \kappa_T \), where the subscript \( T \) is used because \( \kappa \) comes from the bottom boundary condition of the upper layer, which coincides with the thermocline interface) the dimensional form of the velocity (24) at the bottom of the layer \((z = 0)\) reads

\[
u^*_E |_{z=0} = \frac{\tau^*_{w0} D^*_E}{\rho^*_E \nu^*_E} \left[ \frac{1}{4(1 + 3\kappa_T)} + \frac{\phi_T \cos(n\pi y) \tanh(\chi_E)}{\chi_E \tanh(\chi_E) + 1} \right] \kappa_T \phi_w \cos(n\pi y) + 2(1 + 3\kappa_T) + \kappa_T \chi_E \sinh(\chi_E) + \cosh(\chi_E).
\]

(68)

Substituting from (67) and (68) into (65)-(66), an equation is obtained with the two unknowns \( \tau^*_{T0} \) and \( \phi_T \), which can be solved considering the terms constant in \( y \) and those proportional to \( \cos(n\pi y) \). Thus, the average shear stress and the excess ration at the interface turn out to be linearly proportional to the average wind stress:

\[
\tau^*_{T0} = \sigma \tau^*_{w0}, \quad \phi_T = \frac{\phi_w}{\sigma \Psi_T},
\]

(69)
where

\[ \sigma = \frac{-1}{2(1 + 3\kappa T + 3\psi/4)}, \]

(70)

\[ \Psi_0 = \psi \frac{\tanh(\chi H)}{\chi H} + \kappa_T, \]

(71)

\[ \Psi_1 = \cosh(\chi E) + \Psi_0 \chi_E \sinh(\chi E). \]

(72)

In this way the solution for the dimensionless velocity in the epilimnion, \( u_E \), and in the hypolimnion, \( u_H \), can be obtained as in (30) and (31); note that the latter relationship is expressed in terms of \( \tilde{u}_H = u_H \psi \sigma \), using the scale \( U^*_E \) also for the hypolimnion.

The average tilting of the thermocline can be determined as follows. For this sake, we introduce \( s_T \) as the top surface slope of the hypolimnion and \( s_E \) for the free surface slope (top of the epilimnion); thus \( s_s = s_E - s_T \) in the hypolimnion. The general solution (64) for \( \theta \) can be written as \( \theta_H = 3/2 \) (no-slip conditions at the hypolimnion bottom) and \( \theta_E = 3(2 + \psi + 4\kappa_T)/[4 + 3(\psi + 4\kappa_T)] \) for the layer \( H \) and \( E \), respectively. Recalling that in this case \( g_{sH} = \rho^*_E/\rho^*_H \sim 1 \) for the hypolimnion and \( g_{sE} \approx 0 \) for the epilimnion, the condition (63) gives

\[ s_T = \frac{3 \sigma \tau_{w0}^*}{2 \rho_H g^* D_H^* - g_{sH} s_E} \left[ \frac{1}{1 - g_{sH}} \right], \]

(73)

\[ s_E = \frac{3(2 + \psi + 4\kappa_T)}{[4 + 3(\psi + 4\kappa_T)] \rho^*_E g^* D_E^*} \tau_{w0}^*. \]

(74)

Then, substitution from (74) into (73) gives

\[ s_T = - \left[ 1 + \frac{D_E^*}{D_H^* (2 + \psi + 4\kappa_T)} \right] \frac{s_E}{(\rho^*_H/\rho^*_E - 1)}, \]

(75)

where it clearly appears that the thermocline tilt is opposite to free surface slope and that it can become very large when \( \rho^*_H \approx \rho^*_E \).

**Notation.**

The main symbols are listed below in alphabetical order. Note: (*) indicates that a superscript star * in the text denotes the dimensional version of the variable.
E, T, H subscripts denoting variables in epilimnion, thermocline, and hypolimnion,

$D_0, L_x, L_y, F^2$ vertical, longitudinal and lateral dimensions of the idealized basin,

$g^*$ gravity acceleration,

$g_s$ modified gravity acceleration (*), see equation (11c),

$h$ elevation of the top surface of the main layer (*),

$H$ elevation of the top surface of the overlying layer (in a two layer system) (*),

$n$ lateral mode of wind forcing, see equation (19),

$s_h$ longitudinal slope of the free surface of the main layer,

$s_s$ additional longitudinal gradient of the free surface due to an overlying layer,

$u, v, w$ components of the velocity vector (*),

$u_b$ friction velocity at the layer bottom (*),

$u_0$ vertical structure of the plane-flow velocity,

$\tilde{u}_1$ vertical structure of the velocity lateral correction,

$U$ dimensionless layer-averaged velocity,

$U^*_0$ scale of velocity, see equation (10),

$x, y, z, t$ longitudinal, lateral, vertical and temporal coordinates (*),

$\alpha$ dimensionless parameter combining geometry and anisotropy, see equation (16a),

$\gamma^*_b$ drag coefficient, see equation (5),

$\delta$ width-to-depth ratio, see equation (16a),

$\Delta \rho$ density variation within the main layer (*), see equation (11d),

$\eta$ turbulence anisotropy (lateral over vertical viscosity), see equation (13),

$\theta$ dimensionless barotropic gradient term, see equation (16c),

$\kappa$ dimensionless drag coefficient (inverse of $\gamma^*_b$), see equation (15),

$\nu_x, \nu_y, \nu_z$ principal components of the eddy viscosity tensor (*),

$\rho^*, \rho^*_0$ fluid density, and its reference value in the main layer,

$\rho^*_s$ reference density of the overlying layer,

$\sigma, \Psi_0, \Psi_1$ dimensionless parameters, see equations (70), (71), (72) in Appendix B.2,

$\tau_w$ shear stress applied at the top of the main layer (*),

$\phi_w$ excess shear stress ratio, see equation (19),

$\psi$ dimensionless ratio between scales of epilimnion and hypolimnion, see equation (29),

$\omega$ circulation number, see equation (39).

References


Figure 1: Sketch of a simplified lake, illustrating the basic notation, the wind forcing on the surface and the two types of circulation.

Figure 2: Lateral view of the two-layer case: equations (1)-(2) are solved for the main layer, but take into account the possible effects of an overlying layer on the pressure gradients.

Figure 3: The dimensionless velocity field in the cross-section of a single-layer system for $\alpha = 0.3$ and $\phi_w = 0.4$, for the first lateral mode of wind forcing ($n = 1$).
Figure 4: The dimensionless velocity field in the cross-section of a single-layer system for $\alpha = 3$, $\phi_w = 0.4$, $n = 1$.

Figure 5: Contour plot of the circulation number $\omega$ given by (39) in the $\alpha$-$\phi_w$ plane for a single layer with no-slip bottom boundary condition and for the first lateral mode of wind forcing ($n = 1$). The tendency towards horizontal/vertical circulation is highlighted. Note that larger values of $\alpha$ are associated with stronger turbulence anisotropy or deeper lakes, and $\phi_w$ represents the wind excess ratio.
Figure 6: The dimensionless velocity field in the cross-section of a two-layer system for $\alpha_E = 0.27$, $\alpha_H = 0.18$, $\psi = 21.1$, $\kappa_T = 22.2$ and $\phi_w = 0.4$, for the first lateral mode of wind forcing ($n = 1$).

Figure 7: Contour plot of the circulation number $\omega$ given by (39) for the epilimnion of a two-layer system, for the first lateral mode of wind forcing ($n = 1$): $\omega$ is plotted in the $\alpha_E$-$\phi_w$ plane in the case of large $\kappa_T$, equation (36). Note that larger values of $\alpha_E$ are associated with stronger turbulence anisotropy or deeper epilimnion, and $\phi_w$ represents the wind excess ratio.
Figure 8: The dimensionless velocity field in the cross-section for the second mode of lateral variation of the wind stress ($n = 2$) and $\phi_w = -0.4$: (a) single layer with $\alpha = 0.3$; (b) upper and (c) lower layer of a two-layer system with $\alpha_E = 0.3$, $\alpha_H = 0.3$, $\psi = 10$, $\kappa_T = 10$.

Figure 9: Example of the seasonal behavior of the dimensionless velocity field in the cross-section, following the development of thermal stratification, for the first mode of lateral variation of the wind stress ($n = 1$) and $\phi_w = 0.2$: (a) homogenous lake (single layer, $\alpha = 0.05$); (b) stratification with a weak thermocline (two layers, $\alpha_E = 0.2$, $\alpha_H = 0.2$, $\psi = 5$, $\kappa_T = 1$); (c) stratification with a strong thermocline (two layers, $\alpha_E = 0.2$, $\alpha_H = 0.2$, $\psi = 5$, $\kappa_T = 100$). The depths of epilimnion and hypolimnion are assumed equal to half the total depth; in plots (b) and (c) the dimensionless vertical coordinate $z$ is scaled using the whole depth.